# On orthogonal mappings and their dimensions

by

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If $f_1\colon X\to Y_1$ and $f_2\colon X\to Y_2$ are mappings, the *diagonal* of $f_1$ and $f_2$ is understood to mean the mapping

$$f_1\triangle f_2\colon X\to Y_1\times Y_2$$

defined by $(f_1\triangle f_2)(x)=\big(f_1(x),f_2(x)\big)$ for $x\in X$. As usual, the dimension $\dim f$ of a mapping $f\colon X\to Y$ is given by the formula

$$\dim f=\sup_{y\in Y}\dim f^{-1}(y)\,,$$

where $\dim X$ denotes the ordinary covering dimension of a space $X$. We shall say that the mappings $f_1$ and $f_2$ of a space $X$ are *orthogonal*, written as $f_1\perp f_2$, if

$$\dim(f_1\triangle f_2)=0\,;$$

and this is, of course, equivalent to the condition that the set $f_1^{-1}(y_1)\cap f_2^{-1}(y_2)$ is empty or 0-dimensional for every $y_i\in Y_i$ $(i=1,2)$. Thus, for instance, the projections of a square onto its sides are orthogonal mappings. Similarly, the projections of an arbitrary Cartesian product onto the axes are orthogonal mappings. If $\dim f_1=0$, then $f_1\perp f_2$ for any $f_2$.

The concept of orthogonality, as proposed above, will be shown in the present note to have some connections with other aspects of dimension theory. In particular, by means of orthogonal mappings we can estimate the so-called strong dimension, introduced by Katětov and Smirnov for mappings of metric spaces (not necessarily separable). There is also a relation between the existence of certain orthogonal mappings and the problem recently raised by Šersnev (see 3.2 below).

All spaces considered throughout are assumed to be metric. The distance between points $x_1$ and $x_2$ is denoted by $\varrho(x_1,x_2)$. The diameter of a set $A$ is denoted by $\delta(A)$.

**1. Preliminaries on closed mappings.** Recall that a mapping $f\colon X\to Y$ is said to be *closed* if $f$ maps closed subsets of $X$ onto closed subsets of $Y$.

1.1. *The diagonal* $f_1 \triangle f_2$ *is a closed mapping provided that* $f_1$ *and* $f_2$ *are closed mappings.*

This readily follows from the definition of the diagonal and from the fact that the metric spaces satisfy the first axiom of countability.

1.2. *Let* $f\colon X \to Y$ *be a closed mapping. If* $y \in Y$ *and* $U$ *is an open neighbourhood of* $f^{-1}(y)$ *in* $X$, *then there exists an open neighbourhood* $V$ *of* $y$ *in* $Y$ *such that* $f^{-1}(V) \subset U$.

The spaces $X$ and $Y$ in 1.2 could even be non-metrizable. This is a simple consequence of well-known theorems concerning the quotient topology (see [2], p. 95 and 97).

1.3. *Let* $f_1\colon X \to Y_1$ *and* $f_2\colon X \to Y_2$ *be closed mappings. If* $y_i \in Y_i$ *and* $U$ *is an open neighbourhood of* $f_1^{-1}(y_1) \cap f_2^{-1}(y_2)$ *in* $X$, *then there exist open neighbourhoods* $V_i$ *of* $y_i$ *in* $Y_i$ $(i = 1, 2$; *respectively) such that* $f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \subset U$.

Proof. Since we have

$$(f_1 \triangle f_2)^{-1}(V_1 \times V_2) = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \tag{1}$$

for arbitrary subsets $V_i$ of $Y_i$ $(i = 1, 2)$, it is sufficient to apply 1.1 and 1.2.

1.4. (the Hurewicz inequality) *If* $f\colon X \to Y$ *is a closed mapping, then*

$$\dim X \leqslant \dim Y + \dim f .$$

Proof. By the Stone theorem on paracompactness of metric spaces (see [2], p. 160), a generalization of the original Hurewicz inequality, due to Morita (see [4], p. 161), gives

$$\dim X \leqslant \operatorname{Ind} Y + \dim f ,$$

where $\operatorname{Ind} Y$ denotes the "big" inductive dimension, defined by means of neighbourhoods for closed subsets. But we have $\operatorname{Ind} Y = \dim Y$, according to the Katětov theorem (see [1], p. 361 and 362).

1.5. *If* $f\colon X \to Y$ *is a closed mapping,* $\dim f = 0$ *and* $B \subset Y$, *then*

$$\dim f^{-1}(B) \leqslant \dim B .$$

Proof. Clearly, the partial mapping

$$f' = f|f^{-1}(B)\colon\ f^{-1}(B) \to B$$

is closed and $\dim f' \leqslant \dim f = 0$. So 1.4 yields the desired inequality.

1.6. *If* $f\colon X \to Y$ *is a closed mapping,* $\varphi\colon Y \to Z$ *and* $\dim \varphi = 0$, *then*

$$\dim f = \dim \varphi f .$$



Proof. The inequality $\dim f \leqslant \dim \varphi f$ trivially holds, the dimension being a monotone function for metrizable spaces. Let $z \in Z$ and consider the mapping

$$f'' = f|(\varphi f)^{-1}(z)\colon\ (\varphi f)^{-1}(z) \to \varphi^{-1}(z) ,$$

which is closed and has dimension $\dim f'' \leqslant \dim f$. It follows from 1.4 that

$$\dim (\varphi f)^{-1}(z) \leqslant \dim \varphi^{-1}(z) + \dim f'' \leqslant \dim f ,$$

because $\dim \varphi^{-1}(z) \leqslant \dim \varphi = 0$.

Remark. The hypothesis that the mapping $f$ is closed cannot be omitted in 1.6. A suitable counter-example arises if we take $f = p$ and a constant mapping $\varphi$ on the Cantor set in the example which is given at the beginning of the next section.

**2. Properties of orthogonal mappings.** We start with an old example of a space.

2.1. (the Knaster-Kuratowski example) *There exist a subset* $K$ *of the plane and a mapping* $p\colon K \to C$ *of* $K$ *onto the Cantor set* $C$ *such that* $\dim K = 1$ *and* $\dim p = 0$.

Indeed, let $K$ be a set obtained from the Knaster-Kuratowski biconnected set (see [3], p. 241) by removing the vertex $(\frac{1}{2}, \frac{1}{2})$. Then of course $K$ is 1-dimensional and the projection $p$ of $K$ from the point $(\frac{1}{2}, \frac{1}{2})$ onto the Cantor set on the $x$-axis has point-inverses homeomorphic either to rationals or to irrationals of an interval. Hence $\dim p = 0$.

Now put $p_1(x) = 1$ for $x \in K$, and $p_2 = p$. Then the mappings $p_1$ and $p_2$ of $K$ are orthogonal, and the sets $B_1 = \{1\}$ and $B_2 = C$ are 0-dimensional, but their inverses have the 1-dimensional intersection $p_1^{-1}(B_1) \cap p_2^{-1}(B_2) = K$. Here only one mapping, namely $p_1$, is closed. This is in contrast to what follows.

2.2. *If* $f_1\colon X \to Y_1$ *and* $f_2\colon X \to Y_2$ *are orthogonal closed mappings,* $B_i \subset Y_i$ *and* $\dim B_i = 0$ $(i = 1, 2)$, *then*

$$\dim\big(f_1^{-1}(B_1) \cap f_2^{-1}(B_2)\big) \leqslant 0 .$$

Proof. Let $f = f_1 \triangle f_2$ and $B = B_1 \times B_2$. Since the diagonal $f$ is a closed mapping by 1.1, and $\dim f = 0$, we have

$$\dim\big(f_1^{-1}(B_1) \cap f_2^{-1}(B_2)\big) = \dim f^{-1}(B) \leqslant \dim B \leqslant \dim B_1 + \dim B_2 = 0 ,$$

according to (1) and 1.5.

2.3. *If* $f$ *and* $g$ *are orthogonal mappings of a space* $X$, $g$ *is closed,* $\psi$ *is a mapping of the space* $g(X)$, *and* $\dim \psi = 0$, *then* $f$ *and* $\psi g$ *are also orthogonal mappings.*

Proof. Let $y \in f(X)$ and $z \in \psi g(X)$. The intersection

(2) $$X' = f^{-1}(y) \cap (\psi g)^{-1}(z)$$

being a closed subset of $X$, the partial mapping $g' = g|X'$ is closed. On the other hand, we have

$$\dim g' \leqslant \dim g|f^{-1}(y) \leqslant \dim(f \triangle g) = 0\,,$$

according to the orthogonality $f \perp g$. Therefore 1.4 implies that

$$\dim X' \leqslant \dim g'(X')\,,$$

and the inequality

$$\dim(f \triangle \psi g) \leqslant \dim \psi$$

follows, since

$$X' = (f \triangle \psi g)^{-1}(y, z) \quad \text{and} \quad g'(X') \subset g\big((\psi g)^{-1}(z)\big) = \psi^{-1}(z)\,,$$

according to (1) and (2). By $\dim\psi = 0$, we get $f\psi \perp g$.

2.4. *If $f$ and $g$ are orthogonal mappings of a space $X$ and $g$ is closed, then*

$$\dim f \leqslant \dim g(X)\,.$$

Proof. Let $y \in f(X)$ and $g'' = g|f^{-1}(y)$. Then $g''$ is a closed mapping and $\dim g'' \leqslant \dim(f \triangle g) = 0$. Consequently, 1.4 gives

$$\dim f^{-1}(y) \leqslant \dim g''\big(f^{-1}(y)\big) \leqslant \dim g(X)\,.$$

Remarks. The hypothesis that the mapping $g$ is closed cannot be omitted in 2.3. In fact, taking as $f$ a constant mapping on $K$ in the Knaster-Kuratowski example (see 2.1), as $g$ the projection $p$, and as $\psi$ a constant mapping on $C$, we obtain $f \perp g$, but $f \perp \psi g$ does not hold. The same mappings are good suitable for showing that $g$ must be closed in 2.4. The inequality from 2.4 will be strengthened in the next section (see 3.1) under the assumption that the mappings $f$ and $g$ are both closed and that the image $f(X)$ is compact.

### 3. Orthogonality and strong dimension.

In view of the Knaster-Kuratowski example, the effort made to generalize the Hurewicz inequality for non-closed mappings has led to the Katětov-Smirnov modification of the dimension of a mapping [1]. A mapping $f\colon X \to Y$ is said to be *strongly 0-dimensional*, written $\mathrm{Dim} f = 0$, if for each number $\varepsilon > 0$ there exists a number $\eta > 0$ such that the inverse image $f^{-1}(B)$ of any set $B \subset Y$ with the diameter $\delta(B) < \eta$ can be represented as the union

$$f^{-1}(B) = \bigcup_{t \in T} G_t$$

of a collection $\{G_t\}_{t \in T}$ of mutually disjoint open subsets $G_t$ of $f^{-1}(B)$ with diameters $\delta(G_t) < \varepsilon$ for $t \in T$ (see [1], p. 353). The strong dimension $\mathrm{Dim} f$ of the mapping $f$ is now defined to mean the minimum integer $k = 0, 1, 2, \ldots$ (or infinity if $k$ does not exist) such that the space $X$ is the union

$$X = X_0 \cup X_1 \cup \ldots \cup X_k$$

of $k+1$ sets with $f$ strongly 0-dimensional on each of them, that is

$$\mathrm{Dim} f|X_i = 0$$

for $i = 0, 1, \ldots, k$. The mapping $f$ is called *strongly $n$-dimensional*, if $\mathrm{Dim} f = n$ (see [5], p. 209).

Strongly 0-dimensional mappings are 0-dimensional (see [1], p. 360), but the projection $p$ in the Knaster-Kuratowski example shows that a 0-dimensional mapping need not be strongly 0-dimensional. In general, we have the inequality

$$\dim f \leqslant \mathrm{Dim} f$$

(see [5], p. 214). Observe that even homeomorphisms need not be strongly 0-dimensional mappings. Indeed, taking the homeomorphism $h$ of the real line, defined by $h(x) = \arctan x$, we get $\mathrm{Dim} h = 1$.

3.1. Theorem. *If $f$ and $g$ are orthogonal closed mappings of a space $X$ and $f(X)$ is compact, then*

$$\mathrm{Dim} f \leqslant \dim g(X)\,.$$

Proof. Let $\dim g(X) = k < \infty$. Since the space $g(X)$ is metrizable, there exists a decomposition

$$g(X) = Z_0 \cup Z_1 \cup \ldots \cup Z_k$$

such that $\dim Z_i = 0$ for $i = 0, 1, \ldots, k$ (see [1], p. 361 and 362). Evidently, it is enough to prove that

$$\mathrm{Dim} f|g^{-1}(Z_i) = 0$$

for $i = 0, 1, \ldots, k$. We are going to do so, assuming that the index $i = 0, 1, \ldots, k$ is established for the rest of this proof.

By $\dim Z_i = 0$, there exists for each number $n = 1, 2, \ldots$ an open covering $\{W_t^n\}_{t \in T_n}$ of the space $Z_i$ such that

(3) $$\delta(W_t^n) < 1/n$$

[1] There is a difference between Katětov's terminology [1] and that used by Russian topologists (see [5], p. 208). Katětov has called his mappings *uniformly* 0-dimensional. We adopt here the terminology from Šersnev's work [5].

for $t \in T_n$ and $W^n_{t'} \cap W^n_{t''} = \emptyset$ for $t', t'' \in T_n$, $t' \neq t''$. Put $\bar{f} = f|g^{-1}(Z_i)$. For any set $B \subset f(X)$ and any number $n = 1, 2, \ldots$ we have

$$\bar{f}^{-1}(B) = f^{-1}(B) \cap g^{-1}(Z_i) = \bigcup_{t \in T_n} \big(f^{-1}(B) \cap g^{-1}(W^n_t)\big),$$

where all sets $g^{-1}(W^n_t)$ are open in $g^{-1}(Z_i)$, and therefore all terms in the union are mutually disjoint open subsets of $f^{-1}(B)$. It is thus sufficient to prove that for each number $\varepsilon > 0$ there exists a positive integer $n_0$ satisfying the condition: if $\delta(B) < 1/n_0$ and $t \in T_{n_0}$, then the set

$$A = f^{-1}(B) \cap g^{-1}(W^{n_0}_t)$$

can be represented as the union of mutually disjoint open subsets $G_\tau$ of $A$ with diameters $\delta(G_\tau) < \varepsilon$.

Suppose, on the contrary, that there exists a number $\varepsilon_0 > 0$ such that for each number $n = 1, 2, \ldots$ there exist a set $B_n \subset f(X)$ and an index $t_n \in T_n$ such that

(4) $$\delta(B) < 1/n$$

and the set

(5) $$A_n = f^{-1}(B_n) \cap g^{-1}(W^n_{t_n})$$

cannot split into mutually disjoint sets $G_\tau$, open in $A_n$ and satisfying the inequality $\delta(G_\tau) < \varepsilon_0$.

The last statement implies that the set $g^{-1}(W^n_{t_n})$ cannot be contained in the inverse image $f^{-1}(F)$ of any finite subset $F$ of $f(X)$. Indeed, since $W^n_{t_n} \subset Z_i$ and $\dim Z_i = 0$, we have $\dim A_n \leqslant 0$, according to (5) and 2.2. It follows that there exists an infinite sequence $y_1, y_2, \ldots$ of different points of $f(X)$ such that the sets $f^{-1}(y_n)$ and $g^{-1}(W^n_{t_n})$ intersect for $n = 1, 2, \ldots$ Choose a point

$$x_n \in f^{-1}(y_n) \cap g^{-1}(W^n_{t_n})$$

for $n = 1, 2, \ldots$ and notice that the sequence $x_1, x_2, \ldots$ must contain a convergent subsequence, since the sequence $y_1, y_2, \ldots$ contains one, by the compactness of $f(X)$, and $f$ is a closed mapping. Hence we can assume that the whole sequence $x_1, x_2, \ldots$ converges to a point $x \in X$.

For the same reason, the sequence $B_1, B_2, \ldots$ has a subsequence of sets containing points which converge to a point $y \in f(X)$, and we can assume that

(6) $$\{y\} = \operatorname*{Lim}_{n\to\infty} B_n,$$



by virtue of (4). Putting $z = g(x)$ we see that the points $g(x_n) \in W^n_{t_n}$ converge to $z$, and (3) gives

(7) $$\{z\} = \operatorname*{Lim}_{n\to\infty} W^n_{t_n}.$$

Since $f \perp g$, there exists an open covering $\{U_\tau\}_{\tau \in T}$ of the set $f^{-1}(y) \cap g^{-1}(z)$, consisting of its mutually disjoint open subsets $U_\tau$ with $\delta(U_\tau) < \varepsilon_0/3$ for $\tau \in T$. Let $u \in U_\tau$. There exists a number $\lambda_u$ such that

$$0 < \lambda_u < \varepsilon_0 \quad \text{and} \quad \lambda_u < \varrho(u, x)$$

for any $x \in f^{-1}(y) \cap g^{-1}(z) \setminus U_\tau$. The sets

$$U^*_\tau = \bigcup_{u \in U_\tau} \{x\colon\ x \in X,\ \varrho(u, x) < \lambda_u/3\}$$

are open in $X$, mutually disjoint, and $\delta(U^*_\tau) < \varepsilon_0$ for $\tau \in T$. Moreover, their union

$$U = \bigcup_{\tau \in T} U^*_\tau$$

is a neighbourhood of $f^{-1}(y) \cap g^{-1}(z)$ in $X$. Hence, according to 1.3, there exist open neighbourhoods $V$, $W$ of $y$, $z$ in $f(X)$, $g(X)$, respectively, such that $f^{-1}(V) \cap g^{-1}(W) \subset U$. By (6) and (7), we can find an index $n$ satisfying

$$B_n \subset V \quad \text{and} \quad W^n_{t_n} \subset W,$$

whence $A_n \subset U$, by (5). Thus the set $A_n$ splits into the sets $G_\tau = A_n \cap U^*_\tau$, which contradicts our supposition concerning $A_n$.

Remark. It has been stated in the paragraph preceding 3.1 that there are mappings $f\colon X \to Y$ for which two notions of dimension do not coincide, i.e. for which $\dim f < \operatorname{Dim} f$. Combining the mappings from those examples, namely $p$ or $h$, with constant mappings, one can easily verify that each hypothesis in 3.1 is necessary. However, in all known examples of this kind either the mapping $f$ is not closed or the image $f(X)$ is not compact. By a theorem of Šersnev (see [5], p. 215), the conditions $\dim f = 0$ and $\operatorname{Dim} f = 0$ are equivalent provided that $f$ is closed and $f(X)$ is compact. Šersnev's problem whether

$$\dim f = \operatorname{Dim} f$$

for closed $f$, and compact $f(X)$, still remains unsolved. But, in view of Theorem 3.1, the following problem is suggested.

3.2. Problem. *Does there exist, for every closed mapping $f$ of a space $X$ onto a compact space $f(X)$, a closed mapping $g$ of $X$ such that $f$ and $g$ are orthogonal and*

$$\dim f = \dim g(X)?$$

An affirmative solution of 3.2 would imply, by 3.1, an affirmative solution of Šersnev's problem. It is not known if the reverse implication holds, i.e. if these two problems are equivalent.

Note that Problem 3.2 can be rewritten, without using the notion of orthogonal mappings, as follows: do there exist, for every closed mapping $f$ of $X$ onto compact $f(X)$, a space $Y$ and a closed mapping $\varphi\colon X\to f(X)\times Y$ such that

$$\dim f = \dim Y, \qquad \dim\varphi = 0\,,$$

and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & f(X) \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\pi} & \\ & f(X)\times Y & \end{array}$$

is commutative, $\pi$ being the Cartesian projection.

Remarks. (I) Problem 3.2 reduces to a question concerning the finitely dimensional cubes $I^n$ or the Hilbert cube $I^{\aleph_0}$. Namely, if 3.2 has an affirmative solution for $f(X)\subset I^n$ $(n=1,2,\ldots,\aleph_0)$, the same is true for any compact $f(X)$ with $\dim f(X)\leqslant n$. In fact, consider a mapping $\varphi\colon f(X)\to I^n$ such that $\dim\varphi=0$ (see [1], p. 361). Since $f$ is closed and $f(X)$ is compact, $\varphi f$ is closed. But $\varphi f\perp g$ clearly implies $f\perp g$, and we have $\dim f=\dim\varphi f$, according to 1.6.

(II) Problem 3.2, in the case of compact $X$ with $\dim X\leqslant n$, reduces to 3.2 for $X=I^m$, where $m\geqslant 2n+1$ ($m$ may be $\aleph_0$). In fact, all at most $n$-dimensional compacta $X$ are, by the Menger-Nöbeling theorem, embedable in $I^{2n+1}$. So we can assume that $X\subset I^m$. Take a mapping $f^*$ of $I^m$, determined by the upper semi-continuous decomposition of $I^m$ into the sets $f^{-1}(y)$, where $y\in f(X)$, and the single points from $I^m\setminus X$. Let $g^*$ be a suitable mapping of the cube $I^m$. Then $g=g^*|X$ is closed, and $f^*\perp g^*$ implies $f\perp g$. But $\dim g(X)\leqslant\dim g^*(I^m)=\dim f^*=\dim f$, whence $\dim f=\dim g(X)$, according to 2.4.

(III) Problem 3.2, after such a modification that the inclusion $g(X)\subset I^{\dim f}$ is required, reduces to 3.2, $g(X)$ being a compact space. In fact, if $g(X)$ is compact and $\dim g(X)=\dim f$, then there exists, by the Hurewicz theorem, a mapping $\psi\colon g(X)\to I^{\dim f}$ for which $\dim\psi=0$. Thus $\psi g$ is closed, and $f\perp g$ implies $f\perp\psi g$, according to 2.3.

(IV) Problem 3.2 can also be considered as a conjecture concerning some strengthened form of the Hurewicz theorem. Namely, suppose $f$ is a mapping of a space $X$. By a generalization of the original Hurewicz theorem, due to Katětov (see [1], p. 361), there exists, for any point $y\in f(X)$, a mapping $\varphi_y\colon f^{-1}(y)\to I^{\dim f}$ such that $\dim\varphi_y=0$. Now, if $f$ is closed and $f(X)$ is compact, an affirmative solution of 3.2 would allow us to choose these mappings $\varphi_y$ so that, taking them all together, one would obtain a continuous mapping of the whole space $X$ into $I^{\dim f}$. In fact, let $g$ be a closed mapping of $X$ such that $f\perp g$ and $\dim g(X)=\dim f$. By the same result of Katětov, there exists a $\psi\colon g(X)\to I^{\dim f}$ such that $\dim\psi=0$. It suffices to put $\varphi_y=\psi g|f^{-1}(y)$, since $f\perp\psi g$, according to 2.3.

(V) Problem 3.2 has a trivial solution for two particular cases: (i) $\dim f=0$, and (ii) $\dim f=\dim X$. In fact, we can define $g$ as a constant mapping in case (i), and the identity mapping in case (ii). Moreover, case (i) yields a new proof of Šersnev's result quoted above, in the remark preceding 3.2, since Theorem 3.1 gives here $\operatorname{Dim} f=0$.



## References


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