Embedding of graphs in the projective plane

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1. Introduction. In this paper we give necessary and sufficient conditions for a graph to be embeddable in the real projective plane. It is well known that Kuratowski [1] solved the corresponding problem for the Euclidean plane. However his characterization by excluded figures does not seem appropriate in the case of the projective plane, the number of excluded figures becoming rather large.

Another characterization of plane graphs has been given by Mac Lane [2]. He proved that a graph $G$ with nullity $\geq 1$ is plane, if and only if there is a number of circuits in $G$ forming a base of the cycles modulo 2, with the property that every edge of $G$ is on at most two of these circuits.

Our characterization of graphs embeddable in the projective plane is an analogon of the theorem of Mac Lane. However we do not work with cycles modulo 2, but with integral cycles.

By $Z$ we denote the (additive) group of integers, and by $Z_n$ the group of integers modulo 2. The first homology group $H_1(G, Z)$, respectively $H_1(G, Z_2)$ of a graph $G$ is isomorphic with the group of integral cycles, respectively cycles modulo 2, the dimension of $G$ being one. If the group $H_1(G, Z_2)$ has a base of Mac Lane type, the graph $G$ can be embedded in the Euclidean plane $E$. It is well known that in this case the set of integral circuits contained in the boundaries of the bounded components of $E \setminus G$ is a base in the group $H_1(G, Z)$. Let

$$\xi: H_1(G, Z) \rightarrow H_1(G, Z_2)$$

be the homomorphism derived from the natural map $Z \rightarrow Z_2$ (the map $\xi$ can be obtained by choosing a base for $H_1(G, Z)$, and replacing the integral coefficients of the base by their cosets modulo 2). As $G$ does not contain torsion in dimension zero and one, the homomorphism $\xi$ is an epimorphism (cf. [3], p. 219). Hence if $\{s_1, \ldots, s_n\}$ is a base for $H_1(G, Z)$, the set $\{\xi(s_1), \ldots, \xi(s_n)\}$ contains a base of Mac Lane type in the group $H_1(G, Z_2)$. It follows that in the theorem of Mac Lane the cycles modulo 2 can be replaced by integral cycles. As we do not consider in this
Embedding of graphs

induces a homomorphism \( i_*: H_i(G) \to H_i(G) \) of the integral homology groups. Clearly \( H_i(G) \) is isomorphic with the infinite cyclic group \( \mathbb{Z} \); let \( \zeta \) be a generating element of \( H_1(G) \) and let \( z = i_*(\zeta) \). Then we call \( z \) a cycle corresponding to the circuit \( C \). It is clear that \( -z \) is also a cycle corresponding to the circuit \( C \). We choose this terminology to make a clear distinction between the circuit \( G \) as a geometric object and the corresponding cycle \( z \) as an algebraic object, which is an element of \( H_1(G) \).

An edge \( e \) of \( G \) is called a circuit edge if and only if \( e \) is on some circuit of \( G \). Let \( B \) be the set of all edges of \( G \) and \( B_1 \) the set of all circuit edges of \( G \). We define an equivalence relation \( T \) on the set \( B \) as follows: for every two elements \( e_1 \) and \( e_2 \) of \( B \), the relation \( e_1 e_2 \) holds if and only if \( G \) contains a circuit \( C \) with the property that \( e_1 \) and \( e_2 \) are both on \( C \). It is easy to prove that \( T \) is an equivalence relation. A subgraph \( H \) of \( G \) is called a leaf of \( G \) if the set of edges of \( H \) is an equivalence class of \( T \) and if the set of vertices of \( H \) consists of precisely those vertices of \( G \) that are incident with at least one edge in that equivalence class. It is clear that a leaf of \( G \) does not contain a cutpoint. In fact a leaf of \( G \) is a maximal subgraph without cutpoints (cf. [4], p. 83).

Consider a graph \( G \) that can be embedded into the projective plane \( \mathbb{P} \). Let \( f: G \to \mathbb{P} \) be a homeomorphism of \( G \) into \( \mathbb{P} \) and let \( f_*: H_1(G) \to H_1(\mathbb{P}) \) be the induced homomorphism of the first homology groups.

**Proposition (2.1).** The graph \( G \) is plane if and only if there exists an embedding \( f: G \to \mathbb{P} \) such that \( f_*(H_1(G)) = 0 \).

**Proof.** The proof of this proposition is nearly a direct consequence of some well known facts. At first it is well known that \( f \) induces a homomorphism \( f_*: \pi_1(G) \to \pi_1(\mathbb{P}) \) of the first homotopy groups of \( G \) and \( \mathbb{P} \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\pi_1(G) & \xrightarrow{f_*} & \pi_1(\mathbb{P}) \\
\downarrow h(G) & & \downarrow h(\mathbb{P}) \\
H_1(G) & \xrightarrow{f_*} & H_1(\mathbb{P})
\end{array}
\]

(where \( h(G) \) and \( h(\mathbb{P}) \) are the canonical homomorphisms; \( h(\mathbb{P}) \) is an isomorphism \( \pi_1(\mathbb{P}) \cong H_1(\mathbb{P}) \cong \mathbb{Z} \). From \( f_*(H_1(G)) = 0 \) it follows that \( f_*(\pi_1(G)) = 0 \).

As the two-dimensional sphere \( S^2 \) is an universal covering space of the projective plane \( \mathbb{P} \), we conclude by theorem 17.3 in [5], p. 96, that the map \( f: G \to \mathbb{P} \) can be lifted over \( S^2 \). If \( j: S^2 \to \mathbb{P} \) is the natural map.
of \( S^0 \) onto \( P \), we know that there exists a continuous map \( \varphi : G \to S^0 \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\varphi & \to & j \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & P
\end{array}
\]

As \( f \) is a homeomorphism and \( \varphi \sim fp \) we conclude that \( \varphi \) is a continuous 1-1 map of \( G \) onto \( S^0 \), hence a homeomorphism. This means that \( G \) is a plane graph.

If on the other hand \( G \) is a plane graph, one can find an embedding of \( G \) into \( P \) which factors through the plane \( B \). In that case \( I_r(B_2(G)) = 0 \) follows directly from \( H_r(B) = 0 \), and the proposition is proved.

**Proposition (2.2).** If \( G \) is a non-plane graph, \( f \) an embedding of \( G \) in the projective plane \( P \), and \( U \) a component of \( P \setminus f(G) \), then the closure \( \overline{U} \) of \( U \) with respect to \( P \) is a plane set.

Proof. Let \( i : U \to P \) be the natural embedding of \( U \) into \( P \), and let \( \epsilon : \pi_2(U) \to \pi_2(P) \) be the induced homomorphism of the first homotopy groups. If \( \epsilon_2(\pi_2(U)) = 0 \) the map \( i \) can be lifted over \( S^0 \), hence in that case it follows that \( \overline{U} \) is a plane set. So let us assume \( \epsilon_2(\pi_2(U)) \neq 0 \). We choose \( \alpha \in \pi_2(U) \) with \( \epsilon_2(\alpha) \neq 0 \). Let \( I \) denote the unit interval \([0, 1]\) of the set of real numbers, and choose a continuous map \( g : I \to \overline{U} \) with \( g \circ \alpha \). We shall prove that \( g \) can be chosen in such a way that \( g(I) \) is a circuit, having at most one point in common with the boundary \( B = \partial U \) of \( U \). Assume that \( B \cap g(I) \neq \emptyset \) and take a point \( p \in B \cap g(I) \). We shall choose a map \( h : \alpha \to h(I) = p \). As \( B \subseteq f(G) \), we know that \( B \) is locally connected. Every point \( y \in h(I) \) is contained in an Euclidean neighborhood \( V(y) \), so that \( V(y) \cap U \) is connected. The family \( \{V(y)\} \) with \( y \in h(I) \) is a covering of the set \( h(I) \). As \( h(I) \cap h(I) \) is an open set in \( h(I) \), it follows that the components of the set \( h^{-1}(V(z) \cap h(I)) = h(I) \) are elements of an open covering of \( I \). From this it follows that we can find a finite sequence of real numbers \( y_0 = 0 < y_1 < \ldots < y_n = 1 \) so that:

1. \( 0 = y_0 < y_1 < \ldots < y_{n-1} < y_n = 1 \);
2. \( h(y_i) \) and \( h(y_{i+1}) \) (\( 0 < i < n-1 \)) are contained in one of the sets \( V(z) \), say \( V(z_{i+1}) \).

Let \( p_{i+1} = h(y_i) \) \( (0 < i < n-1) \). As the set \( V(z_{i+1}) \cap U \) is connected we can join the points \( p_0 \) and \( p_n \) by an arc \( L_{i+1} \subseteq V(z_{i+1}) \cap (U \cup p_{i+1} \cup p_{i+1}) \). Let \( h_i \) be a homeomorphism of the interval \([0, 1]\) onto the arc \( L_{i+1} \) so that \( h_i(0) = p_0 \) and \( h_i(y_n) = p_n \). Then we define the map \( h_i \) as follows:

\[
h_i(y) = \begin{cases} 
  h_i(y) & \text{if } 0 < y < y_i , \\
  h(y) & \text{if } y_i < y < 1 .
\end{cases}
\]

It is clear that \( h_i \) is homotopic with \( h_i \), so \( h_i \in c_a \). Moreover we have \( B \cap h(I) \subseteq h([z_1, 1]) \). Suppose that for some \( i (1 < i < n) \) we have constructed a map \( h_i : I \to \overline{U} \) so that:

1. \( h_i(x) = u_i \);
2. \( h_i(0) = p_{n-1} \); \( h_i(y_n) = p_{n-1} \) and \( h_i(0, y_n) \subseteq U \);
3. \( B \cap h(I) \subseteq h([z_1, 1]) \).

As \( p_{n-1} \) and \( p_{n-1} \) are contained in \( V(z_{n-1}) \), we can choose a point \( z_i \in (z_{i-1}, y_{i-1}) \) so that \( h_i(z_i) \in V(z_{i+1}) \). We join the points \( h_i(z_i) \) and \( p_{i+1} \) by an arc \( L_{i+1} \subseteq V(z_{i+1}) \cap (U \cup p_{i+1}) \) so that \( L_{i+1} \cap L_{i+1} \cap h([0, 1]) = h_i(z_i) \). We choose a homeomorphism \( h_{i+1} \) of the interval \([z_i, y_{i+1}]\) onto the arc \( L_{i+1} \), with \( h_{i+1}(z_i) = h_i(z_i) \) and \( h_{i+1}(y_{i+1}) = p_{i+1} \). Then we define a map \( h_{i+1} : I \to \overline{U} \) as follows:

\[
h_{i+1}(y) = \begin{cases} 
  h_i(y) & \text{if } 0 < y < z_i , \\
  h_{i+1}(y) & \text{if } z_i < y < y_{i+1} , \\
  h(y) & \text{if } y_{i+1} < y < 1 .
\end{cases}
\]

It is clear that the map \( h_{i+1} \) is homotopic with \( h_i \), hence \( h_{i+1} \in c_a \). Moreover \( B \cap h_i(I) \subseteq h_{i+1}([z_i, 1]) \). From this construction it follows that the map \( h_i : I \to \overline{U} \) is an element of \( c_a \), and that \( B \cap h(I) = p_0 \). Let us assume that the map \( g : c_a \) was chosen in such a way that:

1. \( g(I) \) is a circuit in \( U \);
2. \( B \cap g(I) \) contains at most one point.

Now consider the natural map \( g : P \to P \setminus f(G) \). As \( g \) is a circuit in \( P \) that is not homotopic with \( 0 \) in \( P \), it is well known that \( P \setminus f(G) \) is non-plane and hence \( \epsilon_2(\pi_2(U)) \) equals to zero, and \( U \) is a plane set, which proves the proposition.

From the preceding proposition we conclude:

**Corollary (2.3).** If the graph \( G \) is embeddable in the projective plane, \( G \) contains at most one non-plane leaf.

3. In this section we give a proof of:

**Theorem (3.1).** A graph with nullity \( \geq 1 \) can be embedded in the projective plane if and only if \( G \) contains a set of circuits \( C_1, \ldots, C_n \) so that the following conditions are satisfied:

1. every edge of \( G \) is on at most two of the circuits \( C_i \) with \( 1 < i < n-1 \);
2. let \( z_i (1 < i < n) \) be the cycle corresponding to the circuit \( C_i \), the set of cycles \( z_1, \ldots, z_n \) is a generating set of the first homology group \( H_1(G) \) of \( G \);
3. the set \( \{z_1, \ldots, z_n\} \) is a maximally independent set in \( H_1(G) \).
Proof. First we show these three conditions to be necessary. Let $G$ be a graph of nullity $\geq 1$ that can be embedded in the projective plane. If $G$ is a plane graph we know from the theorem of Mac Lane, that $G$ contains a set of circuits $C_1, \ldots, C_n$ satisfying condition (1), so that the corresponding cycles $c_1, \ldots, c_n$ form a base in $H_2(G)$. It is clear that these cycles $c_1, \ldots, c_n$ together with some cycle $c_{n+1}$ satisfy also conditions (2) and (3).

Now let $G$ be a non-plane graph. By (2.3) we know that at most one leaf of $G$ is non-plane. Let $G_1, G_2, \ldots, G_m$ be the leaves of $G$; then

$$G = \bigcup_{i=1}^m G_i,$$

where $G_{i+1}$ is the subgraph of $G$ generated by the non-circuit edges of $G_i$. We assume that only $G_1$ is non-plane. Let $f$ be the embedding of $G_1$ into the projective plane $P$. By (2.2) we learn that the components $E_1, E_2, \ldots, E_{n-1}$ of $P \setminus f(G_1)$ are plane sets. As $G_1$ does not contain a cut point, it follows that the boundary $C_1$ of $E_1$ ($1 \leq i \leq n-1$) is a circuit. As $G_1$ is compact it is clear that $G_1 = E_1 \setminus E_2 \cup E_1$. The 2-cells $E_1, \ldots, E_{n-1}$ together with the edges and vertices of $f(G_1)$ form a cellular decomposition of $P$. The embedding $f: G_1 \to P$ induces a homomorphism $\lambda: H_2(G_1) \to H_2(P)$. Let $M$ be the kernel of $\lambda$. If $z_i$ ($1 < i < n-1$) is a cycle corresponding to the circuit $C_i$, it is clear that $z_i \in M$. Moreover the cycles $z_i$ form a base in $M$. To prove this we first consider a linear combination $\lambda_1 z_1 + \ldots + \lambda_n z_n$. We assume that the 2-cells $E_i$ to be oriented in such a way that $z_i = \partial(E_i)$ ($1 < i < n-1$). From $\sum_{i=1}^{n-1} \lambda_i (E_i) = 0$ we conclude $\sum_{i=1}^{n-1} \lambda_i (E_i) = 0$. It follows that $\partial(\sum_{i=1}^{n-1} \lambda_i E_i) = 0$. Hence

$$\sum_{i=1}^{n-1} \lambda_i E_i$$

is a two-dimensional cycle in the projective plane. It follows that

$$\lambda_1 = 0 = \lambda_2 = \ldots = \lambda_{n-1}.$$

Hence the cycles $z_i$ ($1 < i < n-1$) are linearly independent. If $z$ is an element of $M$, we have $f_i(z) = 0$. Hence we can choose $\lambda_i \in \mathbb{Z}$ so that

$$z = \sum_{i=1}^{n-1} \lambda_i z_i.$$

It follows that the set $(z_1, \ldots, z_{n-1})$ is a base in $M$. As $G_1$ is not a plane graph, $M \neq H_2(G_1)$ by (3.1); hence we can choose a circuit $C_n$ in $G_1$ with corresponding cycle $c_n \in H_2(G_1)$ so that $f_i(z_n) \neq 0$. If $z$ is an element of $H_2(G_1)$, then we have $f_i(z) = f_i(z_n)$. Hence $z = z_n \in M$. In the same way we have for $z \in H_2(G_1) \setminus M$ that $2: f_i(z) = f_i(2z) = 0$. Hence we can find $\lambda_i \in \mathbb{Z}$ with

$$2z = \sum_{i=1}^{n-1} \lambda_i z_i.$$

It follows that the set $(z_1, \ldots, z_{n-1})$ is maximally independent in $H_2(G)$. As every edge is on at most two of the circuits $C_i$ ($1 < i < n-1$) we conclude that the circuits $C_1, C_2, \ldots, C_n$ satisfy the three conditions of the theorem with respect to $G_1$. As the leaves $G_2, \ldots, G_m$ are all plane graphs, we can find in $G_i$ ($2 < i < m$) a set of circuits $C_1^i, \ldots, C_{n_i}^i$ forming a base of Mac Lane in $G_i$. As

$$H_2(G) = H_2(G_1) \oplus H_2(G_2) \oplus \ldots \oplus H_2(G_m)$$

it is clear that the set $(C_1^1, \ldots, C_{n_1}^1, C_1^2, \ldots, C_{n_2}^2, \ldots, C_1^m, \ldots, C_{n_m}^m)$ satisfies the conditions of the theorem, which proves these conditions to be necessary.

Now we shall prove our conditions to be sufficient. Let $G$ be a graph satisfying the conditions of the theorem; we prove that $G$ can be embedded in the projective plane. If $G$ is a plane graph it can be embedded in the projective plane. Thus we assume that $G$ is a non-plane graph.

Let $G_1, G_2, \ldots, G_m$ be the leaves of $G$. We remark that the circuit $C_n$ is contained in only one of the leaves $G_i$ ($1 < i < m$). Assume that $C_n$ is contained in $G_1$. It follows that every element $z$ of $H_2(G_1) \oplus H_2(G_2) \oplus \ldots \oplus H_2(G_m)$ is independent of $z_n$. Hence the set $(z_1, z_2, \ldots, z_{n-1})$ contains a base of the group $H_2(G_1) \oplus H_2(G_2) \oplus \ldots \oplus H_2(G_m)$. According to the theorem of Mac Lane it follows that the graph $G \setminus G_1$ is a plane graph. Moreover it is clear that every component of $G \setminus G_1$ has at most one vertex in common with $G_1$. Hence $G$ can be embedded in the projective plane if and only if $G_1$ can be embedded in $P$. Let $(C_1, C_2, \ldots, C_{n_1})$ be the set of those circuits $C_i$ that are contained in $G_1$ ($1 < i < m$). Let

$$E = \bigcup_{i=1}^{n_1} E_i$$

be the union of $n_1$ pairwise disjoint two-dimensional closed discs $E_i$ ($1 < i < r$), and let $g$ be a continuous map which maps the boundary $B_i$ of $E_i$ homeomorphically onto the circuit $C_0$ ($1 < i < r$).

Now consider the adjunction space $X$ of the spaces $G_1$ and $E$ with respect to the map $g$ (cf. [5], p. 9). We remark that $X$ is a two-dimensional polyhedron with 2-cells $E_1, \ldots, E_{n_1}$ whose vertices and edges are the vertices respectively edges of $G_1$. Every one-dimensional cycle $z$ of $X$ is also a one-dimensional cycle of $G_1$. According to the conditions (2) and (3) of (3.1), we know that there are integers $\lambda_1, \ldots, \lambda_n, \lambda_n$ so that

$$z = \lambda_1 z_1 + \ldots + \lambda_n z_n.$$

Moreover there are integers $\mu_1, \mu_2, \ldots, \mu_n, \mu_n$ so that

$$\lambda_1 \mu_1 + \ldots + \lambda_n \mu_n = 0.$$
of \( X \) into the one-sphere \( S^1 \), and the induced homomorphism \( h: H_1(X) \to H_1(S^1) \). As \( x_0 \sim (1 < j < r) \in X \), we have \( h_*(x_0) = 0 \). From (a) we conclude:

\[
\mu h_*(x_0) = \sum_{j=1}^r \mu h_*(x_0) = 0.
\]

Hence \( h_*(x_0) = 0 \). From (a) it follows:

\[
h_*(z) = \sum_{j=1}^r \lambda_j h_*(x_0) = 0.
\]

So \( h \) maps every one-dimensional cycle of \( X \) with degree zero onto \( S^1 \). It follows that \( f \) is homotopic with zero (cf. [3], p. 317). Hence \( \nu(X) = 0 \). Because \( X \) is a locally connected continuum, this means that \( X \) is unicoherent (cf. [3], p. 317). It follows that every edge of \( G_1 \) is on at least one of the circuits \( G_0 \) (1 < j < r). Assume \( k \) to be an edge of \( G_1 \) that is on none of the circuits \( G_0 \) (1 < j < r). As \( G_1 \) is non-separable, \( k \) is on at least one circuit \( C \). Let \( z \) be a cycle corresponding to \( C \). Let \( h \) be a map of \( G_1 \) into the one-sphere \( S^1 \) that maps all edges of \( G_1 \) different from \( k \) into the point 0 of \( S^1 \), and that winds \( k \) one time around the circuit \( S^1 \). Then \( h \) maps the cycle \( z \) with degree 1 into \( S^1 \). As every \( z_0 \) (1 < j < r) is mapped with degree zero, the map \( h \) can be extended to a map \( h_0 \) of \( X \) into \( S^1 \). However \( h_0 \) maps \( z \) with degree 1 into \( S^1 \), hence \( h_0 \) is not homotopic with zero. This contradicts the fact that \( \nu(X) = 0 \). So every edge of \( G_1 \) is on at least one of the circuits \( G_0 \) (1 < j < r).

In the next step we prove that every edge of \( G_1 \) is on precisely two of the circuits \( G_0 \) (1 < j < r). We write \( G_1 = G_j \) and \( s_j = s_j \). Two circuits \( G_j \) and \( G_k \) are said to be connected by a regular chain if there exist a sequence \( G_{j_0} = G_j, G_{j_1}, \ldots, G_{j_r} = G_k \) so that \( G_{j_i} \) and \( G_{j_{i+1}} \) (0 < j_0 < s_i < 1) have at least one edge in common. It is easily proved that connectedness by a regular chain is an equivalence relation in the set consisting of the circuits \( G_j \); let \( N = \{ G_1, G_2, \ldots, G_n \} \) be an equivalence class. We define the subset \( N' \) of \( X \) as follows:

1. every 2-cell \( E \) having its boundary in \( N \) belongs to \( N' \);
2. every edge on at least one element of \( N \) belongs to \( N' \);
3. every vertex on at least one element of \( N \) belongs to \( N' \).

It is clear that \( N' \) is a subpolyhedron in \( X \). Let \( N'' \) be the polyhedron formed by the edges respectively 2-cells of \( X \) that are not in \( N' \); it is clear that \( \dim(N' \cap N'') < 0 \). If this dimension would be zero, the set \( N' \cap N'' \) would be a zero-dimensional separating set of \( X \). Because \( X \) is a two-dimensional unicoherent continuum without cutpoints, no region of \( X \) can be separated by a zero-dimensional set (cf. [8], p. 338). It follows that \( N' \cap N'' = 0 \). This means that \( N'' = X \). Hence the set \( \{ G_1, \ldots, G_n \} \) is regularly connected. Analogously we show that for every vertex \( p \) of \( X \) the set of circuits \( G_i \) (1 < i < r) containing \( p \), is regularly connected (i.e. any two elements of that set are connected by a regular chain). Otherwise \( p \) would be a local cutpoint of \( X \). However \( X \) cannot have local cutpoints as \( G_1 \) is non-separable. It follows that \( X \) is a two-dimensional variety.

Let \( B = \partial X \) and suppose that \( B = F \). Every point of \( B \) is on an edge of \( G_1 \) that is incident with at most one 2-cell of \( X \).

Let \( C \subset B \) be a circuit and consider a closed disc \( D \) with boundary \( C \) so that \( D \times C \subset C \). Let \( x \) be a cycle corresponding to the circuit \( C \). As the set \( \{ z_1, \ldots, z_n \} \) is a maximally independent set in \( H_1(G_1) \) it follows that there exist integers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) so that

\[
\lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n = 0
\]

As \( C \subset B \) and as the set \( \{ z_1, \ldots, z_n \} \) is regularly connected it follows that

\[
|z_i| = |\lambda_i| = \cdots = |\lambda_n|.
\]

We write

\[
z = a_1 z_1 + \cdots + a_n z_n
\]

with \( a_i = \lambda_i / |\lambda_i| = \pm 1 \) (1 < i < r).

It follows that \( C = \partial X \), so \( X \subset D \) is an orientable manifold. As \( X \subset D \) is unicoherent we conclude that \( X \subset D \) is a two-sphere. This however contradicts the fact that \( G_1 \) is a non-plane graph.

Hence we have shown that \( \partial X = \emptyset \). Now \( X \) being a unicoherent variety without boundary, it must be the sphere or the projective plane. Because \( G_1 \) is a non-plane graph, \( X \) is not a sphere. Hence \( X \) is the projective plane. So \( G_1 \) can be embedded in a projective plane. It follows that the same is true for \( G \). Hence we have proved theorem (3.1).

*Added in proof.* As the proof of theorem (3.1) depends on the unicoherence of the projective plane, there is no immediate generalization of this theorem using homology groups. However, a generalization of this theorem for arbitrary orientable surfaces can be proved, replacing homology groups by homotopy groups; this will be published soon.

**References**


