Embedding of graphs in the projective plane

by

J. Ch. Boland (Amsterdam)

1. Introduction. In this paper we give necessary and sufficient conditions for a graph to be embeddable in the real projective plane.

It is well known that Kuratowski [1] solved the corresponding problem for the Euclidean plane. However his characterization by excluded figures does not seem appropriate in the case of the projective plane, the number of excluded figures becoming rather large.

Another characterization of plane graphs has been given by Mac Lane [2]. He proved that a graph G with nullity $\geqslant 1$ is plane, if and only if there is a number of circuits in G forming a base of the cycles modulo 2, with the property that every edge of G is on at most two of these circuits.

Our characterization of graphs embeddable in the projective plane is an analogon of the theorem of Mac Lane. However we do not work with cycles modulo 2, but with integral cycles.

By $\boldsymbol{Z}$ we denote the (additive) group of integers, and by $\boldsymbol{Z}_2$ the group of integers modulo 2. The first homology group $H_1(G, \boldsymbol{Z})$, respectively $H_1(G, \boldsymbol{Z}_2)$ of a graph G is isomorphic with the group of integral cycles, respectively cycles modulo 2, the dimension of G being one. If the group $H_1(G, \boldsymbol{Z}_2)$ has a base of Mac Lane type, the graph G can be embedded in the Euclidean plane E. It is well known that in this case the set of integral circuits contained in the boundaries of the bounded components of $E \setminus G$ is a base in the group $H_1(G, \boldsymbol{Z})$. Let

$$\xi_2\colon\ H_1(G, \boldsymbol{Z}) \to H_1(G, \boldsymbol{Z}_2)$$

be the homomorphism derived from the natural map $\boldsymbol{Z} \to \boldsymbol{Z}_2$ (the map ξ_2 can be obtained by choosing a base for $H_1(G, \boldsymbol{Z})$, and replacing the integral coefficients of the base by their cosets modulo 2). As G does not contain torsion in dimension zero and one, the homomorphism ξ_2 is an epimorphism (cf. [3], p. 219). Hence if $\{z_1, \ldots, z_n\}$ is a base for $H_1(G, \boldsymbol{Z})$, the set $\{\xi_2(z_1), \ldots, \xi_2(z_n)\}$ contains a base of Mac Lane type in the group $H_1(G, \boldsymbol{Z}_2)$. It follows that in the theorem of Mac Lane the cycles modulo 2 can be replaced by integral cycles. As we do not consider in this paper coefficients groups different from $\mathbf{Z}$, we write $H_1(G)$ instead of $H_1(G, \mathbf{Z})$.

The main theorem of this paper reads as follows:

THEOREM. *A graph G of nullity $\geqslant 1$ can be embedded in the projective plane if and only if there exists a set of circuits $z_1, \ldots, z_n$ in G with the following properties*:

a) *the set $\{z_1, \ldots, z_n\}$ is a generating set in $H_1(G)$*;

b) *the set $\{z_1, \ldots, z_{n-1}\}$ is a maximally independent set in $H_1(G)$*;

c) *every edge of G is on at most two of the circuits $\{z_1, \ldots, z_{n-1}\}$*.

This theorem will be proved in section 3. In section 2 some definitions and theorems needed in section 3 are given.

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2. By a graph we mean a set A, together with a binary symmetric relation R defined on A. A graph $G = [A, R]$ is called *finite* if the set A is finite. In this paper we consider only finite graphs. Hence the word "graph" will stand for a finite graph. We obtain a geometric representation of a graph $G = [A, R]$ by representing the set A by a pointset in the Euclidian three dimensional space $\mathbf{R}_3$. This pointset we denote also by A. We join the points x and y of A by an arc $L[x, y]$ of $\mathbf{R}_3$ with endpoints x and y if and only if the pair (x, y) satisfies the relation R, such that for $(x_1, y_1) \neq (x_2, y_2)$ the arcs $L[x_1, y_1]$ and $L[x_2, y_2]$ have at most an endpoint of both in common. The points of A will be called the *vertices* of G, and the arcs $L[x, y]$ the *edges* of G. The graph G is completely determined by the set A of its vertices and the set B of its edges. We say that a graph G can be *embedded* in a topological space X if and only if the geometric representation of G can be mapped homeomorphically into X.

A subgraph of G is a graph whose set of vertices and set of edges are contained in the vertexset of G, respectively in the set of edges of G.

The local degree of a graph G at a vertex x of G is the number of edges that are incident with x. The notion of connectedness of a graph will be taken in the usual sense. A vertex $a \in A$ of a graph $G = [A, R]$ is called a *cutpoint* of G if the set $A \backslash a$ can be seperated into two sets A_1 and A_2 such that no point of A_1 is in the relation R with a point of A_2. It is clear that a cutpoint of G is also a cutpoint in the topological sense of the geometric representation of G. As the homology groups of G are free with respect to $\mathbf{Z}$ we define the nullity of a graph to be the rank of its first homology group $H_1(G)$.

A circuit is a connected graph with local degree two at every vertex. If a circuit C is a subgraph of a graph G the natural inclusion $i\colon C \to G$

induces a homomorphism $i_*\colon H_1(C) \to H_1(G)$ of the integral homology groups. Clearly $H_1(C)$ is isomorphic with the infinite cyclic group Z; let ζ be a generating element of $H_1(C)$ and let $z = i_*(\zeta)$. Then we call z a *cycle corresponding to the circuit C*. It is clear that $-z$ is also a cycle corresponding to the circuit C. We choose this terminology to make a clear distinction between the circuit C as a geometric object and the corresponding cycle z as an algebraic object, which is an element of $H_1(G)$.

An edge e of G is called a *circuit edge* if and only if e is on some circuit of G. Let B be the set of all edges of G and B_1 the set of all circuitedges of G. We define an equivalence relation T on the set B_1 as follows: for every two elements e_1 and e_2 of B_1 the relation $e_1 T e_2$ holds if and only if G contains a circuit C with the property that e_1 and e_2 are both on C. It is easy to prove that T is an equivalence relation. A subgraph H of G is called a *leaf* of G if the set of edges of H is an equivalence class of T and if the set of vertices of H consists of precisely those vertices of G that are incident with at least one edge in that equivalence class. It is clear that a leaf of G does not contain a cutpoint. In fact a leaf of G is a maximal subgraph without cutpoints (cf. [4], p. 82).

Consider a graph G that can be embedded into the projective plane P. Let $f\colon G \to P$ be a homeomorphism of G into P and let $f_*\colon H_1(G) \to H_1(P)$ be the induced homomorphism of the first homology groups.

PROPOSITION (2.1). *The graph G is plane if and only if there exists an embedding $f\colon G \to P$ such that $f_*(H_1(G)) = 0$.*

Proof. The proof of this proposition is nearly a direct consequence of some well known facts. At first it is well known that f induces a homomorphism $f'_*\colon \pi_1(G) \to \pi_1(P)$ of the first homotopy groups of G and P such that the following diagram is commutative

$$\begin{array}{ccc} \pi_1(G) & \xrightarrow{f'_*} & \pi_1(P) \\ \downarrow{\scriptstyle h(G)} & & \downarrow{\scriptstyle h(P)} \\ H_1(G) & \xrightarrow{f_*} & H_1(P) \end{array}$$

(where $h(G)$ and $h(P)$ are the canonical homomorphisms); $h(P)$ is an isomorphism $\big(\pi_1(P) \cong H_1(P) \cong \mathbf{Z}_2\big)$. From $f_*\big(H_1(G)\big) = 0$ it follows that $f'_*\big(\pi_1(G)\big) = 0$.

As the two-dimensional sphere S^2 is an universal covering space of the projective plane P, we conclude by theorem 17.3 in [5], p. 96, that the map $f\colon G \to P$ can be lifted over S^2. If $j\colon S^2 \to P$ is the natural map of S^2 onto P, we know that there exists a continuous map $\varphi\colon G\to S^2$ such that the following diagram is commutative

$$\begin{array}{ccc} & & S^2 \\ & \overset{\varphi}{\nearrow} & \downarrow j \\ G & \xrightarrow{f} & P. \end{array}$$

As f is a homeomorphism and $f = j\varphi$ we conclude that φ is a continuous 1-1 map of G into S^2, hence a homeomorphism. This means that G is a plane graph.

If on the other hand G is a plane graph, one can find an embedding of G into P which factors through the plane E. In that case $f_*(H_1(G)) = 0$ follows directly from $H_1(E) = 0$, and the proposition is proved.

PROPOSITION (2.2). *If G is a non-plane graph, f an embedding of G in the projective plane P, and U a component of $P\setminus f(G)$, then the closure $\overline{U}$ of U with respect to P is a plane set.*

Proof. Let $i\colon \overline{U}\to P$ be the natural embedding of $\overline{U}$ into P, and let $i_*\colon \pi_1(\overline{U})\to\pi_1(P)$ be the induced homomorphism of the first homotopy groups. If $i_*(\pi_1(\overline{U})) = 0$ the map i can be lifted over S^2; hence in that case it follows that $\overline{U}$ is a plane set. So let us assume $i_*(\pi_1(\overline{U}))\neq 0$. We choose $\alpha\in\pi_1(\overline{U})$ with $i_*(\alpha)\neq 0$. Let I denote the unit interval $[0,1]$ of the set of real numbers, and choose a continuous map $g\colon I\to\overline{U}$ with $g\in\alpha$. We shall prove that g can be chosen in such a way that $g(I)$ is a circuit, having at most one point in common with the boundary $B = \overline{U}\setminus U$ of U. Assume that $B\cap g(I)\neq\emptyset$ and take a point $p_0\in B\cap g(I)$. We shall choose a map $h\in\alpha$ so that $h(0) = h(1) = p_0$. As $B\subset f(G)$, we know that B is locally connected. Every point $x\in h(I)$ is contained in an Euclidean neighborhood $V(x)$, so that $V(x)\cap U$ is connected. The family $\{V(x)\}$ with $x\in h(I)$ is a covering of the set $h(I)$. As $V(x)\cap h(I)$ is an open set in $h(I)$, it follows that the components of the sets $h^{-1}(V(x)\cap h(I))$, $x\in h(I)$ are elements of an open covering of I. From this it follows that we can find a finite sequence of real numbers $y_0 = 0, y_1, \ldots, y_n = 1$ so that:

1) $0 = y_0 < y_1 < \ldots < y_{n-1} < y_n = 1$;

2) $h(y_j)$ and $h(y_{j+1})$ $(0\leqslant j\leqslant n-1)$ are contained in one of the sets $V(x)$, say $V(x_{j+1})$.

Let $p_i = h(y_i)$ $(0\leqslant i\leqslant n-1)$. As the set $V(x_1)\cap U$ is connected we can join the points p_0 and p_1 by an arc $L_1\subset V(x_1)\cap\{U\cup p_0\cup p_1\}$. Let h_1' be a homeomorphism of the interval $[0, y_1]$ onto the arc L_1 so that $h_1'(0) = p_0$ and $h_1'(y_1) = p_1$. Then we define the map h_1 as follows:

$$h_1(y) = \begin{cases} h_1'(y) & \text{if} \quad 0\leqslant y\leqslant y_1, \\ h(y) & \text{if} \quad y_1\leqslant y\leqslant 1. \end{cases}$$

It is clear that h_1 is homotopic with h, so $h_1\in\alpha$. Moreover we have $B\cap h_1(I)\subset h_1([y_1, 1])$. Suppose that for some i $(1\leqslant i\leqslant n)$ we have constructed a map $h_i\colon I\to\overline{U}$ so that:

1) $h_i\in\alpha$;

2) $h_i(0) = p_0$; $h_i(y_i) = p_i$ *and* $h_i(0, y_i)\subset U$;

3) $B\cap h(I)\subset h_i([y_i, 1])$.

As p_i and p_{i+1} are contained in $V(x_{i+1})$, we can choose a point $z_i\in(y_{i-1}, y_i)$ so that $h_i(z_i)\in V(x_{i+1})$. We join the points $h_i(z_i)$ and p_{i+1} by an arc $L_{i+1}\subset V(x_{i+1})\cap(U\cup p_{i+1})$ so that $L_{i+1}\cap h_i([0, z_i]) = h_i(z_i)$. We choose a homeomorphism h_{i+1}' of the interval $[z_i, y_{i+1}]$ onto the arc L_{i+1} with $h_{i+1}'(z_i) = h_i(z_i)$ and $h_{i+1}'(y_{i+1}) = p_{i+1}$. Then we define a map $h_{i+1}\colon I\to\overline{U}$ as follows:

$$h_{i+1}(y) = \begin{cases} h_i(y) & \text{if} \quad 0\leqslant y \;\; z_i, \\ h_{i+1}'(y) & \text{if} \quad z_i\leqslant y\leqslant y_{i+1}, \\ h(y) & \text{if} \quad y_{i+1}\leqslant y\leqslant 1. \end{cases}$$

It is clear that the map h_{i+1} is homotopic with h_i; hence $h_{i+1}\in\alpha$. Moreover $B\cap h_{i+1}(I)\subset h_{i+1}([y_{i+1}, 1])$. From this construction it follows that the map $h_n\colon I\to\overline{U}$ is an element of α, and that $B\cap h_n(I) = p_0$. Let us assume that the map $g\in\alpha$ was chosen in such a way that:

1) $g(I)$ is a circuit in $\overline{U}$;

2) $B\cap g(I)$ contains at most one point.

Now we consider the natural map $g_1\colon P\to P/ig(I)$. As $ig(I)$ is a circuit in P that is not homotopic with 0 in P, it is well known that $P/ig(I)$ is homeomorphic to S^2. Moreover if $g_1' = g_1|f(G)$, it is clear that $g_1'f\colon G\to S^2$ is a homeomorphism. It follows that G is a plane graph. This however contradicts the assumption that G would be non-plane and hence $i_*(\pi_1(\overline{U}))$ equals to zero, and $\overline{U}$ is a plane set, which proves the proposition.

From the preceeding proposition we conclude:

COROLLARY (2.3). *If the graph G is embeddable in the projective plane, G contains at most one non-plane leaf.*

3. In this section we give a proof of:

THEOREM (3.1). *A graph with nullity $\geqslant 1$ can be embedded in the projective plane if and only if G contains a set of circuits $C_1, \ldots, C_n$ so that the following conditions are satisfied:*

(1) *every edge of G is on at most two of the circuits C_i with $1\leqslant i\leqslant n-1$;*

(2) *let z_i $(1\leqslant i\leqslant n)$ be the cycle corresponding to the circuit C_i; the set of cycles $\{z_1, \ldots, z_n\}$ is a generating set of the first homology group $H_1(G)$ of G;*

(3) *the set $\{z_1, \ldots, z_{n-1}\}$ is a maximally independent set in $H_1(G)$.*

Proof. First we show these three conditions to be necessary. Let G be a graph of nullity $\geqslant 1$ that can be embedded in the projective plane. If G is a plane graph we know from the theorem of Mac Lane, that G contains a set of circuits $C_1, \ldots, C_{n-1}$ satisfying condition (1), so that the corresponding cycles $z_1, \ldots, z_{n-1}$ form a base in $H_1(G)$. It is clear that these cycles $z_1, \ldots, z_{n-1}$ together with some cycle z_n satisfy also conditions (2) and (3).

Now let G be a non-plane graph. By (2.3) we know that at most one leaf of G is non-plane. Let $G_1, G_2, \ldots, G_m$ be the leaves of G; then

$$G = \bigcup_{i=1}^{m+1} G_i,$$

where G_{m+1} is the subgraph of G generated by the non-circuit edges of G. We assume that only G_1 is non-plane. Let f be the embedding of G_1 into the projective plane P. By (2.2) we learn that the components $E_1, E_2, \ldots, E_{n-1}$ of $P \setminus f(G_1)$ are plane sets. As G_1 does not contain a cut point, it follows that the boundary C_i of E_i $(1 \leqslant i \leqslant n-1)$ is a circuit. As G_1 is compact it is clear that $C_i = \bar{E}_i \setminus E_i \subset f(G_1)$. The 2-cells $E_1, \ldots, E_{n-1}$ together with the edges and vertices of $f(G_1)$ form a cellular decomposition of P. The embedding $f\colon G_1 \to P$ induces a homomorphism $f_*\colon H_1(G_1) \to H_1(P)$. Let M be the kernel of f_*. If z_i $(1 \leqslant i \leqslant n-1)$ is a cycle corresponding to the circuit C_i, it is clear that $z_i \in M$. Moreover the cycles z_i form a base in M. To prove this we first consider a linear combination $\lambda_1 z_1 + \ldots + \lambda_{n-1} z_{n-1} = 0$, with $\lambda_i \in \boldsymbol{Z}$. We assume the 2-cells E_i to be oriented in such a way that $z_i = \partial(E_i)$ $(1 \leqslant i \leqslant n-1)$. From $\sum_{i=1}^{n-1} \lambda_i z_i = 0$ we conclude $\sum_{i=1}^{n-1} \lambda_i \partial(E_i) = 0$. It follows that $\partial\big(\sum_{i=1}^{n-1} \lambda_i E_i\big) = 0$. Hence $\sum_{i=1}^{n-1} \lambda_i E_i$ is a two-dimensional cycle in the projective plane. It follows that $\lambda_1 = 0 = \lambda_2 = \ldots = \lambda_{n-1}$. Hence the cycles z_i $(1 \leqslant i \leqslant n-1)$ are linearly independent. If z is an element of M, we have $f_*(z) = 0$. Hence we can choose $\lambda_i \in \boldsymbol{Z}$ so that

$$z = \partial \sum_{i=1}^{n-1} \lambda_i E_i = \sum_{i=1}^{n-1} \lambda_i \partial(E_i) = \sum_{i=1}^{n-1} \lambda_i z_i .$$

It follows that the set $\{z_1, \ldots, z_{n-1}\}$ is a base in M. As G_1 is not a plane graph, $M \neq H_1(G_1)$ by (2.1); hence we can choose a circuit C_n in G_1 with corresponding cycle $z_n \in H_1(G_1)$ so that $f_*(z_n) \neq 0$. If z is an element of $H_1(G_1) \setminus M$ we have $f_*(z - z_n) = 0$. Hence $z - z_n \in M$. It follows that the set $\{z_1, \ldots, z_n\}$ is a generating set in $H_1(G_1)$. In the same way we have for $z \in H_1(G_1) \setminus M$ that $2 \cdot f_*(z) = f_*(2z) = 0$. Hence we can find $\lambda_i \in \boldsymbol{Z}$ with $2z = \sum_{i=1}^{n-1} \lambda_i z_i$. It follows that the set $\{z_1, \ldots, z_{n-1}\}$ is maximally independent in $H_1(G)$. As every edge is on at most two of the circuits C_i $(1 \leqslant i \leqslant n-1)$ we conclude that the circuits $C_1, C_2, \ldots, C_n$ satisfy the three conditions of the theorem with respect to G_1. As the leaves $G_2, \ldots, G_m$ are all plane graphs, we can find in G_i $(2 \leqslant i \leqslant m)$ a set of circuits $C_1^{(i)}, \ldots, C_{n_i}^{(i)}$ forming a base of Mac Lane in G_i. As

$$H_1(G) = H_1(G_1) \oplus H_1(G_2) \oplus \ldots \oplus H_1(G_m)$$

it is clear that the set $\{C_j^{(i)}\}_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n_i}$ with $C_j^{(1)} = C_j$ satisfies the conditions of the theorem, which proves these conditions to be necessary.

Now we shall prove our conditions to be sufficient. Let G be a graph satisfying the conditions of the theorem; we prove that G can be embedded in the projective plane. If G is a plane graph it can embedded in the projective plane. Thus we assume that G is a non-plane graph. Let $G_1, G_2, \ldots, G_m$ be the leaves of G. We remark that the circuit C_n is contained in only one of the leaves G_i $(1 \leqslant i \leqslant m)$. Assume that C_n is contained in G_1. It follows that every element z of $H_1(G_2) \oplus H_1(G_3) \oplus \ldots \oplus H_1(G_m)$ is independent of z_n. Hence the set $\{z_1, z_2, \ldots, z_{n-1}\}$ contains a base of the group $H_1(G_2) \oplus H_1(G_3) \oplus \ldots \oplus H_1(G_m)$. According to the theorem of Mac Lane it follows that the graph $G \setminus G_1$ is a plane graph. Moreover it is clear that every component of $G \setminus G_1$ has at most one vertex in common with G_1. Hence G can be embedded in the projective plane if and only if G_1 can be embedded in P. Let $\{C_{i_1}, C_{i_2}, \ldots, C_{i_r}\}$ be the set of those circuits C_i that are contained in G_1 $(1 \leqslant i \leqslant n)$. Let $E = \bigcup_{i=1}^{r} E_i$ be the union of r pairwise disjoint two-dimensional closed discs E_i $(1 \leqslant i \leqslant r)$, and let g be a continuous map which maps the boundary B_j of E_j homeomorphically onto the circuit C_{i_j} $(1 \leqslant j \leqslant r)$. Now consider the adjunction space X of the spaces G_1 and E with respect to the map g (cf. [5], p. 9). We remark that X is a two-dimensional polyhedron with 2-cells $E_1, \ldots, E_r$, whose vertices and edges are the vertices respectively edges of G_1. Every one-dimensional cycle z of X is also a one-dimensional cycle of G_1. According to the conditions (2) and (3) of (3.1), we know that there are integers $\lambda_{i_1}, \ldots, \lambda_{i_r}, \lambda_n$ so that

$$(\alpha) \qquad z = \lambda_{i_1} z_{i_1} + \ldots + \lambda_{i_r} z_{i_r} + \lambda_n z_n .$$

Moreover there are integers $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_r}, \mu_n$ so that

$$(\beta) \qquad \mu_n z_n = \mu_{i_1} z_{i_1} + \ldots + \mu_{i_r} z_{i_r} .$$

From equations (α) and (β) we conclude that the first cohomotopy group of X, $\pi^1(X) = 0$. To prove this we consider a continuous map $h\colon X \to S^1$

of X into the one-sphere S^1, and the induced homomorphism $h_*\colon H_1(X)\to H_1(S^1)$. As $z_{i_j}\sim 0$ $(1\leqslant j\leqslant r)$ in X, we have $h_*(z_{i_j})=0$ $(1\leqslant j\leqslant r)$. From (β) we conclude:

$$\mu_n h_*(z_n)=\sum_{j=i}^{n}\mu_{i_j}h_*(z_{i_j})=0\,.$$

Hence $h_*(z_n)=0$. From (α) it follows:

$$h_*(z)=\sum_{j=1}^{r}\lambda_{i_j}h_*(z_{i_j})+\lambda_n h_*(z_n)=0\,.$$

So H maps every one-dimensional cycle of X with degree zero onto S^1. It follows that f is homotopic with zero (cf. [3], p. 517). Hence $\pi^1(X)=0$. Because X is a locally connected continuum, this means that X is unicoherent (cf. [3], p. 292). It follows that every edge of G_1 is on at least one of the circuits C_{i_j} $(1\leqslant j\leqslant r)$. Assume k to be an edge of G_1 that is on none of the circuits C_{i_j} $(1\leqslant j\leqslant r)$. As G_1 is non-separable, k is on at least one circuit C. Let z be a cycle corresponding to C. Let h be a map of G_1 into the one-sphere S^1 that maps all edges of G_1 different from k into the point 0 of S^1, and that winds k one time around the circuit S^1. Then h maps the cycle z with degree 1 into S^1. As every z_{i_j} $(1\leqslant j\leqslant r)$ is mapped with degree zero, the map h can be extended to a map h_1 of X into S^1. However h_1 maps z with degree 1 into S^1, hence h_1 is not homotopic with zero. This contradicts the fact that $\pi^1(X)=0$. So every edge of G_1 is on at least one of the circuits C_{i_j} $(1\leqslant j\leqslant r)$.

In the next step we prove that every edge of G_1 is on precisely two of the circuits C_{i_j} $(1\leqslant j\leqslant r)$. We write $C'_j=C_{i_j}$ and $z'_j=z_{i_j}$. Two circuits C'_i and C'_k are said to be *connected by a regular chain* if there exists a sequence $C'_{i_0}=C'_i, C'_{i_1},\ldots,C'_{i_s}=C'_k$ so that C'_{i_j} and $C'_{i_{j+1}}$ $(0\leqslant j\leqslant s-1)$ have at least one edge in common. It is easily proved that connectedness by a regular chain is an equivalence relation in the set consisting of the circuits C'_j; let $N=\{C'_{i_0},C'_{i_1},\ldots,C'_{i_k}\}$ be an equivalence class. We define the subset N' of X as follows:

(1) every 2-cell E_i having its boundary in N belongs to N';
(2) every edge on at least one element of N belongs to N';
(3) every vertex on at least one element of N belongs to N'.

It is clear that N' is a subpolyhedron in X. Let N'' be the polyhedron formed by the edges respectively 2-cells of X that are not in N'; it is clear that $\dim(N'\cap N'')\leqslant 0$. If this dimension would be zero, the set $N'\cap N''$ would be a zero-dimensional separating set of X. Because X is a two-dimensional unicoherent continuum without cutpoints, no region of X can be separated by a zero-dimensional set (cf. [6], p. 338). It follows that $N'\cap N''=\emptyset$. This means that $N'=X$. Hence the set $\{C'_1,\ldots,C'_r\}$ is regularly connected. Analogously we show that for every vertex p of X the set of circuits C'_i $(1\leqslant i\leqslant r)$ containing p, is regularly connected (i.e. any two elements of that set are connected by a regular chain). Otherwise p would be a local cutpoint of X. However X cannot have local cutpoints as G_1 is non-separable. It follows that X is a two-dimensional variety.

Let $B=\partial X$ and suppose that $B\neq\emptyset$. Every point of B is on an edge of G_1 that is incident with at most one 2-cell of X.

Let $C\subset B$ be a circuit and consider a closed disc D with boundary C so that $D\cap X\subset C$. Let z be a cycle corresponding to the circuit C. As the set $\{z'_1,\ldots,z'_r\}$ is a maximally independent set in $H_1(G_1)$ it follows that there exist integers $\lambda,\lambda_1,\ldots,\lambda_r$ so that

$$\lambda z=\lambda_1 z'_1+\ldots+\lambda_r z'_r\,.$$

As $C\subset\partial X$ and as the set $\{z'_1,\ldots,z'_r\}$ is regularly connected it follows that $|\lambda|=|\lambda_1|=\ldots=|\lambda_n|$. We write

$$z=\alpha_1 z'_1+\ldots+\alpha_r z'_r \quad \text{with } \alpha_i=\lambda_i/\lambda=\pm 1 \quad (1\leqslant i\leqslant r)\,.$$

It follows that $C=\partial X$, so $X\cup D$ is an orientable manifold. As $X\cup D$ is unicoherent we conclude that $X\cup D$ is a two-sphere. This however contradicts the fact that G_1 is a non-plane graph.

Hence we have shown that $\partial X=\emptyset$. Now X being a unicoherent variety without boundary, it must be the sphere or the projective plane. Because G_1 is a non-plane graph, X is not a sphere. Hence X is the projective plane. So G_1 can be embedded in a projective plane. It follows that the same is true for G. Hence we have proved theorem (3.1.)

Added in proof. As the proof of theorem (3.1) depends on the unicoherence of the projective plane, there is no immediate generalization of this theorem using homology groups. However, a generalization of this theorem for arbitrary orientable surfaces can be proved, replacing homology groups by homotopy groups; this will be published soon.

References

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