# Locally equiconnected spaces and absolute neighborhood retracts *

by

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**1. Introduction.** It is well known, and easy to prove, that if a metric space is an ANR, then it is locally equiconnected (cf. [10], [11]). The purpose of this paper is to determine some conditions under which the converse is true.

In the first part (§ 2) we derive some equivalent formulations of local equiconnectedness; it is an easy consequence of one of these that the equiconnected spaces are precisely the contractible locally equiconnected ones, a result that is apparently new.

In the second part (§ 3) we characterize the locally equiconnected spaces that are ANRs; one application is given, which leads to a slight extension of a result due to Milnor ([6], p. 279).

**2. Equiconnected spaces.** Unless otherwise explicitly stated, all spaces will be metric (not necessarily separable), and $I$ will denote the unit interval. A metric space $Y$ is *locally equiconnected* if there exists a neighborhood $U$ of the diagonal $\Delta \subset Y\times Y$ and a continuous map $\lambda\colon U\times I\to Y$ such that $\lambda(a,b,0)=a$, $\lambda(a,b,1)=b$, and $\lambda(a,a,t)=a$ for all $(a,b)\in U$, $t\in I$; the map $\lambda$ is called an *equiconnecting function*. The space $Y$ is equiconnected if $\lambda$ is defined on $Y\times Y$.

2.1. THEOREM. *$Y$ is (locally) equiconnected if and only if the diagonal $\Delta$ is a strong (neighborhood) deformation retract* (¹) *in* $Y\times Y$.

Proof. Assume that $Y$ is locally equiconnected, and let $\lambda\colon U\times I\to Y$ be an equiconnecting function. Define $\varrho\colon U\times I\to Y\times Y$ by $\varrho[(a,b),t] = [\lambda(a,b,t),b]$; then $\varrho$ is easily verified to be a strong deformation retraction of $U$ into $\Delta$. For the converse, let $\varrho\colon U\times I\to Y\times Y$ be a strong deformation retraction of $U\subset Y\times Y$ into $\Delta$, where $\varrho(u,0)=u$ for all

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(¹) A closed $A\subset X$ is a strong neighborhood deformation retract in $X$ if there exists an open $U\supset A$ and a homotopy $h\colon U\times I\to X$ such that $h(u,0)=u$, $h(u,1)\in A$ and $h(a,t)=a$ for every $u\in U$, $a\in A$ and $t\in I$; $h$ is called a *strong deformation retraction* of $U$ into $A$.

$u \in U$. Letting $p_1$: $Y \times Y \to Y$ be the projection $(a, b) \to a$ and $p_2$: $Y \times Y \to Y$ the projection $(a, b) \to b$, define $\lambda$: $U \times I \to Y$ by

$$\lambda(a, b, t) = \begin{cases} p_1 \circ \varrho[(a, b), 2t], & 0 \leqslant t \leqslant \frac{1}{2}, \\ p_2 \circ \varrho[(a, b), 2-2t], & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is evident that $\lambda$ is continuous, and an equiconnecting function. The proof for equiconnectedness is entirely analogous.

Though it is equally simple to prove directly, we find from 2.1 that

2.2. COROLLARY. *Every* ANR *is locally equiconnected, and every* AR *is equiconnected.*

Proof. If $Y$ is an AR (ANR), then $\Delta$, being homeomorphic to $Y$, is also an AR (ANR), and each AR (ANR) is a strong (neighborhood) deformation retract in any metric-space containing it as a closed set ([2], p. 239, [4], p. 325).

To get a relation between equiconnectedness and local equiconnectedness, and for future reference, we state explicitly the trivial

2.3. LEMMA. *Let* $\lambda$: $U \times I \to Y$ *be an equiconnecting function. Then for each* $y_0 \in Y$ *and neighborhood* $W$ *of* $y_0$, *there is a neighborhood* $V$, $y_0 \in V \subset W$, *such that* $\lambda(V, V, I) \subset W$. *In particular, each locally equiconnected space is locally contractible, and each equiconnected space is both contractible and locally contractible.*

Proof. Since $\lambda^{-1}(W)$ is open in the open $U \times I \subset Y \times Y \times I$, it is open in $Y \times Y \times I$, and because $y_0 \times y_0 \times I \subset \lambda^{-1}(W)$, it follows ([5], p. 86) that there is a neighborhood $V \times V \supset y_0 \times y_0$ such that $V \times V \times I \subset \lambda^{-1}(W)$. In particular, defining $\varrho$: $V \times I \to Y$ by $\varrho(y, t) = \lambda[y, y_0, t]$ we obtain a contraction of $V$ over $W$ to $y_0$, keeping $y_0$ fixed throughout the entire deformation.

2.4. THEOREM. *The equiconnected spaces are precisely the contractible locally equiconnected ones.*

Proof. In view of 2.3, we need prove only that a contractible locally equiconnected space is equiconnected. It is known ([3], XV 8.2) that if a closed set $A$ in a metric space $X$ is a strong neighborhood deformation retract, and if $X$ can be deformed [2] into $A$ in such a way that the points of $A$ remain in $A$ during the entire deformation, then $A$ is a strong deformation retract of $X$. Because of 2.1, we therefore need only construct a deformation of $Y \times Y$ into $\Delta$ that keeps $\Delta$ in $\Delta$ and, if $\varrho$: $Y \times I \to Y$ is a contraction of $Y$ to $y_0$, then the map $\hat{\varrho}$: $Y \times Y \times I \to Y \times Y$ given by $\hat{\varrho}[(a, b), t] = [\varrho(a, t), \varrho(b, t)]$ is such a deformation.

(2) $X$ is deformable into $A \subset X$ if there is an $h$: $X \times I \to X$ such that $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$.



Because of 2.4, we can confine our attention to locally equiconnected spaces. These are also characterized by a homotopy property which we need later. If $Y$ is any space and $\mathfrak{W}$ any open covering, then two maps $f, g$: $X \to Y$ of a space $X$ into $Y$ are called $\mathfrak{W}$-close whenever $f(x)$ and $g(x)$ belong to a common set $W \in \mathfrak{W}$ for each $x \in X$; $f$ and $g$ are $\mathfrak{W}$-homotopic if there is a homotopy $\Phi$: $f \simeq g$ such that $\Phi(x, I) \subset$ some $W \in \mathfrak{W}$ for each $x \in X$. A homotopy $\Phi$: $f \simeq g$ is called *stationary* if $\Phi(x, I)$ is constant whenever $f(x) = g(x)$.

2.5. THEOREM. *$Y$ is locally equiconnected if and only if for each open covering $\mathfrak{W}$ of $Y$ there exists a refinement $\mathfrak{V}$ such that any two $\mathfrak{V}$-close maps of any space $X$ into $Y$ are stationarily $\mathfrak{W}$-homotopic.*

Proof. Assume that $Y$ is locally equiconnected, and let $\lambda$ be an equiconnecting function. Given $\mathfrak{W}$, select for each $y \in Y$ a neighborhood $V_y$ such that $\lambda(V_y, V_y, I) \subset$ come $W \in \mathfrak{W}$ containing $y$ (cf. 2.3) and let $\mathfrak{V} = \{V_y | y \in Y\}$. If $f, g$: $X \to Y$ are $\mathfrak{V}$-close, then $\Phi(x, t) = \lambda[f(x), g(x), t]$ defines a stationary $\mathfrak{W}$-homotopy of $f$ to $g$. Conversely, if the condition is satisfied, choose $\mathfrak{W}$ to consist of one set, $Y$, and let $\mathfrak{V}$ be a refinement having the stated property. Define $U = \bigcup \{V \times V | V \in \mathfrak{V}\} \subset Y \times Y$; then the maps $f, g$: $U \to Y$ given by $(a, b) \to a$, $(a, b) \to b$ respectively, are $\mathfrak{V}$-close, and the stationary $\mathfrak{W}$-homotopy is an equiconnecting function.

## 3. Relation to ANR.

The following result is well known, at least for separable metric spaces:

3.1. THEOREM. *A finite-dimensional metric space* (3) *is locally equiconnected if and only if it is an* ANR.

Proof. Since a finite-dimensional locally contractible metric space is an ANR ([2], p. 244), the converse of 2.2 follows from 2.3.

In particular, the properties in 2.1 and 2.5 characterize the ANR among the finite-dimensional metric spaces. To consider the general case, we recall some terminology. Let $Y$ be a space, $\mathfrak{W}$ an open covering of $Y$, and $P$ a polytope (4). A partial realization of $P$ in $\mathfrak{W}$ is a continuous map $f$: $Q \to Y$, of some subpolytope $Q \subset P$ that contains the zero-skeleton $P^0$ of $P$, such that $f(Q \cap \bar{\sigma})$ is contained in some $W \in \mathfrak{W}$ for each closed simplex $\bar{\sigma}$ of $P$. It is known ([2], p. 240) that a metric space $Y$ is an ANR if and only if for each open covering $\mathfrak{W}$ of $Y$ there is a refinement $\mathfrak{V}$ such that any partial realization of any polytope $P$ in $\mathfrak{V}$ extends to a full realization in $\mathfrak{W}$. The locally equiconnected spaces that are ANRs are characterized by a weaker version of this partial realization property:

3.2. THEOREM. *Let $Y$ be locally equiconnected. Then $Y$ is an* ANR *if and only if for each open covering $\mathfrak{W}$ of $Y$ there exists a refinement $\mathfrak{V}$*

(3) We use the covering definition of dimension.

(4) All polytopes are taken to be rectilinear, and with the CW-topology ([8], p. 223); they are not required to be finite dimensional, nor locally finite.

*such that every partial realization $f\colon P^0 \to Y$ in $\mathfrak{V}$ of the zero-skeleton of any polytope $P$, extends to a full realization of $P$ in $\mathfrak{W}$.*

Proof. In view of the preceding remarks, we need to prove only that a locally equiconnected space having the stated property is an ANR. We will show that for each open covering $\mathfrak{W}$ of $Y$ there is a polytope $P$ that $\mathfrak{W}$-dominates (5) $Y$; this suffices ([2], p. 243, [4], p. 359) to establish that $Y$ is an ANR.

Given $\mathfrak{W}$, let $\mathcal{H}$ be a refinement satisfying 2.5. Let $\mathcal{H}^*$ be a star-refinement (6) of $\mathcal{H}$ and let $\mathcal{S}$ be a refinement of $\mathcal{H}^*$ having the partial realization property in the statement of the theorem relative to $\mathcal{H}^*$. Finally, let $\mathfrak{V}$ be a neighborhood-finite star-refinement of $\mathcal{S}$. Let $P$ be the nerve (7) of $\mathfrak{V}$ and let $\varkappa\colon Y \to P$ be the canonical map of $Y$ into the nerve of $\mathfrak{V}$ ([1], p. 355).

Let $g^0\colon P^0 \to Y$ be the map sending each vertex $v$ to a point of the corresponding set $V$. This is a partial realization of $P$ in $\mathcal{S}$: for, if $(v_0, \ldots, v_n)$ is any simplex of $P$, then $\bigcap_0^n V_i \neq \emptyset$, consequently $\bigcup_0^n V_i \subset$ some $S \in \mathcal{S}$. By the hypothesis, $g^0$ therefore extends to a full realization $g\colon P \to Y$ in $\mathcal{H}^*$.

We now show that for each $y \in Y$, $g \circ \varkappa(y)$ and $y$ belong to a common $H \in \mathcal{H}$. Let $y$ belong to $V_0, \ldots, V_n$ and only these sets; then $\varkappa(y)$ belongs to the closed simplex $\overline{(v_0, \ldots, v_n)}$ and therefore $g \circ \varkappa(y)$ lies in some $H_0^*$ containing $\bigcup_0^n g(v_i)$. On the other hand, $y \in \bigcup_0^n V_i \subset$ some $H_1^*$, consequently $H_0^* \cap H_1^* \neq \emptyset$ so that $y$ and $g \circ \varkappa(y)$ lie in a single $H \in \mathcal{H}$. Because of 2.5, we conclude that $Y$ is $\mathfrak{W}$-dominated by $P$, and the proof is complete.

It is easy to see that a finite-dimensional locally equiconnected space always has the partial realization property of 3.2 (which yields another proof of 3.1); it is not known whether this is also true for infinite-dimensional locally equiconnected spaces.

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(5) If $\mathfrak{W}$ is an open covering of $Y$, a space $P$ $\mathfrak{W}$-dominates $Y$ whenever there are continuous maps $\varkappa\colon Y \to P$ and $g\colon P \to Y$ such that $g \circ \varkappa$ is $\mathfrak{W}$-homotopic to the identity map of $Y$.

(6) A refinement $\mathcal{H}^*$ of $\mathcal{H}$ is called a *star-refinement* of $\mathcal{H}$ if $\bigcup \{H^* \mid H^* \cap H_0^* \neq \emptyset\} \subset$ some $H \in \mathcal{H}$ for each $H_0^* \in \mathcal{H}^*$. A covering $\mathcal{G}$ of a space is called *neighborhood-finite* if each point of the space has a neighborhood meeting at most finitely many $G \in \mathcal{G}$. Every open covering of a metric space has an open neighborhood-finite star-refinement ([7], p. 980). In this paper, a refinement of an open covering is understood to be an open covering.

(7) We realize the nerve of a covering $\mathfrak{V}$ as a rectilinear polytope in a real vector space spanned by linearly independent vectors in a fixed one-to-one correspondence with the non-empty sets $V \in \mathfrak{V}$; the vertex of the nerve corresponding to $V \in \mathfrak{V}$ is the unit point on the corresponding vector, and is denoted by the corresponding lower-case letter.



As an application of 3.2, we derive a sufficient (but not necessary) condition for a locally equiconnected space to be an ANR that is based directly on the behaviour of some given equiconnecting function. Let $\lambda\colon U \times I \to Y$ be an equiconnecting function, and let $W \subset Y$ be an open set. For any $A \subset W$, define the sets $A^n$, $n \geqslant 1$, inductively (8) by $A^1 = \lambda(A, A, I)$, $A^{n+1} = \lambda(A, A^n, I)$. If all $A \times A^n \subset U$ and all $A^n \subset W$, we say that $A$ is *$\lambda$-stable* in $W$. If $A$ is $\lambda$-stable in $W$, then it is clear that $A \subset A^1 \subset A^2 \subset \ldots \subset W$ and, if $A^\infty = \bigcup_1^\infty A^i$, then $\lambda(A, A^\infty, I) = A^\infty$. The following proof follows the lines of one due to Milnor ([6], p. 279).

3.3. Lemma. *Let $Y$ be locally equiconnected, and let $\lambda$ be a given equiconnecting function. Assume that an open covering $\mathfrak{W}$ of $Y$ has a refinement $\mathfrak{V}$ such that each $V \in \mathfrak{V}$ is $\lambda$-stable in some $W \in \mathfrak{W}$. Then every partial realization $g\colon P^0 \to Y$ in $\mathfrak{V}$ of the zero-skeleton of any polytope $P$, extends to a full realization of $P$ in $\mathfrak{W}$.*

Proof. We define an extension of $g$ over $P$ by induction on the skeletons of $P$. Well-order the vertices of $P$, and assume that $g$ has been extended to a continuous map $g^n\colon P^n \to Y$ ($P^n$ denotes the $n$-skeleton of $P$) in such a way that for each closed simplex $\bar{\sigma}^n = \overline{(p_0, \ldots, p_n)}$ we have $g^n(\bar{\sigma}^n) \subset \bigcap \{V^n \mid V \supset \bigcup_0^n g(p_i)\}$. Let $\bar{\sigma}^{n+1}$ be any closed $(n+1)$-simplex, with vertices $p_0 < p_1 < \ldots < p_{n+1}$, and note that each $x \in \bar{\sigma}^{n+1}$ can be written uniquely as $x = (1-t)p_0 + ty$ where $y \in \bar{\sigma} = \overline{(p_1, \ldots, p_{n+1})}$ and $t \in I$. Now, if $V$ is any set containing $\bigcup_0^{n+1} g(p_i)$ (such sets exist because $g$ is a partial realization in $\mathfrak{V}$) then by the inductive hypothesis we have $g(\bar{\sigma}) \subset V^n$ and therefore

$$g^{n+1}(x) = \lambda[g(p_0), g^n(y), t] \qquad (x \in \bar{\sigma}^{n+1})$$

is well-defined, and gives an extension of $g^n$ over $\bar{\sigma}^{n+1}$; since $g^{n+1}(\bar{\sigma}^{n+1}) \subset V^{n+1}$, where $V$ is any set containing $\{g(p_0), \ldots, g(p_{n+1})\}$, we have $g^{n+1}(\bar{\sigma}^{n+1}) \subset \bigcap \{V^{n+1} \mid V \supset \bigcup_0^{n+1} g(p_i)\}$. Extending over each $\bar{\sigma}^{n+1}$ of $P$ in this manner, gives a continuous $g^{n+1}\colon P^{n+1} \to Y$ and completes the induction. It is evident that the map $G\colon P \to Y$ obtained is a realization of $P$ in $\mathfrak{W}$.

3.4. Theorem. *Let $Y$ be locally equiconnected. If $Y$ has an equiconnecting function $\lambda$ with the property that for each $y_0 \in Y$ and neighborhood $W$ of $y_0$, there is a neighborhood $V \subset W$ of $y_0$ that is $\lambda$-stable in $W$, then $Y$ is an ANR.*

Proof. It is clear that with the given hypothesis, every open covering $\mathfrak{W}$ of $Y$ has a refinement $\mathfrak{V}$ satisfying 3.3, so an application of 3.2 completes the proof.

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(8) An equiconnecting function need not satisfy the condition $\lambda(a, b, I) = \lambda(b, a, I)$.

Let $Y$ be locally equiconnected, and let $\lambda\colon U\times I\to Y$ be an equiconnecting function. In his paper ([6], p. 279) Milnor proved that if $Y$ has an open covering $\mathfrak{W}$ by $\lambda$-convex sets (that is, $W\times W\subset U$ and $\lambda(W, W, I) = W$ for each $W\in\mathfrak{W}$), then $Y$ belongs to the homotopy type of an ANR. We remark that, in view of 3.3, his method applies equally well to show that if some open covering $\mathfrak{W}$ of $Y$ satisfying $W\times W\subset U$ for each $W\in\mathfrak{W}$ has a refinement $\mathfrak{V}$ such that each $V$ is $\lambda$-stable in some $W$, then $Y$ belongs to the homotopy type of an ANR.

**4. Borsuk's space.** As indicated after the proof of 3.2, no example of a locally equiconnected non-ANR is known. If such a space exists, then it must be infinite-dimensional and, according to 2.3, also locally contractible. In this section we will show that the evident candidate, Borsuk's [9] locally contractible non-ANR, is not locally equiconnected.

Regard the Hilbert cube $H$ as the cartesian product $\prod_1^\infty I_i$ of a countable family of unit intervals (⁹), and for each $k = 1, 2, \ldots$ let $C_k$ be the $k$-cube

$$C_k = \{x\in H\,|\,1/(k+1)\leqslant [x]_1\leqslant 1/k \quad\text{and}\quad [x]_i = 0 \quad\text{for all } i>k\}.$$

Let $B_k$ be the boundary $(k-1)$-sphere of $C_k$ and let $B_0 = \{x\in H\,|\,[x]_1 = 0\}$. Borsuk's locally contractible non-ANR is the compact subspace $B = \bigcup_{i=0}^\infty B_i \subset H$. Recall that for each integer $N>0$ there is a retraction $\varrho_N\colon B\to B_N$ given by

$$[\varrho_N(x)]_1 = \begin{cases} 1/(k+1) & \text{if} \quad [x]_1\leqslant 1/(k+1),\\ [x]_1 & \text{if} \quad 1/(k+1)\leqslant [x]_1\leqslant 1/k,\\ 1/k & \text{if} \quad 1/k\leqslant [x]_1,\end{cases}$$

$$[\varrho_N(x)]_i = \begin{cases} [x]_i & \text{if} \quad 2\leqslant i\leqslant N,\\ 0 & \text{if} \quad i>N.\end{cases}$$

4.1. THEOREM. *$B$ is not locally equiconnected.*

Proof. We argue by contradiction. Assume that $B$ were locally equiconnected. By 2.5, there would exist an open covering $\mathfrak{V}$ such that any two $\mathfrak{V}$-close maps of $B$ into itself are homotopic. Since $B$ is compact, we can assume $\mathfrak{V}$ to be a finite covering, say $\mathfrak{V} = \{V_1, \ldots, V_s\}$ and also that each $V_i$ is a set of the form $B\cap\langle U_{i(1)}, \ldots, U_{i(n_i)}\rangle$, where $U_{i(q)}$ is a set open in the $i(q)$-factor $I$.

(⁹) We denote the $i$th coordinate of $x\in H$ by $[x]_i$ and, for open sets $U_{a_i}\subset I_{a_i}$, $i = 1, \ldots, s$, $\langle U_{a_1}, \ldots, U_{a_s}\rangle$ denotes the basic open set $\{x\in H\,|\,[x]_{a_i}\in U_{a_i},\ i = 1, \ldots, s\}$.



Letting $N = \max\{i(q)\,|\,1\leqslant q\leqslant n_i;\ 1\leqslant i\leqslant s\}$, the largest index for which a coordinate is restricted, we would define a continuous map $\varphi$ of $B$ into itself by

$$[\varphi(x)]_i = \begin{cases} [x]_i & \text{if} \quad i\leqslant N,\\ 0 & \text{if} \quad i>N.\end{cases}$$

Due to the choice of $N$, it is clear that $\varphi$ would be $\mathfrak{V}$-close to the identity map of $B$, and so $\varphi$ would be homotopic to the identity map of $B$. It now follows that $\varrho_{N+1}\circ(\varphi|B_{N+1})\colon B_{N+1}\to B_{N+1}$ would be homotopic to the identity map of the $N$-sphere $B_{N+1}$ on itself. This is the desired contradiction: for, $\varrho_{N+1}\circ\varphi(B_{N+1})$ is clearly a proper subset of $B_{N+1}$ and consequently $\varrho_{N+1}\circ(\varphi|B_{N+1})$ is nullhomotopic.

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