On some problems of G. Gratzer and E. T. Schmidt

by

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This paper solves problems 1, 13 and 15 posed by G. Gratzer and E. T. Schmidt in their paper [2]. Defining the standard element s of a lattice L to be one which satisfies the equality $x(s+y) = xs+xy$ for all x, y in L and a standard ideal as a standard element of the lattice of ideals of L, G. Gratzer and E. T. Schmidt prove in [2] the following fundamental characterization theorems on standard elements and standard ideals.

THEOREM 1. *The following conditions on an element s of the lattice are equivalent.*

(α) *s is a standard element.*

(β) *The equality $u = us+ut$ holds whenever $u \leqslant s+t$ for all u, t in L.*

(γ) *The relation θ_s defined by $x \equiv y(\theta_s)$ if and only if $xy+s_1 = x+y$ for some $s_1 \leqslant s$ is a congruence relation on L.*

(δ) *For all x, y in L, $s+xy = (s+x)(s+y)$ and $sx = sy$; $s+x = s+y$ imply $x = y$.*

THEOREM 2. *The following seven conditions for an ideal S of the lattice L are equivalent.*

(α') *S is a standard ideal.*

(α'') *The equality $I(S+K) = IS+IK$ holds if I and K are principal ideals.*

(β') *For any ideal I, the elements of $S+I$ are of the form $s+x$; $s \in S$ and $x \in I$.*

(β'') *For any principal ideal I, the elements of $S+I$ are of the form $s+x$; $s \in S$ and $x \in I$.*

(γ') *The relation θ_s of $I(L)$ defined by $I \equiv K(\theta_s)$ if and only if $IK+S_1 = I+K$ with a suitable $S_1 \subset S$ is a congruence relation on $I(L)$.*

(γ'') *The relation $\theta(s)$ of L defined by $x \equiv y(\theta(S))$ if and only if $xy+s = x+y$ with a suitable s in S is a congruence relation.*

(δ') *For all I and K of $I(L)$, $S+IK = (S+I)(S+K)$ and $SI = SK$; $S+I = S+K$ imply $I = K$.*

They next ask the question (problem 1 of [2]). Is the standardness of S equivalent to

(δ'') *for all principal ideals* I *and* K *of* L

(i) $S+IK = (S+I)(S+K)$

and

(ii) $SI = SK$; $S+I = S+K$ *imply* $I = K$.

We give a counter-example and give a solution to the above problem in the negative. We however give modified conditions under which this could be so.

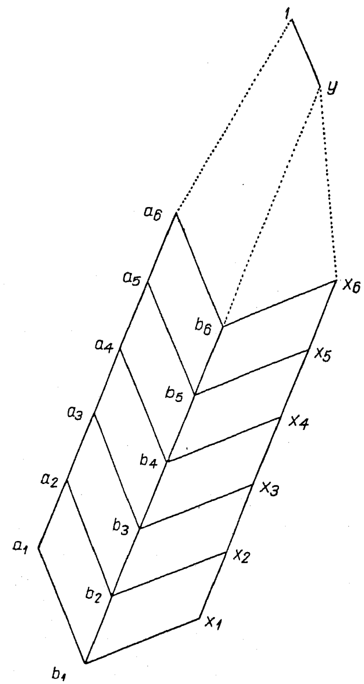

Fig. 1.

Consider the lattice L of figure 1. The lattice L consists of three sequences (a_i), (b_i) and (x_i). Each of these is an ascending sequence. Further $a_i \nearrow 1$; $b_i \nearrow y$ and $x_i \nearrow y$. Also $a_i > b_i$ for each i, and $x_i > b_i$ for each i. The other lattice operations are clear from the figure.

Consider the ideals S of L consisting of the two sequences (a_i) and (b_i). S is an ideal of L satisfying conditions (i) and (ii) of (δ''). But S is not a standard ideal of L. For if A = the principal ideal generated by y and B = the ideal consisting of the sequences (x_i) and (b_i) then $A \gneq B$; as $y \in A$ and $y \notin B$. But $S+A = S+B = L$ (the whole lattice) and $SA = SB$ = the ideal consisting of the sequence (b_i). Thus conditions (i) and (ii) of (δ'') are not equivalent to the standardness of the ideal S.

Though conditions (i) and (ii) of (δ'') are not equivalent to the standardness of the ideal in general but for the following two particular cases they are equivalent.

1. If L is a modular lattice then condition (δ'') implies the standardness of the ideal S.

This is so, since condition (i) alone of (δ'') implies the neutrality of the ideal S, when L is modular.

2. If S is a principal ideal of L then (δ'') implies the standardness of the ideal S.

Proof follows from condition (δ) of theorem 1.

Next we give modified conditions for the standardness of the ideal thus:

THEOREM 3. *Any ideal S of a lattice L is standard if and only if* (a) $S+AB = (S+A)(S+B)$ *for all principal ideals A, B of L and* (b) $A \geqslant B$; $S+A = S+B$ *and* $SA = SB$ *imply* $A = B$ *for all principal ideals A of L and all ideals B of L.*

Proof. Let S be any ideal of L satisfying the two conditions above. As S satisfies (a), S is a distributive ideal of L. Let θ_S be the distributive congruence generated by S. We shall show that θ_S is a standard congruence on L using condition (b).

(P) *Let* $x \equiv y(\theta_S) \Longleftrightarrow xy \equiv x+y(\theta_S)$ *implies* $xy+a = x+y+a$ *for some a in S.*

Let A and C be the principal ideals generated by $(x+y)$ and xy respectively. Let $B = SA+C$; then $A \geqslant B$ and $S+A = S+B$ (as each equals $S+C$ by (P)). Also $SA = SB$ (as $SA \geqslant SB \geqslant SA$). Further A is principal, therefore, by condition (b), $A = B$. This implies that B is a principal ideal generated by $(x+y)$. Therefore $x+y = xy+s$ for some s in S. This implies that θ_S is a standard congruence and hence S is a standard ideal of L.

Hence onwards we are interested in those congruences on L which have standard ideals as their kernels and give a solution to problem 13 of [2].

THEOREM 4. *The kernel of every congruence relation on a lattice L is a standard ideal if and only if L satisfies the following condition:*

(η) *If* (a, b) *is a lattice translate* (cf. [4]) *of an interval* $p = (x, 0)$ *then there exists a* y *in* L *such that* $0 \equiv y$ (mod *the congruence generated by* p, θ_p *(say)) and* $a = b+y$.

Proof. Necessity: The kernel of every congruence is a standard ideal and so is the kernel of θ_p. Let S be the kernel of θ_p, then $\theta_p = \theta_S$, i.e., θ_p is a standard congruence. Now as (a, b) is a lattice translate of p, $a \equiv b\ (\theta_p)$ implies $a = b+y$ for some y in S, implies $a = b+y$ for some y with $0 \equiv y(\theta_p)$. Thus L satisfies condition (η).

Sufficiency. Let S be any congruence ideal of L. Let θ_S be the congruence generated by S. Let $a \equiv b(\theta_S)$, $a > b$; then there exists a finite chain $b = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = a$ such that (x_i, x_{i-1}) is a lattice translate of some interval (b_i, a_i) $(a_i < b_i)$ of S for each i. This implies (x_i, x_{i-1})

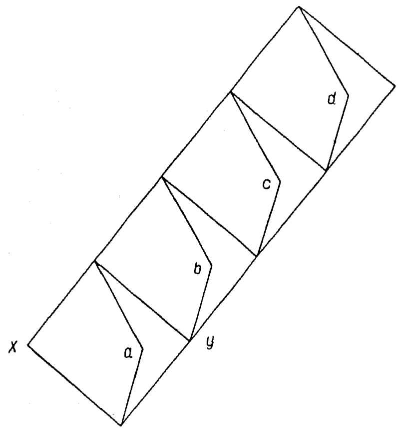

Fig. 2

is a lattice translate of $J_i = (b_i, 0)$ for each i; implies $x_{i-1}+y_i = x_i$ for y_i in S as $\theta_{J_i} \subset \theta_S$ for each i. Implies $b+y = a$; where $y = \sum_{i=1}^{n} y_i$ belongs to S; S being an ideal. This implies θ_S is a standard congruence on L and so S is a standard ideal of L. Making use of this we get:

THEOREM 5. *There is a* 1-1 *correspondence between all standard ideals and congruences of a lattice with zero if and only if* L *satisfies the following property.*

(i) *Given an interval* $p = (v, u)$ *in* L *there exists a* x *in* L *such that* $0 \equiv x(\theta_p)$ *and* $v = u+x$.

Proof. Let there be a 1-1 correspondence between standard ideals and congruences on L and let $p = (v, u)$ be an interval of L. Consider θ_p, the congruence generated by p. Then θ_p is a standard congruence. Thus $u \equiv v(\theta_p)$ implies $u+x = v$ for some x in $I(\theta_p)$, implies $u+x = v$ with $x \equiv 0(\theta_p)$. Thus L satisfies condition (i). Hence condition (i) is necessary.

Conversely let L satisfy the condition of the theorem. Now this condition implies condition (η) of theorem 4. Hence the kernel of every congruence on L is a standard ideal. Also this condition implies that every congruence is determined by its zero class. Hence the sufficiency of the condition.

COROLLARY. *There is a* 1-1 *correspondence between standard ideals and congruences of a weakly complemented lattice* (cf. [2] also).

Next we tackle problem 15 of [2] and show that the degree of non-distributivity of a modular lattice as defined thereof is inconsistent. For, consider the modular lattice of figure 2 and the sets of elements $S_1 = (a, x, y)$ and $S_2 = (a, b, c, d)$ of L. Now both these sets satisfy the condition; that each is maximal with respect to the property that any three elements of it generate a nondistributive sublattice of L. Also the cardinality of these sets is obviously different. Hence the degree of nondistributivity of the modular lattice as defined in problem 15 of [2] has no meaning.

References

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