Zero-dimensional sets blocking connectivity functions *

by

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In [2], p. 262, Stallings described a compact 0-dimensional set $K \subset I^2$ (closed 2-cell) such that the graph of every continuous function f: $I \to I$ intersected K. He asked this question: If f is a connectivity function on I into I (definition below), then is it necessarily true that such a set K must intersect the graph of f? This question is answered in the present paper. Example 1 is a compact 0-dimensional set $K_1 \subset I^2$ which blocks (i.e., intersects the graph of) every continuous function but does not block every connectivity function. The compact 0-dimensional set K_2 of Example 2 blocks every connectivity function. A theorem is abstracted from the argument needed to prove that Example 1 has the desired properties.

DEFINITION. A function f: $A \to B$ (topological spaces) is a *connectivity function* if and only if for every connected set $C \subset A$ the graph of $f|C$ is connected.

EXAMPLE 1. Description. First, define the Cantor set C on $I = [0,1]$ as $\bigcap_{n=1}^{\infty} C_n$, as follows. We get C_1 from $[0,1]$ by taking out the open interval of length $1/4$ with center at $1/2$. In general, C_n is the union of 2^n closed intervals, and C_{n+1} is obtained from C_n by taking out of each of these 2^n intervals a concentric open interval of length $1/2^{2n+2}$. Thus the sum of the lengths of the intervals taken out is $\sum_{0}^{\infty} 2^n/2^{2n+2} = 1/2$, so we also have $m(C) = 1/2$, where m denotes Lebesgue linear measure. Now define F: $I \to I \times I$ as follows:

$$(*) \qquad \begin{aligned} F(t) &= \big(x(t), y(t)\big), \\ x(t) &= 2m(C \cap [0,t]), \\ y(t) &= 4m(C \cap [0,t]) - t = 2x(t) - t, \end{aligned}$$

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for $0 \leqslant t \leqslant 1$. Let $M = F(I)$ and

$$K_1 = F(C).$$

Figure 1 shows an approximation to M. The set K_1 is obtained from M by taking out all open vertical intervals. These occur when the abscissa is $1/2, 1/4, 3/4, 1/8, \ldots$

Now F is a homeomorphism and M is an arc with end-points $(0,0)$ and $(1,1)$. The set K_1 is a topological Cantor set and is a subset of M.

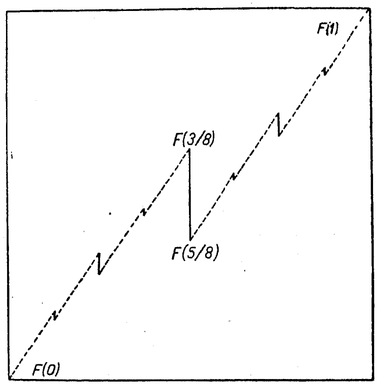

Fig. 1

The set $M - K_1$ is a countable union of disjoint open vertical intervals, one on each vertical line with abscissa of the form $k/2^n$ (k and n positive integers, $k < 2^n$, k odd). To see that $M \subset I \times I$, first note that, from (*), $x(0) = 0$, $x(1) = 1$, and x does not decrease as t increases. Also $y(0) = 0$, $y(1) = 1$. To prove that $y(t) > 0$ for all $t > 0$, it is sufficient to show that the average metric density of C on $[0, t]$ (i.e., $m(C \cap [0, t])/t$) is greater than 1/4. In fact, a calculation shows that the minimum average metric density of C on $[0, t]$ occurs for $t = 5/8$, and is equal to 2/5. Thus $y(t) \geqslant 0$ on $0 \leqslant t \leqslant 1$. That $y(t) \leqslant 1$ follows from symmetry ($y(1-t) = 1 - y(t)$).

We now prove that K_1 blocks every continuous f: $I \to I$. Assume that G, the graph of a continuous f, does not intersect K_1. Then $f(0) > 0$ and $f(1) < 1$, so from formulas (*) we have $y(0) < f(x(0))$, but $y(1) > f(x(1))$. Thus there exists a smallest w ($0 < w < 1$) such that $y(w) \geqslant f(x(w))$. Then $(x(w), y(w)) \in M \cap G$, but is not an element of K_1, so $w \in I - C$ and there exists $w_1 < w$ such that $[w_1, w] \subset I - C$. But then $y(w_1) > y(w)$ and $x(w_1) = x(w)$, so $y(w_1) > f(x(w_1))$, a contradiction.

Finally, we show that K_1 does not block every connectivity function f: $I \to I$. We use the theorem, below, and our immediate objective is to identify the terms used in the theorem and to show that the hypothesis is satisfied. Obviously K_1 is a compact 0-dimensional subset of I^2. Let D be the set of all x of the form $k/2^n$, k and n positive integers, $k < 2^n$, k odd, and for $x \in D$ let s_x be the component of $M - K_1$ in I_x. (For $0 \leqslant x \leqslant 1$, I_x is the vertical interval from $(x, 0)$ to $(x, 1)$.)

Now suppose that N is a continuum in I^2 such that (ii) and (iii) are false. Then there exist points $p \in N$, $q \in N$, $x \in D$ such that $x_p < x < x_q$ and s_x is not a subset of N. Choose y so that $(x, y) \in s_x - N$. For every $\varepsilon > 0$ let T_ε be the arc which is the union of two vertical intervals and one horizontal interval connecting the following points in the indicated order: $(x-\varepsilon, 0)$, $(x-\varepsilon, y)$, $(x+\varepsilon, y)$, $(x+\varepsilon, 1)$. For sufficiently small ε, T_ε separates p from q in I^2, hence intersects N. But for sufficiently small ε, $T_\varepsilon \cap K_1 = \emptyset$ and (since N is closed) the interval from $(x-\varepsilon, y)$ to $(x+\varepsilon, y)$ does not intersect N. Thus $N - K_1$ intersects at least one of the two vertical intervals having abscissas $x - \varepsilon$ and $x + \varepsilon$, so (i) is true. We have shown that the hypothesis of the following theorem is satisfied, and thus it follows that K_1 does not block every connectivity function.

THEOREM. Hypothesis. *K is a compact 0-dimensional subset of I^2, $D \subset I$ is a countable set and for every $x \in D$ there is an open vertical interval $s_x \subset I_x$ such that $s_x \cap K = \emptyset$ and such that if N is any continuum in I^2, separating I^2, then at least one of the following is true:*

(i) *$N - K$ intersects the vertical interval I_x for every x in some set having the cardinality of the continuum,*

(ii) *N is a subset of a single vertical interval, or*

(iii) *N contains s_x for some $x \in D$.*

Conclusion. *There exists a connectivity function f: $I \to I$ whose graph G does not intersect K.*

Proof. Let $\mathfrak{N}$ be the set of all continua $N \subset I^2$ such that N separates I^2 and (i) is true. Since each $N \in \mathfrak{N}$ is closed and I^2 has a countable base, it follows that $|\mathfrak{N}| \leqslant |I|$, where $|\ |$ denotes cardinality. Obviously $|\mathfrak{N}| \geqslant |I|$ so $|\mathfrak{N}| = |I|$. It follows that there exists a smallest ordinal Ω such that $|\mathfrak{N}| = |\Omega| = |I|$. We define f: $I \to I$ as follows:

Step 1. For all $x \in D$ define $f(x) \in I$ so that $(x, f(x))$ is the midpoint of the vertical interval s_x. Then $(x, f(x)) \notin K$.

Step 2. We may write $\mathfrak{N} = \{N_\alpha : \alpha < \Omega\}$, and $\Omega = (0, 1, \ldots; \omega, \omega + 1, \ldots)$. From (i) in the hypothesis, and the fact that D is countable, there exists $x_0 \notin D$ such that $N_0 - K$ intersects the vertical interval with abscissa x_0, and we define $f(x_0)$ so that $(x_0, f(x_0)) \in N_0 - K$. In general, we want the following:

(a) $x_a \in I - D - \{x_\gamma:\ \gamma < a\}$, and

(b) $(x_a, f(x_a)) \in N_a - K$.

Assume $\beta < \Omega$ and for all $a < \beta$, x_a and $f(x_a)$ have been defined and (a) and (b) are true. Let $A_\beta = \{x:\ (N_\beta - K) \cap I_x \neq \emptyset\}$ and let $B_\beta = D \cup \cup \{x_a:\ a < \beta\}$. From cardinality considerations it is clear that $A_\beta - B_\beta \neq \emptyset$ and we choose $x_\beta \in A_\beta - B_\beta$ and define $f(x_\beta)$ so that $(x_\beta, f(x_\beta)) \in N_\beta - K$. Thus we may assume that x_a and $f(x_a)$ have been defined for all $a < \Omega$ and (a) and (b) are true.

Step 3. Set $C = I - D - \{x_a:\ a < \Omega\}$. If $x \in C$ define $f(x)$ so that $(x, f(x)) \in I^2 - K$. This completes the definition of $f\colon I \to I$.

Assertion 1. *$(x, f(x)) \notin K$ for all $x \in I$.*

For the proof of Assertion 2 (below) we need the following

Lemma. *If A and B are mutually separated sets in I^2 and $a \in A$, $b \in B$, then there exists a continuum $N \subset I^2 - (A \cup B)$ such that N separates a and b in I^2.*

Proof. By [1], Theorem 73, p. 150, there exists a closed set N such that (i) $N \subset I^2 - (A \cup B)$, (ii) N separates a and b in I^2, and (iii) N is irreducible with respect to properties (i) and (ii). Let D_a be the complementary domain of N (relative to I^2) which contains a. Then $\overline{D}_a$ and $I^2 - D_a$ are continua whose union is I^2 and whose intersection is N. Since I^2 is unicoherent, it follows that N is connected.

Assertion 2. *G, the graph of f, is a connected set.*

Proof. Assume that G is not connected. Then by the lemma, there exists a continuum N such that $N \subset I^2 - G$ and N separates I^2. It follows that (ii) and (iii) of the hypothesis are false for this N. Therefore (i) is true and $N \in \mathfrak{N}$, so there exists $a < \Omega$ such that $N = N_a$, and $(x_a, f(x_a)) \in N \cap G$, a contradiction. Thus G is connected.

Assertion 3. *f is a connectivity function.*

Proof. Suppose that C is a connected subset of I such that the graph of $f|C$ is not connected. Then it easily follows that there exists a closed interval $C_1 = [c, d]$ with $c < d$ such that $C_1 \subset C$ and T, the graph of $f|C_1$, is not connected. Then $T = A_1 \cup B_1$, mutually separated, with $(c, f(c)) \in A_1$ (a matter of notation). If $(d, f(d)) \in A_1$, write

$$G = [A_1 \cup \{(x, f(x)):\ x \notin C_1\}] \cup B_1,$$

mutually separated sets. If $(d, f(d)) \in B_1$, write

$$G = [A_1 \cup \{(x, f(x)):\ x < c_1\}] \cup [B_1 \cup \{(x, f(x)):\ x > d_1\}],$$

mutually separated sets. In either case we have a contradiction to the fact that G is connected. This completes the proof of our theorem.

Example 2. For each positive integer n, let D_n be the set of all numbers $k/2^n$, where k is an odd positive integer less than 2^n. Thus $D_1 = \{1/2\}$, $D_2 = \{1/4, 3/4\}$, etc. Set $D = \bigcup_{n=1}^{\infty} D_n$. (This is the same D as in Example 1.)

For each n $(n = 1, 2, \ldots)$ we will define, for each $x \in D_n$, a disjoint collection $\mathfrak{V}(x)$ of 2^n vertical intervals all in I_x, and a collection $\mathfrak{T}(n)$

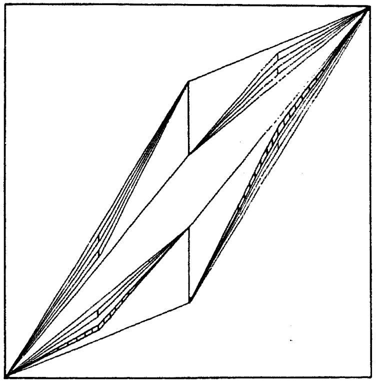

Fig. 2

of 4^n closed 2-simplexes. Figure 2 shows the 4 elements of $\mathfrak{T}(1)$, the 16 elements of $\mathfrak{T}(2)$, the two elements of $\mathfrak{V}(1/2)$, and the four elements (each) of $\mathfrak{V}(1/4)$ and $\mathfrak{V}(3/4)$. If $n = 1$ and $x \in D_1$ then $x = 1/2$ and (definition) $\mathfrak{V}(1/2) = \{[(1/2, 1/5), (1/2, 2/5)], [(1/2, 3/5), (1/2, 4/5)]\}$. Each of the four elements of $\mathfrak{T}(1)$ has an element of $\mathfrak{V}(1/2)$ as one side, and the third vertex is $(0, 0)$ or $(1, 1)$.

Auxiliary definitions. Suppose that T is a closed 2-simplex with vertices a, p, and q such that $T \subset I^2$, $x_p = x_q$ (i.e., $[p, q]$ is a vertical interval) and $y_p < y_q$. Let $x = (x_a + x_p)/2$ and let $[s, t]$ denote the vertical interval $I_x \cap T$, with $y_s < y_t$. Divide $[s, t]$ into 5 equal subintervals and let $\mathfrak{W}(T)$ be the disjoint set consisting of the second and the fourth of these subintervals. Let $\mathfrak{S}(T)$ be the collection of 4 closed 2-simplexes determined as follows:

Case 1. $x_a < x_p$. Each element of $\mathfrak{S}(T)$ has an element of $\mathfrak{W}(T)$ as one side, the third vertex being a or q.

Case 2. $x_a > x_p$. Each element of $\mathfrak{S}(T)$ has an element of $\mathfrak{W}(T)$ as one side, the third vertex being a or p.

Inductive definition. Let $\mathfrak{T}(n+1) = \bigcup_{T\in\mathfrak{T}(n)} \mathfrak{S}(T)$. Suppose $x \in D_{n+1}$, $T \in \mathfrak{T}(n)$ and $T \cap I_x \neq \emptyset$. Then T is a triangle with vertices a, p, and q, with $x_p = x_q$, and $x = (x_a + x_p)/2$. Thus the intervals of $\mathfrak{W}(T)$ (Auxiliary definition) are subintervals of I_x. Let $\mathfrak{V}(x) = \bigcup \mathfrak{W}(T)$, where the union is over all $T \in \mathfrak{T}(n)$ such that $T \cap I_x \neq \emptyset$. This completes the inductive definition of $\mathfrak{T}(n)$ and of $\mathfrak{V}(x)$, $x \in D_n$, $n = 1, 2, \ldots$

For each n let M_n be the union of elements of $\mathfrak{T}(n)$, and let

$$K_2 = \bigcap_{n=1}^{\infty} M_n .$$

It is clear from the definitions that $M_n \supset M_{n+1}$, K_2 is compact, and for $x \in D_n$, $I_x \cap M_{n+1}$ consists of the finite set of end points of the elements of $\mathfrak{V}(x)$. Also, $I_0 \cap M_n = \{(0, 0)\}$ and $I_1 \cap M_n = \{(1, 1)\}$. If $x \notin D$ and $x \neq 0$, $x \neq 1$, then $I_x \cap M_n$ is the union of a disjoint collection of 2^n vertical intervals each of length less than 5^{-n}, and $I_x \cap K_2$ is a topological Cantor set. Now M_2 has two components, lying respectively in the sets $\{(x, y): 0 \leqslant x \leqslant 1/2\}$ and $\{(x, y): 1/2 \leqslant x \leqslant 1\}$. Similarly, for each n, each component of M_{n+1} has an extent in the x-direction equal to 2^{-n}. Since K_2 contains no vertical interval, it therefore follows that K_2 is totally disconnected.

We now prove that K_2 blocks every connectivity function from I into I. Let $f: I \to I$ be a function, G its graph, such that $G \cap K_2 = \emptyset$. In the following construction the notation, for simplicity, does not show the dependence of $V(x)$, N_i, and N on f. For all i $(i = 1, 2, \ldots)$, and $x \in D_i$, we define sets $V(x)$ and N_i such that

(a) $V(x) \in \mathfrak{V}(x)$ and $(x, f(x)) \notin V(x)$,

(b) N_i is a continuum which is a subset of the union of M_i and all vertical intervals $V(x)$ for $x \in D_j$, $j < i$, and

(c) the number of components of $N_i \cap M_i$ is 2^{i-1}, and each of these has an extent in the x-direction of 2^{-i+1}.

First, if $x \in D_1$ then $x = 1/2$, and $V(1/2)$ (definition) is one of the two elements of $\mathfrak{V}(1/2)$ such that $(1/2, f(1/2)) \notin V(1/2)$. Of the four elements of $\mathfrak{T}(1)$, let T_0 and T_1 be the two which have $V(1/2)$ as a side, with $(0, 0) \in T_0$ and $(1, 1) \in T_1$. Set $N_1 = T_0 \cup V(1/2) \cup T_1 = T_0 \cup T_1$.

Next, let $V(1/4)$ be one of the two elements of $\mathfrak{V}(1/4)$ which are subsets of T_0 such that $(1/4, f(1/4)) \notin V(1/4)$. Of the 16 elements of $\mathfrak{T}(2)$ there are two which are subsets of T_0 and have $V(1/4)$ as a side. These may be labelled T_{00} and T_{01} so that $(0, 0) \in T_{00}$ and the upper end point of $V(1/2)$ is in T_{01}. Sets $V(3/4)$, T_{10} and T_{11} may be defined similarly, with $(1, 1) \in T_{11}$, so that when we set $N_2 = T_{00} \cup T_{01} \cup V(1/2) \cup T_{10} \cup T_{11}$ requirements (a), (b), and (c) are satisfied $(i \leqslant 2)$.

It is clear that this process can be continued. Set $N = \bigcap_{i=1}^{\infty} N_i$. Then N is a continuum (in fact, an arc) containing $(0, 0)$ and $(1, 1)$. Suppose $p \in N$. If $p \in V(x)$ for some $x \in D$ then $p \notin G$, by (a). If p is not in any $V(x)$ then for all i (by (b)), $p \in M_i$, hence $p \in K_2$ and therefore $p \notin G$. Then $N \cap G = \emptyset$, so G is not connected. Thus f is not a connectivity function.

References

[1] R. L. Moore, *Foundations of Point Set Theory*, New York 1932.

[2] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959), pp. 249-263.

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