On regular extensions of operator systems

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The aim of this note is to propose a slightly improved version of the extension theorem given by W. Słowikowski in this journal, [8].

1. An operator system is an ordered pair \((S, X)\), where
   a. \(S\) is a semigroup with the unit element,
   b. \(X\) is a commutative group,
   c. with each element \(A \in S\) there is associated a subgroup \(G_A \subseteq X\), and
      and with each pair \(A \in S, x \in G_A\) there is associated an element \(y \in X\),
      which is called the composition of \(A\) and \(x\), \(y = Ax\), and
      \(Ax \in X\) is a homomorphism of \(G_A\) onto the whole group \(X\).

We do not assume the cancellation law for \(S\).

An operator system \((S, X)\) is linear, if \(X\) is a linear space, and
\(x \in G_A\) implies \(Ax \in G_A\), and \(D(\lambda x) = \lambda Dx\) for every scalar \(\lambda\).

We may always reduce the theory of linear operator systems \((S, X)\) to the theory of ordinary operator systems considering \(X\) a group, and enlarging \(S\) to include all the operators of multiplication by a scalar.

An operator system is regular, if
1. \(G_A = X\) for every \(A \in S\), and
2. \(D(\lambda x) = (\lambda x)\), which means that if either member of this equation makes sense then the other does too and both are equal.

In regular operator systems composition is always feasible, and the semigroup operation is compatible with the operation of superposition of the maps \(A: G_A \to X\).

The notion of operator system was first introduced by Słowikowski in [6] and then described in detail in [8], but it lurks in all the papers cited. We follow the terminology of Słowikowski, but we do not require here \(a \quad \text{a priori} \quad \text{a posteriori} \quad \text{superposition of its elements considered as maps} \quad \text{as it is done in [5]. Dropping these two requirements we avoid some inconvenient conditions on domains} \quad \text{an important example of a linear operator}
system is described in [1]. It is \((D, C)\), where \(C\) is the space of all real continuous functions \(\pi(t)\) defined for real \(t\), and \((D)\) is the semigroup of formal differential operators generated by \(D = d/dt\). The domain \(\mathbb{D}_{\mathbb{R}}\) is the set of functions with continuous \(k\)-th derivatives. This operator system is not regular. The smallest regular operator system containing it is \((D, Y)\), where \(Y\) is the linear space of distributions of finite order. Likewise, in abstract theory of the operator systems, one can develop a theory of extensions to regular operator systems. This is the main point of all the papers cited. We shall state here one more extension theorem, which we claim is still more handy.

II. Let \((S, X)\) be an operator system. An ideal (or a regularizer) of this operator system is a function \(\mathcal{Z}\) which associates every element \(A \in S\) with a subgroup \(\mathcal{Z}(A) \subseteq X\), so that

1. \(\mathcal{Z}(B) \subseteq \mathcal{Z}(AB)\),
2. \(\mathcal{Z}(AB) \cap A^{-1}(X) = A^{-1}(\mathcal{Z}(B))\),
3. \(\mathcal{Z}(AB) = \mathcal{Z}(BA)\).

If in addition
4. \(\mathcal{Z}(I) = \{0\}\),
then the ideal \(\mathcal{Z}\) is called an extensor (cf. [8], p. 254). In the special case of the operator system \((D, C)\) defined above, the function which associates \(D_X\) with the set on which \(D_X\) vanishes is an extensor.

We distinguish a special class of operator systems \((S, X)\) which satisfy the following condition:

\((\ast)\) For every \(A, B \in S\) and \(x \in X\),
\[B^{-1}(A^{-1}x) \cap (AB)^{-1}x \cap (BA)^{-1}x \neq 0.\]

This condition amounts to saying that for every \(A, B \in S\) and \(x \in X\), there exists \(u \in G_{AB} \cap G_{BA} \cap G_A\) such that
\[x = (AB)u = (BA)u = B(Au)\].[5]

**Lemma I.** If \((S, X)\) is an operator system satisfying \((\ast)\) and if \(\mathcal{Z}\) is an extensor for \((S, X)\), then
\[A(Bu) - B(Av) = 0 \quad \text{implies} \quad u - v \in \mathcal{Z}(AB).\]

**Proof.** Suppose that \(A(Bu) = B(Av) = y\). It follows from \((\ast)\) that there exist elements \(s, t, w\) such that
\[y = (AB)s = (BA)s,\]
\[y = (AB)t = A(Bt),\]
\[y = (BA)w = B(Aw).\]

We have
\[A(Bu) - A(Bv) = 0 \quad \text{implies} \quad B(u - t) \in \mathcal{Z}(A)\] and \(u - t \in \mathcal{Z}(A)\),
\[B(Av) - B(Aw) = 0 \quad \text{implies} \quad A(v - w) \in \mathcal{Z}(B)\] and \(v - w \in \mathcal{Z}(AB)\),
\[(AB)
\[B(Av) - (BA)w = 0 \quad \text{implies} \quad t - w \in \mathcal{Z}(AB),\]
\[(BA)w - (BA)v = 0 \quad \text{implies} \quad s - t \in \mathcal{Z}(AB).\]
Therefore \(u - t = (u - t) + (t - s) + (s - w) + (w - v) \in \mathcal{Z}(AB)\), which is what we wanted to prove.

The main point of this paper is

**The Extension Theorem.** Let \((S, X)\) be an operator system satisfying \((\ast)\), and let \(\mathcal{Z}\) be an extensor. Then there exists a unique regular operator system \((D, Y)\) such that
\[n.\]
\[\mathcal{Z}\] is a subgroup of \(Y\),
\[D\] is a commutative semigroup of endomorphisms for which the domains are the entire group \(Y\),
\[c.\] For each \(A \in S\) there exists an \(\tilde{A} \in D\) such that the map \(A: G_{\tilde{A}} \rightarrow Y\) is contained in the map \(\tilde{A}: G_{\tilde{A}} \rightarrow Y\), and the map: \(\tilde{A} \rightarrow \tilde{A}\) makes \(D\) a homomorphic image of \(S\).
\[d.\] For every \(y \in Y\), there exists \(A \in S\) and \(x \in X\) such that \(y = Ax\),
\[e.\] \(A^{-1}(0) \cap X = \mathcal{Z}(A)\).

**Proof.** We consider the Cartesian product \(S \times X\) and a relation of equivalence in it:
\[(A, x) \sim (B, y) \quad \text{iff there are} \quad u, v \in X\] such that \(x = Bu\), \(y = Av\) and \(u - v \in \mathcal{Z}(AB)\).

We shall show that this relation is really a relation of equivalence, i.e., that it is reflexive, symmetric, and transitive. The first two properties hold, it is trivial. We shall only prove that \(\sim\) is transitive. Suppose that
\[(A, x) \sim (B, y), \quad (B, y) \sim (C, z).\]

There exist \(u, v, w\) such that \(x = Bu\), \(y = Av\), and \(u - v \in \mathcal{Z}(AB)\), and there exist \(v', w'\) such that \(y = Cw'\), \(z = Aw'\), and \(v' - w' \in \mathcal{Z}(BC)\).

We set \(w = Cs\) and \(z = Cx\), and then we have
\[Cs - Bu = 0,\]
\[A - Cw = 0,\]
\[A - Bv = 0.\]

Again, setting
\[s = Bw, \quad u = Cw, \quad v = Aw, \quad t = Bw, \quad v' = Aw'.\]
we have
\[ C(\mathbb{u}s^*) - B(\mathbb{v}s^*) = 0, \]
\[ A(\mathbb{u}s^*) - C(\mathbb{v}s^*) = 0, \]
and
\[ A(\mathbb{v}s^*) - B(\mathbb{u}s^*) = 0, \]
and
\[ C(\mathbb{v}s^*) - C(\mathbb{u}s^*) \in \mathcal{Z}(BC), \quad A(\mathbb{v}s^*) - A(\mathbb{u}s^*) \in \mathcal{Z}(BC). \]
By Lemma I it follows that
\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(BC) \subseteq \mathcal{Z}(ABC), \]
[\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(ABC), \]
and that
\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(ABC), \quad v^* - u^* \in \mathcal{Z}(ABC). \]
We have therefore
\[ \mathbb{v}s^* = (\mathbb{u}s^* - \mathbb{v}s^*) - (\mathbb{v}s^* - \mathbb{u}s^*) - (\mathbb{v}s^* - \mathbb{u}s^*) \in \mathcal{Z}(ABC), \]
\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(ABC), \]
\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(ABC). \]
\[ \mathbb{v}s^* - \mathbb{u}s^* \in \mathcal{Z}(ABC), \quad v^* - u^* \in \mathcal{Z}(ABC). \]
We shall prove, for instance, 2. Suppose that \((\mathbb{A}, \mathbb{s}) \sim (\mathbb{B}, \mathbb{y})\). This means that there are \(u, v \in X\) such that
\[ \mathbb{s} = Bu, \quad y = Av, \quad v - u \in \mathcal{Z}(\mathbb{A}), \]
Moreover, it follows from (i) that there are \(u, v \in X\) such that
\[ y = (\mathbb{A})u = \mathbb{A}(\mathbb{u}), \quad v = (\mathbb{B})v = \mathbb{B}(\mathbb{v}). \]
Hence
\[ \mathbb{A}(\mathbb{u} - \mathbb{v}) = 0 \quad \text{and} \quad \mathbb{B}(\mathbb{u} - \mathbb{v}) = 0, \]
and hence
\[ C(\mathbb{u} - \mathbb{v}) \in \mathcal{Z}(\mathbb{A}), \quad C(\mathbb{u} - \mathbb{v}) \in \mathcal{Z}(\mathbb{A}), \]
and hence
\[ (\mathbb{u} - \mathbb{v}) \in \mathcal{Z}(\mathbb{A}), \quad (\mathbb{u} - \mathbb{v}) \in \mathcal{Z}(\mathbb{A}), \]
Therefore, \(\mathbb{u} \in \mathcal{Z}(\mathbb{A}), \mathbb{v} \in \mathcal{Z}(\mathbb{B})\), and thus \((\mathbb{A}, \mathbb{s}) \sim (\mathbb{B}, \mathbb{y})\), which was to be proved.

We consider the family of equivalence classes. We observe that:

**Lemma II.** Any two classes \([\mathbb{A}, \mathbb{s}]\), \([\mathbb{B}, \mathbb{y}]\) can always be represented by pairs with the same first term \(\mathbb{A}\mathbb{B}\):

\[ (\mathbb{A}, \mathbb{s}) \sim (\mathbb{A}\mathbb{B}, \mathbb{s}) \quad \text{and} \quad (\mathbb{B}, \mathbb{y}) \sim (\mathbb{A}\mathbb{B}, \mathbb{y}), \]

for some \(\mathbb{s}, \mathbb{y} \in X\).

**Proof.** There exists an element \(u\) such that \(\mathbb{s} = (\mathbb{A}\mathbb{B})u\). We have \((\mathbb{A}, \mathbb{s}) \sim (\mathbb{A}\mathbb{B}, \mathbb{Au})\). Indeed, the element \(u\) has the property that \((\mathbb{A}\mathbb{B})u = \mathbb{Au} = \mathbb{Au}\), and \(u \in \mathcal{Z}(\mathbb{A}\mathbb{B})\).

Likewise, there exists an element \(v\) such that \(\mathbb{y} = (\mathbb{A}\mathbb{B})v\), and, by the same argument, \((\mathbb{B}, \mathbb{y}) \sim (\mathbb{A}\mathbb{B}, \mathbb{Bv})\). The lemma is therefore proved.

We can provide the family of equivalence classes with a group structure setting:

\[ [\mathbb{A}, \mathbb{s}] + [\mathbb{B}, \mathbb{y}] = [(\mathbb{A}\mathbb{B}, \mathbb{Au}) + (\mathbb{A}\mathbb{B}, \mathbb{Bv})] = [(\mathbb{A}\mathbb{B}, \mathbb{Au} + \mathbb{Bv})]. \]

We denote this group by \(Y\). We have the natural embedding

\[ \mathbb{X} \rightarrow \mathbb{Y}, \quad \mathbb{s} \mapsto [(\mathbb{A}, \mathbb{s})]. \]

For every \(Q \in S\), the map \(\hat{Q}: [(\mathbb{A}, \mathbb{s})] \rightarrow [(\mathbb{A}Q, \mathbb{s})]\) is an endomorphism of \(Y\). The group \(D\) of these endomorphisms is commutative, and it is a homomorphic image of \(S\).

**III.** J. S. o Silva formulated the following theorem (cf. [2], p. 171).

Let \(S\) be a semigroup of homomorphisms \(\mathbb{A} \in \mathcal{S}\) defined on subgroups \(\mathcal{C}_A\) of a given group \(X\), and mapping onto the entire group,

\[ \mathbb{A}(\mathcal{C}_A) = X \quad \text{for each} \quad \mathcal{A}, \quad \text{and let} \quad S \quad \text{contain the identity map}, \quad I \in S. \]

For each \(\mathbb{A}\), there is a given submorphism \(\mathcal{R}(\mathbb{A}) \subseteq X\). In order that there exist a group \(Y \supseteq X\) such that each \(\mathbb{A} \in \mathcal{S}\) can be prolonged to an endomorphism \(\hat{\mathbb{A}}: Y \rightarrow Y\) so that

1. \(\hat{\mathbb{A}}(\mathbb{I}) = \mathbb{I}, \)
2. \(\hat{\mathbb{A}} = \hat{\mathbb{A}} \circ \hat{\mathbb{A}}, \)
3. every element \(x \in X\) has the form \(x = \hat{\mathbb{A}}x, \quad x \in \mathbb{S}, \quad x \in \mathbb{X}, \)
4. \(\mathbb{A}^{-1}(0) \cap X = \mathbb{R}(\mathbb{A}), \)

it is necessary and sufficient that

(a) \(\mathbb{R}(\mathbb{A}) \subseteq \mathbb{R}(\mathbb{A}), \)
(b) \(\mathbb{R}(\mathbb{A}) \subseteq \mathbb{R}(\mathbb{A}), \)
(c) \(\mathbb{R}(\mathbb{A}) \subseteq \mathbb{R}(\mathbb{A}), \)
(d) for every \(x \in \mathcal{S}, \quad (\mathcal{R}(\mathbb{A}))^{-1}x \in \mathbb{R}(\mathbb{A}), \)
(e) \(\mathbb{A}^{-1}x \in \mathbb{R}(\mathbb{A})\) implies \(x \in \mathbb{S}(\mathbb{B}), \)
(f) \(\mathbb{S}(\mathbb{I}) = \{0\}. \)

This extension is always unique.
One can easily prove that the function \( \mathcal{R} : \mathcal{A} \to \mathcal{R}(\mathcal{A}) \) is an extensor for this special operator system. Indeed, the conditions 1, 2, 3 are equivalent to \((x')_0, (x')_0\) and \((x')_0\), respectively. The condition (5) can be formulated as follows

\[
(A(Bu) - B(Ae)) = 0 \implies u - v \in \mathcal{R}(AB) .
\]

This condition imposed on an operator system \((S, X)\) is less convenient that \((\ast)\), since it involves the extensor \(\mathcal{R}\). At first \((\ast\ast)\) seems to be a weaker condition than \((\ast)\), but that is only apparent. We can prove the following

**Lemma III.** If \( S \) is a semigroup of homomorphisms, and if \((S, X)\) is an operator system with an extensor \( \mathcal{R} \) satisfying \((\ast\ast)\), then setting

\[
G_2 = G_1 + \mathcal{R}(A) ,
\]

and extending every \( x \in S \) in the obvious way to the new domain \( G_2 \) one gets an operator system that already satisfies \((\ast)\).

**Proof.** We assume that an operator system \((S, X)\) satisfies the condition of Silva \((\ast\ast)\), and we shall prove that the operator system with the expanded domains \( G_2 \) satisfies condition \((\ast)\).

Let \( x \) be an arbitrary element from \( X \), and \( A, B \in S \). Since \( A \) and \( B \) map their old domains onto \( X \), there exist \( u \in G_B \) and \( v \in G_A \) such that

\[
x = A(Bu) = B(Av) .
\]

We have \( u = v + (u - v) \). Let us calculate \( B(Au) \) in the system with expanded domains:

\[
B(Au) = B(A(v + (u - v))) ,
\]

and, on the other hand, since \( u - v \in \mathcal{R}(AB) \),

\[
x = B(Av) = B(Av) + (BA)(u - v) = (BA)v + (BA)(u - v) ,
\]

and since \( v, u - v \in G_{2A}, \) we have \( u \in G_{2A} \) and

\[
x = (BA)v + (BA)(u - v) = (BA)v + (u - v) .
\]

Therefore \( x = A(Bu) = (BA)u = (BA)u \), condition \((\ast)\) is satisfied.

It follows immediately from this lemma that our extension theorem implies the theorem formulated by Silva.

We know from Lemma I that if \((S, X)\) is an operator system satisfying \((\ast)\), and if \( \mathcal{R} \) is an extensor for \((S, X)\), then condition \((\ast\ast)\) of Silva is always satisfied. The theorem of Silva however does not imply our theorem. If we do not assume that \((S, X)\) is operator system such that the semigroup operation in \( S \) is superposition of homomorphisms, then

condition \((\ast\ast)\) alone is not enough to prove the extension theorem. It is enough only if \( S \) is a semigroup of homomorphisms with the operation of superposition.

Our theorem also implies the Fundamental Theorem of Słowikowski (cf. [5], p. 5, and [6], p. 263). Słowikowski assumes a priori that he has an operator system with a commutative semigroup. In this respect the difference is that we put the commutativity condition into the extensor. However, he proves his Fundamental Theorem for operator systems \((S, X)\) such that \( S \) is commutative and the domains satisfy the condition

\[
G_{AB} \subseteq G_B, \quad B(G_{AB}) \subseteq G_A, \quad \text{and}
\]

\[
(AB)x = A(Bu) \quad \text{for every} \quad x \in G_{AB} .
\]

The advantage of \((\ast)\) is obvious.

**References**


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