Homogeneous operations and homogeneous algebras

by

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Introduction. In algebraic systems usually treated in mathematics, such as groups, fields, vector spaces, etc., single elements have some individual properties and their roles in the systems considered are different. Nevertheless, in some research in universal algebra or the general theory of algebraic systems, and especially in the study of the notion of independence (see [1] and all the other papers in the references) there are examples of interesting abstract algebras which are in a certain sense homogeneous, i.e. in which all elements have the same properties. So are some algebras defined by E. L. Post, denoted here by $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ (cf. Marczewski and Urbanik [3] and [3a]), algebra $\mathfrak{S}$ of Świerczkowski (denoted by $\mathscr{A}_0$ by Świerczkowski, see [4], p. 94) and others.

In this paper I adopt the following definitions of homogeneity (see 1.1): an operation f in a set A is homogeneous whenever the formula

$$x_0 = f(x_1, \ldots, x_k)$$

implies

$$h(x_0) = f\big(h(x_1), \ldots, h(x_k)\big)$$

for every permutation h of A. An algebra is homogeneous if every operation in it is homogeneous (see 3.1).

By replacing permutations in these definitions by all transformations of A into itself, I obtain the definition of strong homogeneity (see 2.3 and 3.1), a notion strictly connected with that of independence (cf. 3.1 (v); let us remark that strong homogeneity is called homogeneity in [3a]).

This paper contains a study of homogeneous operations and homogeneous algebras.

In the first section I give a *representation theorem for homogeneous operations* (1.3 (v)) based on some ideas of Świerczkowski (see [4], especially pp. 95 and 96).

In the second section I consider some classes of homogeneous operations, first of all symmetrical homogeneous operations and strongly homogeneous operations. Symmetry (= commutativity) of an operation may be treated also as a sort of homogeneity, namely as a certain homogeneity of the set of indices (see 2.1 (iii)). It turns out that *operations* *which are simultaneously homogeneous and symmetrical rarely exist: only in two-, three-, four-, and six-elements sets* (2.2 (iii)). Let us remember that strongly homogeneous non-trivial operations are still more exceptional: in view of a result of Świerczkowski [4] such operations exist only in two-element sets (see 2.3 (vii) and (viii)). I gather together all representation and existence theorems in 2.4.

The third section is devoted to homogeneous algebras, especially to their properties connected with the notion of independence. In an earlier paper [2] I considered some numerical constants associated with finite algebras: the minimal number γ of generators, the maximal number ι of independent elements, and others. In 3.3 I give the *complete discussion of these constants for homogeneous algebras.*

Let us remark that in the third section the results concerning symmetrical operations (2.1 and 2.2) are used only in the Appendix (3.4).

Terminology and notation. For each finite or denumerable sequence $a_1, a_2, \ldots$ I denote by $\{a_1, a_2, \ldots\}$ the set of its terms. E.g., $\{a, b, b, a, b\} = \{a, b\}$.

The cardinal of a set A is denoted by $|A|$.

A one-one transformation of A onto A is called a *permutation of* A.

Every mapping $f\colon A^k \to A$ (where $k = 1, 2, \ldots$) is called an *operation of k variables in A*, or shortly a *k-ary operation in A*. For operations in a fixed set A, we use the symbol $\equiv$ in the following sense: $=$ every where in A. E.g. the formula $f(x_1, x_2) \equiv g(x_1)$ means

$$f(x_1, x_2) = g(x_1) \quad \text{for every pair} \quad (x_1, x_2) \in A^2 .$$

Operations of the form

$$e_j^{(k)}(x_1, \ldots, x_k) \equiv x_j ,$$

where $k = 1, 2, \ldots$ and $j = 1, 2, \ldots, k$, are called *trivial*.

An operation f in A is called *quasi-trivial* if $f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$ for every $(x_1, \ldots, x_k) \in A^k$.

All terminology and notation concerning abstract algebras, e.g. such notions as algebraic operations, generators, independence, identity of two algebras, etc. agree with my papers [1] and [2].

1. Homogeneous operations

1.1. Definition and examples. An operation $f\colon A^k \to A$ is called *homogeneous* if

$$h(f(x_1, \ldots, x_k)) \equiv f(h(x_1), \ldots, h(x_k))$$

for every permutation h of A. It is obvious that

(i) *Every trivial operation is homogeneous and quasi trivial.*

(ii) *The identity operation $e_1^{(1)}(x) \equiv x$ is the only quasi-trivial unary operation.*

We will consider the following examples:

r_n (where $n \geqslant 2$)—an $(n-1)$-ary operation defined in every n-element set by the following conditions: if all terms of the sequence $x_1, \ldots, x_{n-1}$ are different, then $r_n(x_1, \ldots, x_{n-1}) \notin \{x_1, \ldots, x_{n-1}\}$, and if there are in it two identical terms, then $r_n(x_1, \ldots, x_{n-1}) = x_1$ (see [2], p. 2).

l_n—an n-ary operation defined in an arbitrary set as follows: $l_n(x_1, \ldots, x_n) = x_1$ if $x_1, \ldots, x_n$ are all different and $l_n(x_1, \ldots, x_n) = x_n$ in the opposite case (see [2], p. 2).

p_* and p^*—two ternary operations defined in every two-element set by the equations

$$p_*(x, y, y) \equiv p_*(y, y, x) \equiv p_*(y, x, y) \equiv x ,$$
$$p^*(x, y, y) \equiv p^*(y, y, x) \equiv p^*(y, x, y) \equiv y .$$

(Operations defined by E. L. Post. See Marczewski [2], Marczewski-Urbanik [3], [3a].)

s—a ternary operation, defined in any four-element set as follows: $s(x, y, z) \notin \{x, y, z\}$ if x, y, z are different and $s(x, y, z) = p_*(x, y, z)$ in the opposite case (Świerczkowski [4], p. 94, cf. Marczewski [2], p. 2).

Let us remark that in the set $A = \{0, 1\}$ the operations p_* and p^* may be described as follows (see Marczewski-Urbanik [3], p. 292):

$$p_*(x, y, z) \equiv x + y + z \pmod 2 ,$$
$$p^*(x, y, z) \equiv xy + yz + xz \pmod 2 .$$

It is obvious that

(iii) *Operation r_n for $n \geqslant 2$ is a homogeneous, non quasi-trivial operation in an n-element set; r_2 is the transposition in a two-element set.*

(iv) *Operation l_n for $n \geqslant 3$ is a homogeneous, quasi-trivial, but non-trivial operation in a set having at least n elements. If the set has less than n elements, then $l_n = e_n^{(n)}$. Moreover, $l_1 = e_1^{(1)}$, i.e. l_1 is the identity operation: $l_1(x) \equiv x$ and $l_2 = e_1^{(2)}$, i.e. $l_2(x, y) \equiv x$.*

(v) *Operations p_* and p^* are homogeneous, quasi-trivial but non-trivial.*

(vi) *Operation s is homogeneous and non quasi-trivial.*

1.2. Properties. At first we will consider the case where a homogeneous operation is non quasi-trivial. The following simple but fundamental proposition shows that this situation is of a somewhat exceptional character and that it is possible only in finite algebras:

(i) *If f is a homogeneous operation in a set A and*

$$(*) \qquad b = f(a_1, \ldots, a_k) \notin \{a_1, \ldots, a_k\} ,$$

then b is the only element of A not belonging to $\{a_1, \ldots, a_k\}$.

Let us denote by c an arbitrary element of $A\setminus\{a_1, \ldots, a_k\}$ and by h a permutation of A such that

$$h(b) = c, \quad h(c) = b, \quad h(x) = x \quad \text{for} \quad x \in A\setminus\{b, c\}.$$

Hence

$$h(a_j) = a_j \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Consequently

$$c = h(b) = h\big(f(a_1, \ldots, a_k)\big) = f\big(h(a_1), \ldots, h(a_k)\big)$$
$$= f(a_1, \ldots, a_k) = b, \quad \text{q.e.d.}$$

Let us note the following corollaries to (i):

(ii) *In an infinite set every homogeneous operation is quasi-trivial.*

(iii) *If* (*) *for a homogeneous operation f in a set A with $|A| = n \geqslant 2$ then A is finite and there are in the sequence $a_1, \ldots, a_k$ precisely $n-1$ different terms.*

(iv) *If* (*) *for a homogeneous operation f and a sequence $a_1, \ldots, a_k$ of different elements, then $|A| = k+1$.*

The following proposition is obvious:

(v) *If for a homogeneous operation f in A and a sequence $a_1, \ldots, a_k$ of different elements*

$$f(a_1, \ldots, a_k) = a_j \quad (\textit{where } 1 \leqslant j \leqslant k)$$

then

$$f(b_1, \ldots, b_k) = b_j$$

for every sequence $b_1, \ldots, b_k$ of different elements of A.

The preceding propositions imply some conclusions concerning the case of a set having only a few elements. In particular, propositions 1.1 (iii) and 1.2 (iv) and (v) imply

(vi) *If f is a non-trivial homogeneous unary operation in a set A, then $|A| = 2$ and $f = r_2$.*

Proposition (v) implies

(vii) *In a two element set there are only two non-trivial homogeneous binary operations g_1 and g_2:*

$$\left.\begin{array}{ll} g_j(x_1, x_2) = x_j & \text{if} \quad x_1 \neq x_2 \\ g_j(x, x) \neq x & \end{array}\right\} \quad \textit{for} \quad j = 1 \textit{ and } 2,$$

whence

(vii') *In a two-element set each quasi-trivial homogeneous binary operation is trivial.*

Propositions (iii) and (v) imply

(viii) *In a three-element set the operation r_3 is the only non-trivial homogeneous binary operation,*

and

(ix) *If $|A| \geqslant 4$, then every homogeneous binary operation in A is trivial.*

Finally let us note the following propositions, easy to prove.

(x) *Every superposition of homogeneous operations is homogeneous.*

(xi) *Every superposition of quasi-trivial operations is quasi-trivial.*

1.3. Representation theorem. A finite class of sets $\delta = \{E_1, \ldots, E_m\}$ is called a *decomposition* of E into m parts whenever they are different, non void, disjoint and

$$E = E_1 \cup \ldots \cup E_m.$$

We denote by $\Delta(k, n)$, where $1 \leqslant k < \infty$ and $1 \leqslant n \leqslant \infty$, the set of all decompositions of $K = \{1, 2, \ldots, k\}$ into at most n parts. Obviously, for every $n \geqslant k$ (and, in particular if n is infinite), $\Delta(k, n)$ is the class of all decompositions of K.

For every sequence $a_1, \ldots, a_k$ we denote by $\delta(a_1, \ldots, a_k)$ the decomposition of $K = \{1, \ldots, k\}$ such that $i, j \in K$ belong to the same element of $\delta(a_1, \ldots, a_k)$ iff $a_i = a_j$.

The following two propositions are obvious:

(i) *$\delta(a_1, \ldots, a_k)$ is a decomposition of K into m parts iff the set $\{a_1, \ldots, a_k\}$ has precisely m elements.*

(ii) *Let A denote an n-element set and δ a decomposition of the set $K = \{1, 2, \ldots, k\}$. There is a sequence $a_1, \ldots, a_k \in A$ such that $\delta = \delta(a_1, \ldots, a_k)$, iff $\delta \in \Delta(k, n)$.*

Two sequences $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are called *similar* if $\delta(a_1, \ldots, a_k) = \delta(b_1, \ldots, b_k)$. Obviously,

(iii) *If $a_1, \ldots, a_k \in A$, h is a permutation of A, and $b_j = h(a_j)$, then $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are similar.*

The following proposition results immediately from the definition of homogeneity and presents a generalization of 1.2 (v):

(iv) *If f is a homogeneous operation in A and two sequences $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are similar, then*

$$\textit{if } f(a_1, \ldots, a_k) = a_j \quad \textit{then} \quad f(b_1, \ldots, b_k) = b_j,$$
$$\textit{if } f(a_1, \ldots, a_k) \notin \{a_1, \ldots, a_k\} \quad \textit{then} \quad f(b_1, \ldots, b_k) \notin \{b_1, \ldots, b_k\}.$$

We denote by $\Phi(k, n)$ the class of all functions φ the domain of which is $\Delta(k, n)$ and for which either (1) $\varphi(\delta) \in \delta$ or (2) $|\delta|+1 = n$ and

$\varphi(\delta) = 0$ (= the empty set). In other words, φ is a set-valued function which for every decomposition $\delta \in \Delta(k, n)$ either distinguishes a set belonging to δ or admits for δ the value 0, but the second case is possible only if n is finite and if δ is a decomposition into precisely $n-1$ parts (cf. Świerczkowski [4], p. 96).

For every function $\varphi \in \Phi(k, n)$ we define a k-ary operation f_φ in any n-element set A by the following conditions: (1) if $\varphi\big(\delta(x_1, \ldots, x_k)\big) \neq 0$, then $f_\varphi(x_1, \ldots, x_k) = x_j$, where $j \in \varphi\big(\delta(x_1, \ldots, x_k)\big)$, and (2) if $\varphi\big(\delta(x_1, \ldots, x_k)\big) = 0$, then $f_\varphi(x_1, \ldots, x_k) \notin \{x_1, \ldots, x_k\}$. The definition is consistent and univocal because (1) if $\varphi\big(\delta(x_1, \ldots, x_k)\big)$ is non-empty and if $i, j \in \varphi\big(\delta(x_1, \ldots, x_k)\big)$, then $x_i = x_j$, and (2) if $\varphi\big(\delta(x_1, \ldots, x_k)\big) = 0$, then by definition of $\Phi(k, n)$ n is finite and $\delta(x_1, \ldots, x_k)$ is a decomposition of K into $n-1$ parts or, in other words, $\{x_1, \ldots, x_k\}$ is a $(n-1)$-element set, whence $A \setminus \{x_1, \ldots, x_k\}$ is a one-element set.

We can now state the following representation theorem for homogeneous operations:

(v) *The correspondence $\varphi \to f_\varphi$ is a one-one correspondence between the class $\Phi(k, n)$ and the class of all homogeneous k-ary operations in an n-element set A.*

Proof. 1. Let us first verify that for $\varphi \in \Phi(k, n)$ the operation f_φ is homogeneous, or, in other words, that if

$$x = f_\varphi(x_1, \ldots, x_k)$$

then, for every permutation h of A,

$$h(x) = f_\varphi\big(h(x_1), \ldots, h(x_k)\big).$$

This is an easy consequence of the preceding definitions, of (iii) and of the remark that if $\{x\} = A \setminus \{x_1, \ldots, x_k\}$, then $\{h(x)\} = A \setminus \{h(x_1), \ldots, (hx_k)\}$.

2. We shall prove that for every homogeneous k-ary operation f in A there is a function $\varphi \in \Phi(k, n)$ such that $f = f_\varphi$. Namely, for every $\delta \in \Delta(k, n)$ we consider such a sequence $a_1, \ldots, a_k \in A$ that $\delta(a_1, \ldots, a_k) = \delta$ (in view of (ii)) and we put

$$\varphi(\delta) = K \cap \{j\colon\ a_j = f(a_1, \ldots, a_k)\}.$$

In view of (iv) this set depends only on δ, i.e. it does not depend on the choice of $a_1, \ldots, a_k$.

For every $x_1, \ldots, x_k$,

$$\textit{if} \quad f(x_1, \ldots, x_k) \notin \{x_1, \ldots, x_k\}, \quad \textit{then} \quad \varphi\big(\delta(x_1, \ldots, x_k)\big) = 0,$$

and consequently

$$f_\varphi(x_1, \ldots, x_k) \notin \{x_1, \ldots, x_k\}.$$

Since, in view of 1.2 (i), $A \setminus \{x_1, \ldots, x_k\}$ is, in this case, a one-element set, we have here $f_\varphi(x_1, \ldots, x_k) = f(x_1, \ldots, x_k)$,

If $f(x_1, \ldots, x_k) = x_j$, then $j \in \varphi\big(\delta(x_1, \ldots, x_k)\big)$, whence $f_\varphi(x_1, \ldots, x_k) = x_j = f(x_1, \ldots, x_k)$.

Hence $f = f_\varphi$.

3. We shall prove finally that if $f_\varphi = f_\psi$, then $\varphi = \psi$. For this purpose, let us consider a decomposition $\delta \in \Delta(k, n)$.

If

$$a = f_\varphi(a_1, \ldots, a_k) = f_\psi(a_1, \ldots, a_k) \notin \{a_1, \ldots, a_k\}$$

then

$$\varphi\big(\delta(a_1, \ldots, a_k)\big) = 0 = \psi\big(\delta(a_1, \ldots, a_k)\big).$$

If

$$a_j = f_\varphi(a_1, \ldots, a_k) = f_\psi(a_1, \ldots, a_k)$$

then

$$j \in \varphi\big(\delta(a_1, \ldots, a_k)\big) \cap \psi\big(\delta(a_1, \ldots, a_k)\big).$$

Consequently

$$\varphi\big(\delta(a_1, \ldots, a_k)\big) = \psi\big(\delta(a_1, \ldots, a_k)\big)$$

for every $a_1, \ldots, a_k \in A$, whence, by (ii), $\varphi(\delta) = \psi(\delta)$ for every $\delta \in \Delta(k, n)$, or, in other words, $\varphi = \psi$.

Representation theorem (v) is thus proved.

The following easy consequence of the definition of f_φ can be treated as a supplement of (v) for quasi-trivial operations:

(vi) *A homogeneous operation f_φ is quasi-trivial iff φ does not vanish (i.e. if $\varphi(\delta) \neq 0$ for every $\delta \in \Delta(k, n)$).*

For trivial operations we first formulate the following obvious proposition:

(vii) *Let $\varphi \in \Phi(k, n)$ and $|A| = n$. In order that $f_\varphi = e_j^{(k)}$ in A it is necessary and sufficient that φ be defined as follows: for each decomposition $\delta = \{K_1, \ldots, K_n\} \in \Delta(k, n)$ we have $\varphi(\delta) = K_p$, where $j \in K_p$.*

For each $\varphi \in \Phi(k, n)$ we denote by $P(\varphi)$ the intersection of all sets $\varphi(\delta)$, where $\delta \in \Delta(k, n)$.

We shall prove that

(viii) *If $\varphi \in \Phi(k, n)$ and $|A| = n$, then the following conditions are equivalent:*

(t) *f_φ is trivial in A.*

(t_1) *φ does not vanish and the class $\{\varphi(\delta)\colon \delta \in \Delta(k, n)\}$ is multiplicative.*

(t_2) $P(\varphi) \neq 0$.

It is easy to see that, in the case $n \geqslant 2$, these conditions are equivalent also to

(t_3) *$P(\varphi)$ is a one-element set.*

(t) $\Rightarrow$ (t_1). If $f_\varphi = e_j^{(k)}$, then for any two decompositions $\delta, \gamma \in \Delta(k, n)$ we have, in view of (vii),

$$j \in \varphi(\delta) \cap \varphi(\gamma).$$

Let us denote by ε any decomposition of K such that $\varphi(\delta)\cap\varphi(\gamma)\in\varepsilon$. Then, by (vii),

$$\varphi(\varepsilon)=\varphi(\delta)\cap\varphi(\gamma), \quad \text{q.e.d.}$$

$(t_1)\Rightarrow(t_2)$. It follows from (t_1) that there is a decomposition δ_0 such that $\varphi(\delta_0)=P(\varphi)$, whence (t_2).

$(t_2)\Rightarrow(t)$. If $j\in P(\varphi)$ then $f_\varphi(x_1,\ldots,x_k)\equiv x_j$, or, in other words, f_φ is trivial, q.e.d.

2. Symmetrical homogeneous operations and strongly homogeneous operations

2.1. Symmetrical homogeneous operations. A k-ary operation f is called *symmetrical* or *commutative* if for each permutation $p\colon j\to p_j$ of the set $K=\{1,\ldots,k\}$

$$f(x_1,\ldots,x_k)\equiv f(x_{p_1},\ldots,x_{p_k}).$$

It is obvious that

(i) *Operations p_*, p^*, r_2, r_3 and s are symmetrical, whereas (in a set having at least two elements) $e_j^{(k)}$ with $k\geqslant 2$, l_k with $k\geqslant 3$ and r_k with $k\geqslant 4$ are not.*

I shall prove that

(ii) *If $2\leqslant k\leqslant n$, then no k-ary homogeneous and quasi-trivial operation f in an n-element set A is symmetrical.*

In fact, if $a_1,\ldots,a_k$ are different elements of A, and if, say,

$$f(a_1,a_2,\ldots)=a_1$$

then, by the homogeneity of f,

$$f(a_1,a_2,\ldots)=a_2,$$

whence f is non-symmetrical.

For every permutation p of K, every $E\subset K$ and every class $\boldsymbol{F}$ of subsets of K we define $p(E)$ and $p(\boldsymbol{F})$ by the formulas

$$p(E)=\{p_j\colon j\in E\},\qquad p(\boldsymbol{F})=\{p(E)\colon E\in\boldsymbol{F}\}.$$

Let us verify that

$$\text{(o)}\qquad p\big(\delta(x_{p_1},\ldots,x_{p_k})\big)=\delta(x_1,\ldots,x_k).$$

In fact, i and j belong to the same part of $p\big(\delta(x_{p_1},\ldots,x_{p_k})\big)$ iff p_i^{-1} and p_j^{-1} belong to the same part of $\delta(x_{p_1},\ldots,x_{p_k})$, or, in other words, iff $x_{p_{p_i^{-1}}}=x_{p_{p_j^{-1}}}$, which is equivalent to $x_i=x_j$.

A function $\varphi\in\Phi(k,n)$ will be called *homogeneous* whenever $\varphi\big(p(\delta)\big)=p\big(\varphi(\delta)\big)$ for every permutation p of K. We may now prove a supplement to the representation theorem which concerns symmetrical operations.

(iii) *A homogeneous k-ary operation f_φ in A (where $\varphi\in\Phi(k,n)$, $|A|=n$) is symmetrical iff φ is homogeneous.*

Let us suppose φ homogeneous and let p denote a permutation of $K=\{1,2,\ldots,k\}$. We have to prove that

$$(+)\qquad f_\varphi(x_1,\ldots,x_k)=f_\varphi(x_{p_1},\ldots,x_{p_k})$$

If $\varphi\big(\delta(x_{p_1},\ldots,x_{p_k})\big)=0$, then, in view of (o) and of the homogeneity of φ,

$$\varphi\big(\delta(x_1,\ldots,x_k)\big)=\varphi\Big(p\big(\delta(x_{p_1},\ldots,x_{p_k})\big)\Big)=p\Big(\varphi\big(\delta(x_{p_1},\ldots,x_{p_k})\big)\Big)=0.$$

Hence, in view of 1.2 (i) and of the definition of f_φ the element $f_\varphi(x_1,\ldots,x_k)$, as well as $f_\varphi(x_{p_1},\ldots,x_{p_k})$ is the only element of A not belonging to $\{x_1,\ldots,x_k\}$. Thus we obtain (+) in the case considered.

If $\varphi\big(\delta(x_{p_1},\ldots,x_{p_k})\big)\neq 0$, i.e. if there exists a j such that $j\in\varphi\big(\delta(x_{p_1},\ldots,x_{p_k})\big)$, or, in other words, $f_\varphi(x_{p_1},\ldots,x_{p_k})=x_{p_j}$, then, in view of (o) and the homogeneity of φ,

$$p_j\in p\Big(\varphi\big(\delta(x_{p_1},\ldots,x_{p_k})\big)\Big)=\varphi\Big(p\big(\delta(x_{p_1},\ldots,x_{p_k})\big)\Big)=\varphi\big(\delta(x_1,\ldots,x_k)\big).$$

Hence

$$f_\varphi(x_1,\ldots,x_k)=x_{p_j}=f_\varphi(x_{p_1},\ldots,x_{p_k}).$$

The formula (+) is thus proved.

Let us suppose now that an operation f_φ is symmetrical. We have to prove that

$$\varphi\big(p(\delta)\big)=p\big(\varphi(\delta)\big)$$

for every permutation p of K and every $\delta\in\Delta(k,n)$.

By 1.3 (ii) there exists a sequence $a_1,\ldots,a_k\in A$ such that $\delta(a_1,\ldots,a_k)=\delta$. It is sufficient to prove that

$$(++)\qquad \varphi\Big(p\big(\delta(a_1,\ldots,a_k)\big)\Big)=p\Big(\varphi\big(\delta(a_1,\ldots,a_k)\big)\Big)$$

or, in view of (o),

$$\varphi\big(\delta(a_{q_1},\ldots,a_{q_k})\big)=p\Big(\varphi\big(\delta(a_1,\ldots,a_k)\big)\Big),$$

where $q\colon j\to q_j$ denotes the inverse of p.

The relation $i\in\varphi\big(\delta(a_{q_1},\ldots,a_{q_k})\big)$ is equivalent consecutively to the following ones:

$$a_{q_i}=f_\varphi(a_{q_1},\ldots,a_{q_k})=f_\varphi(a_1,\ldots,a_k),$$
$$q_i\in\varphi\big(\delta(a_1,\ldots,a_k)\big),\qquad i\in p\Big(\varphi\big(\delta(a_1,\ldots,a_k)\big)\Big),$$

whence we obtain (++).

Proposition (iii) is thus proved.

It easily follows from the definition of homogeneity of φ that

(iv) *A function $\varphi \in \Phi(k, n)$ is homogeneous (or, in other words, f_φ is symmetrical) iff for each two decompositions of $K = \{1, 2, \ldots, k\}$ into $m \leqslant n$ parts:*

$$\delta = \{K_1, \ldots, K_m\} \in \Delta(k, n),$$
$$\delta^* = \{K_1^*, \ldots, K_m^*\} \in \Delta(k, n)$$

with $|K_j| = |K_j^*|$ for $j = 1, \ldots, m$, the equation $\varphi(\delta) = K_l$ implies $\varphi(\delta^*) = K_l^*$.

We can now prove

(v) *If a function $\varphi \in \Phi(k, n)$ is homogeneous, then for every $\delta \in \Delta(k, n)$ we have $|E| \neq |\varphi(\delta)|$ whenever $\varphi(\delta) \neq E \in \delta$ (in other words, the set $\varphi(\delta)$ has a different number of elements from any other set belonging to δ).*

In fact, if for $\delta = \{K_1, \ldots, K_m\}$ we have, say, $\varphi(\delta) = K_1$ and $|K_1| = |K_2|$, then, applying (iv) to the same decomposition written in two forms:

$$\delta = \{K_1, K_2, K_3, \ldots, K_m\},$$
$$\delta = \{K_2, K_1, K_3, \ldots, K_m\},$$

we obtain a contradiction: $\varphi(\delta) = K_1$ and $\varphi(\delta) = K_2$, whereas $K_1 \neq K_2$.

Let us translate (v) into the terms of operations (with the aid of (iii)):

(v′) *If a homogeneous operation is symmetrical and if*

$$f(a_1, \ldots, a_k) = a_s \neq a_t$$

then

$$|\{j\colon 1 \leqslant j \leqslant k \text{ and } a_j = a_s\}| \neq |\{j\colon 1 \leqslant j \leqslant k \text{ and } a_j = a_t\}|.$$

(vi) *If $k \leqslant n$, then no symmetrical homogeneous k-ary operation in an n-element set A is quasi-trivial.*

It is sufficient to consider k different elements of A and to apply (v′).

2.2. Existence of symmetrical homogeneous operations. To every decomposition $\delta = \{K_1, \ldots, K_m\} \in \Delta(k, n)$ (where all K_j are different) corresponds an arithmetical decomposition of the number $k = |K|$:

$$|K| = |K_1| + \ldots + |K_m|.$$

We denote by $D(k, n)$ the class of all such decompositions corresponding to decompositions belonging to $\Delta(k, n)$ or, in other words, the class of all decompositions

$$k = k_1 + \ldots + k_m \tag{*}$$

where k_j are natural numbers and $m \leqslant n$. Obviously two decompositions which differ only in the order of summands are treated as identical. By $D^*(k, n)$ we denote the class of all decompositions (*) belonging to $D(k, n)$ such that $m \neq n-1$.

A decomposition (*) is called *good* if there is a summand which appears in (*) only once. In the opposite case the decomposition is *bad*. A decomposition $\delta \in \Delta(k, n)$ is called respectively good or bad if the corresponding arithmetical decomposition belonging to $D(k, n)$ is such.

We shall prove

(i) *In an n-element set A there exists a homogeneous, symmetrical k-ary operation with $k > 1$ iff every decomposition belonging to $D^*(k, n)$ is good.*

(ii) *In an n-element set A there exists a homogeneous symmetrical and quasi-trivial k-ary operation with $k > 1$ iff every decomposition belonging to $D(k, n)$ is good.*

(Let us remark that, in view of 2.1 (ii), only the case $k > n$ is essential in (ii).)

The necessity in theorems (i) and (ii) follows from 1.3 (v), 1.3 (vi), 2.1 (iii) and 2.1 (v).

In order to prove the sufficiency, let us define $\varphi \in \Phi(k, n)$ as follows. Let $\delta = \{K_1, \ldots, K_m\} \in \Delta(k, n)$.

We may suppose of course that

$$|K_1| \leqslant |K_2| \leqslant \ldots \leqslant |K_m|.$$

If δ is a bad decomposition, then we put $\varphi(\delta) = 0$. If it is good, then there exists a smallest index s such that $|K_s| \neq |K_j|$ for $j \neq s$. We then put $\varphi(\delta) = K_s$.

If every decomposition belonging to $D^*(k, n)$ is good, then $\varphi \in \Phi(k, n)$, whence f_φ is homogeneous by 1.3 (v). It is easy to see that φ is homogeneous, whence f_φ is symmetrical by 2.1 (iii) and theorem (i) is thus proved.

If every decomposition belonging to $D(k, n)$ is good then f_φ is *a fortiori* homogeneous and symmetrical. Moreover, φ does not vanish, whence, by 1.3 (vi), f_φ is quasi-trivial. Theorem (ii) is thus proved.

Theorems (i) and (ii) reduce the problem of the existence of operations simultaneously homogeneous and symmetrical to purely arithmetical questions. I shall prove the following existence theorem for symmetrical homogeneous operations.

(iii) *Symmetrical homogeneous operations of more than one variable in an n-element set exist iff $n = 2, 3, 4$ or 6. For these values of n there is a symmetrical homogeneous k-ary operation (with $k \geqslant 1$) iff k and n are relatively prime and $k \geqslant n-1$.*

We denote by S the set of all pairs (k, n) (with $k \geqslant 1$) such that there exists a k-ary homogeneous and symmetrical operation in an n-element set or, in other words (by (i)) that every decomposition belonging to $D^*(k, n)$ is good.

Theorem (iii) will be proved by the aid of the following lemmas:

(a) *If $(k, n) \in S$, then $k = n-1$ or $k > n$.*

It is enough to consider the following bad decomposition: $k = 1+ +1+\ldots+1$.

(b) *If $(k, n) \notin S$, then $(k, m) \notin S$ for $m > n+1$.*

This is because a bad decomposition of k into l summands where $l \leqslant n$ satisfies at the same time the inequalities $l \leqslant m$ and $l \neq m-1$.

(c) *If k is even, then $(k, 2) \notin S$.*

This is because the decomposition $k = j+j$ is bad.

(c*) *If k is even and $n \geqslant 4$ then $(k, n) \notin S$.*

This follows from (c) and (b).

(d) *If k is odd and $k \geqslant 3$, then $(k, 2) \in S$ and $(k, 4) \in S$.*

Clearly, each decomposition of an odd number $k \geqslant 3$ into two summands is good. Every decomposition into four summands is also good: indeed, if such a decomposition were bad, then every summand would repeat itself two or four times and hence k would be even.

(e) *$(k, 3) \in S$ (for $k > 1$) iff k is not divisible by 3.*

If k is divisible by 3, then $k = j+j+j$ is a bad decomposition. If k is not divisible by 3, then in every decomposition into three summands at least one summand appears only once.

(f) *$(k, 5) \notin S$ for $k = 2, 3, \ldots$*

In view of (b), (c) and (e), it is sufficient to prove the existence of bad decompositions for $k \neq 5$ not divisible by 2 and 3, which follows from the equations:

$$7 = 2+2+1+1+1,$$
$$9+2j = 3+3+3+j+j, \qquad j = 1, 2, \ldots$$

(f*) *$(k, n) \notin S$ for $n \geqslant 7$, $k = 2, 3, \ldots$*

A consequence of (f) and (b).

(g) *If k and 6 are relatively prime and $k \geqslant 5$, then $(k, 6) \in S$.*

There is no bad decomposition of k into 2, 3 and 4 summands, in view of (d) and (e). In a decomposition of k into 6 summands the cases

$$k = u+u+v+v+w+w \quad \text{and} \quad k = u+u+u+v+v+v$$

are impossible since k is not divisible by 2 and by 3, and hence one summand appears precisely once.

The proof of (iii) has been reduced to an easy verification. By (f) and (f*) if $(k, n) \in S$, then $n = 2, 3, 4$ or 6. The case $n = 2$ is treated in lemmas (c) and (d), the case $n = 3$ in lemma (e), the case $n = 4$ in (c*), and (d). Lemma (g) says that if k and 6 are relatively prime and $k \geqslant 5$, then $(k, 6) \in S$. It follows from (c), (e), (b) and (a) that for other k's we have $(k, 6) \notin S$.

Theorem (iii) is thus proved.

Passing now to symmetrical, homogeneous and quasi-trivial operations, we will prove a theorem analogous to (iii):

(iv) *Symmetrical homogeneous quasi-trivial operations of more than one variable in an n-element set exist iff $n = 2, 3$ or 4. For these values of n there exist k-ary operations having the above mentioned properties iff k is not divisible by any number $\leqslant n$.*

Let us denote by S_0 the set of all pairs (k, n) with $k \geqslant 1$ such that there exists a k-ary homogeneous symmetrical and quasi-trivial operation in an n-element set, or, in other words, (by (ii)) that every decomposition belonging to $D(k, n)$ is good.

Theorem (iv) follows from the following lemmas (a_0)-(g_0).

(a_0) *If $(k, n) \in S_0$, then $k > n$.*

(b_0) *If $(k, n) \notin S_0$, then $(k, m) \notin S_0$ for $m \geqslant n$.*

(c_0) *If k is even, then $(k, n) \notin S_0$ for $n \geqslant 2$.*

(d_0) *If k is odd and $k \geqslant 3$, then $(k, 2) \in S_0$.*

(e_0) *If k is devisible by 3, then $(k, n) \notin S_0$ for $n \geqslant 3$.*

These lemmas are easy to verify.

(f_0) *$(k, n) \notin S_0$ for $n \geqslant 5$, $k \geqslant 2$.*

This lemma follows from (f), (b_0) and the relation: $S_0 \subset S$.

(g_0) *If $k > 1$ is odd and not divisible by 3, then $(k, 3) \in S_0$ and $(k, 4) \in S_0$.*

In fact, under the above suppositions, in each decomposition of k into two or three summands the equality of all summands is impossible and consequently one at least of them is different from the others. Each decomposition into four summands is also good. Indeed, each bad decomposition into four summands is of the form $k = u+u+v+v$, which is impossible for odd k.

We finish this paragraph by answering the following question: let (k, n) be a pair of numbers satisfying all conditions of (iii) [or (iv)]; in what manner can one define explicitly a homogeneous symmetrical [and quasi-trivial] k-ary operation f in an arbitrary n-element set? The answer is easy: it is enough to consider the operation f_φ where φ is a function belonging to $\Phi(k, n)$ defined in the proof of 2.2 (i) and (ii).

Let us consider the pair (5, 4) for example. According to this rule, a quintenary operation in a four-element set $\{a, b, c, d\}$ can be defined by the following scheme:

$$5 = 1+1+1+\mathbf{2}, \qquad 5 = \mathbf{1}+2+2,$$
$$5 = 1+1+\mathbf{3}, \qquad 5 = \mathbf{2}+3.$$

In each decomposition belonging to $D(5,4)$ the number $|\varphi(\delta)|$ is here in bold type. Thus, for example,

$$f_\varphi(a, b, c, d, d) = f_\varphi(d, a, b, c, d) = \ldots = f_\varphi(d, d, c, b, a) = d,$$
$$f_\varphi(a, a, b, b, b) = \ldots = f_\varphi(b, b, a, b, a) = a.$$

2.3. Strongly homogeneous operations. A k-ary operation in A is called *strongly homogeneous* if the fundamental equation

$$\text{(o)} \qquad h\big(f(x_1, \ldots, x_k)\big) \equiv f\big(h(x_1), \ldots, h(x_k)\big)$$

is satisfied for every transformation h of A into itself (1). It is easy to verify that

(i) *Every trivial operation is strongly homogeneous.*

(ii) *Operations p_* and p^* are strongly homogeneous.*

(iii) *Every superposition of strongly homogeneous operations is a strongly homogeneous operation.*

I shall prove that

(iv) *Every strongly homogeneous operation is quasi-trivial.*

In fact, if

$$f(a_1, \ldots, a_k) = a \in \{a_1, \ldots, a_k\},$$

then, for the transformation $h\colon A \to A$ defined as

$$h(a) = a_1, \qquad h(x) = x \quad \text{for} \quad x \neq a,$$

we have

$$f\big(h(a_1), \ldots, h(a_k)\big) = f(a_1, \ldots, a_k) = a \neq h(a)$$

and, consequently, (o) does not hold.

Passing to the representation theorem for strongly homogeneous operations let us define the relation $<$ in $\Delta(k, n)$: we write $\delta < \delta^*$ if every set belonging to δ is contained in a certain set belonging to δ^*.

It is easy to see that

(v) *Let $a_j \in A$ and $b_j \in A$ $(j = 1, 2, \ldots, k)$. We have*

$$\delta(a_1, \ldots, a_k) < \delta(b_1, \ldots, b_k)$$

iff there is a mapping $h\colon A \to A$ such that $h(a_j) = b_j$.

A function $\varphi \in \Phi(k, n)$ is called *monotone* whenever $\delta < \delta^*$ implies $\varphi(\delta) \subset \varphi(\delta^*)$ (2).

We shall prove the representation theorem for strongly homogeneous operations:

(vi) *A homogeneous k-ary operation f_φ in A (where $\varphi \in \Phi(k, n)$, $|A| = n$) is strongly homogeneous iff φ is monotone and not vanishing.*

(1) Operations of this kind are called homogeneous in paper [3a] of Marczewski and Urbanik.

(2) Notions essentially due to Świerczkowski. See [4], p. 96.

Let us suppose f_φ strongly homogeneous. Consequently it is quasi-trivial by (iv), whence, in view of 1.3 (vi), φ does not vanish.

In order to prove that φ is monotone, let us put $K = \{1, 2, \ldots, k\}$ and consider two decompositions $\delta < \gamma$ belonging to $\Delta(k, n)$:

$$\delta = \{K_1, \ldots, K_r\}, \qquad \gamma = \{L_1, \ldots, L_s\}, \qquad 1 \leqslant s \leqslant r \leqslant n.$$

In view of 1.3 (ii) there exist in A two sequences $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ such that

$$\delta(a_1, \ldots, a_k) = \delta, \qquad \delta(b_1, \ldots, b_k) = \gamma.$$

Proposition (v) implies the existence of a mapping $h\colon A \to A$ such that $h(a_j) = b_j$ $(j = 1, \ldots, k)$, whence

$$(*) \qquad f_\varphi(b_1, \ldots, b_k) = f_\varphi\big(h(a_1), \ldots, h(a_k)\big) = hf_\varphi(a_1, \ldots, a_k).$$

In order to prove

$$(**) \qquad \varphi(\delta) \subset \varphi(\gamma)$$

let us suppose $j \in \varphi(\delta)$. Hence

$$a_j = f_\varphi(a_1, \ldots, a_k)$$

and consequently, by $(*)$,

$$f_\varphi(b_1, \ldots, b_k) = h(a_j) = b_j,$$

which means that $j \in \varphi(\gamma)$. Formula $(**)$ and the monotoneness of φ are thus proved.

Let us suppose conversely that φ does not vanish and is monotone. The operation f_φ is thus quasi-trivial by 1.3 (vi). If $a_j = f_\varphi(a_1, \ldots, a_k)$ or, in other words, $j \in \varphi\big(\delta(a_1, \ldots, a_k)\big)$, then for each mapping $h\colon A \to A$ we have, by (v),

$$\delta(a_1, \ldots, a_k) < \delta\big(h(a_1), \ldots, h(a_k)\big),$$

whence $j \in \varphi\Big(\delta\big(h(a_1), \ldots, h(a_k)\big)\Big)$, or, in other words

$$f_\varphi\big(h(a_1), \ldots, h(a_k)\big) = h(a_j) = h\big(f_\varphi(a_1, \ldots, a_k)\big).$$

Hence f_φ is strongly homogeneous.

Theorem (vi) is thus proved.

Świerczkowski has proved ([4], p. 96, Lemma (β)) that if $\varphi \in \Phi(k, n)$ with $n \geqslant 3$ is monotone and does not vanish, then $P(\varphi) \neq 0$. Thus, in view of (vi) and 1.3 (viii) there are no strongly homogeneous and non-trivial operations in a set having at least three elements. On the other hand, in a two element set each binary homogeneous and quasi-trivial operation is trivial (1.2 (vii′)). Hence, in view of 1.1 (i), we obtain

(vii) *If f is a k-ary strongly homogeneous non-trivial operation in A, then $|A| = 2$ and $k \geqslant 3$.*

As an example of a k-ary strongly homogeneous, non-trivial operation with $k \geqslant 3$ we may consider

$$f(x_1, \ldots, x_k) \equiv p_*(x_1, x_2, x_3).$$

It is known that the class of all strongly homogeneous operations may be described by the common converse of theorems (i), (ii) and (iii) as follows:

(viii) *The class of all strongly homogeneous operations is identical with the smallest class of operations in a two-element set, containing trivial operations, the operations p_* and p^* and closed with respect to superposition. (In other words, the class of all strongly homogeneous operations can be treated as the class of all algebraic operations in the algebra $\mathfrak{P} = (a, b; p_*, p^*)$.)*

The proof of (viii) is complicated (see Marczewski-Urbanik [3a]).

2.4. Table of representation and existence theorems. We list here the main theorems of sections 1 and 2:

REPRESENTATION THEOREMS. *The correspondence $\varphi \to f_\varphi$ is one-one between $\Phi(k, n)$ and the class of all k-ary homogeneous operations in an n-element set A* (1.3 (v)). *In the above correspondence the following are the corresponding properties of f_φ and φ.*

f_φ	φ	Number of theorem
trivial	$P(\varphi) \neq 0$	1.3 (vii)
quasi-trivial	*not vanishing*	1.3 (vi)
symmetrical	*homogeneous*	2.1 (iii)
strongly homogeneous	*not vanishing and monotone*	2.3 (vi)

EXISTENCE THEOREMS. *The following table gives all pairs (k, n) with $k \geqslant 2$ such that in an n-element set there exists a non-trivial k-ary homogeneous operation having the property formulated in the first column.*

Property of operation	n	k	Number of theorem
symmetrical	2, 3, 4, 6	$k \geqslant n-1$	2.2 (iii)
		k and n relatively prime	
symmetrical and quasi-trivial	2, 3, 4	*k not divisible by numbers $\leqslant n$*	2.2 (iv)
strongly homogeneous and non-trivial	2	$k \geqslant 3$	2.3 (vii)

3. Homogeneous algebras

3.1. Definition and properties. An algebra $(A; \boldsymbol{F})$ is called *homogeneous* [*strongly homogeneous*] if each operation $f \in \boldsymbol{F}$ is homogeneous [strongly homogeneous].

Proposition 1.2 (x) implies

(i) *All algebraic operations in a homogeneous algebra are homogeneous*

and 1.2 (i) implies

(ii) *If G is a set of generators of a homogeneous algebra A, then either $G = A$ or A is finite and $A \setminus G$ is a one-element set.*

In view of (i) we have

(iii) *If S and T are two equivalent subsets of a homogeneous algebra A (i.e. $|S| = |T|$ and $|A \setminus S| = |A \setminus T|$), then S is a set of generators of A iff T is so, and S is a set of independent elements iff T is so.*

Propositions 1.2 (v), (vi), (viii) and (ix) imply

(iv) *Every element of a homogeneous algebra A forms a set of independent elements, and, if $|A| \geqslant 3$, then every pair of elements is a set of independent elements.*

It is easy to see (cf. [3a], p. 200) that

(v) *An algebra A is strongly homogeneous iff the whole set A forms a set of independent elements.*

It is easy to prove, by the aid of (v) and proposition 2.2 (vi) of [1], that

(vi) *Every trivial algebra is strongly homogeneous, the algebras (defined by Post) $\mathfrak{P}_* = (a, b; p^*)$, $\mathfrak{P}_* = (a, b; p^*)$ and $\mathfrak{P} = (a, b; p_*, p^*)$ are strongly homogeneous and the algebra $\mathfrak{S} = (a, b, c, d; s)$ of Świerczkowski is homogeneous but not strongly homogeneous.*

Let us add that $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ are the only strongly homogeneous non-trivial algebras (cf. [3] and [3a]).

An algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is called *quasi-trivial* if each operation $f \in \boldsymbol{F}$ is quasi-trivial.

(vii) *The following conditions for an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ are equivalent:*

(a) *$\mathfrak{A}$ is quasi-trivial.*

(b) *Each algebraic operation in $\mathfrak{A}$ is quasi-trivial.*

(c) *Each non-void subset of A is a subalgebra of $\mathfrak{A}$.*

(d) *A is the only set of generators of $\mathfrak{A}$.*

If $|A| = n$ is finite, then these conditions are equivalent to the following

(e) *Each algebraic n-ary operation is quasi-trivial.*

The implication (a)$\Rightarrow$(b) follows from 1.2 (xi).

The implication (b)$\Rightarrow$(c) and (c)$\Rightarrow$(d) are obvious.

If

$$a = f(a_1, \ldots, a_k) \notin \{a_1, \ldots, a_k\}$$

for an operation $f \in \boldsymbol{F}$, then $A \setminus \{a\}$ is a set of generators of $\mathfrak{A}$. Hence (d)$\Rightarrow$(a).

The implication (b)$\Rightarrow$(e) is obvious and its converse (e)$\Rightarrow$(b) follows from the fundamental properties of algebraic operations (see [1], p. 48, (v) and (vi)).

3.2. Numerical properties of finite algebras. For each algebra $\mathfrak{A} = (A; \boldsymbol{F})$ with $2 \leqslant |A| < \aleph_0$ we define the following integral numbers (see [2] and [5]):

$\alpha = |A|$,

$\gamma^* =$ the smallest number with the following property: every set $G \subset A$ with $|G| = \gamma^*$ is a set of generators,

$\gamma =$ the minimal number of generators,

$\iota =$ the maximal number of independent elements,

$\iota_* =$ the greatest number with the following property: every set $I \subset A$ with $|I| = \iota_*$ is a set of independent elements,

$\tau =$ the greatest number with the following property: every algebraic τ-ary operation is trivial (obviously this number is defined only for non-trivial algebras; if there are in $\mathfrak{A}$ algebraic constants, we put $\tau = -1$ by definition, and if there are no algebraic constants, but there is an algebraic non-trivial unary operation, we put $\tau = 0$).

Let us add that if each element of A is an algebraic constant in A, then the empty set is treated by definition as a set of generators of A, whence we have $\gamma = \gamma^* = 0$.

I will now list some known relations between the numbers defined above. In particular, it is known ([2], (i) and (vii)) that

(a) $\alpha \geqslant \gamma^* \geqslant \gamma \geqslant \iota \geqslant \iota_* \geqslant \tau$ *and* $\tau = \iota_*$ *or* $\tau = \iota_* - 1$.

(b) *If* $\alpha > \gamma$, *then* $\gamma > \tau$ ([2], p. 5, (viii)).

(c) *If in a non-trivial algebra* $\alpha = \iota$, *then* $\mathfrak{A}$ *is identical with* $\mathfrak{P}_*$, $\mathfrak{P}^*$ *or* $\mathfrak{P}$ *and* $\alpha = \gamma^* = \gamma = \iota = \iota_* = \tau = 2$ (see Świerczkowski [4], p. 94, Theorem 1, Marczewski-Urbanik [3] and [3a], Marczewski [2], p. 6).

(d) *If* $\gamma^* = \iota_* \geqslant 3$, *then* $\alpha = 4$ *and* $\mathfrak{A}$ *is identical with* $\mathfrak{S}$ (Świerczkowski [4], p. 94, Theorem 4).

(e) *If* $\iota_* \geqslant 4$, *then* $\tau = \iota_*$ (Świerczkowski [5]).

Let us also note an easy consequence of 3.1 (vii):

(f) $\alpha = \gamma$ *iff the algebra is quasi-trivial.*

Suppose now that the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is homogeneous (and that $2 \leqslant |A| < \aleph_0$). Thus it easily follows from 3.1 (iii) that

(o) $\gamma = \gamma^*$ *and* $\iota = \iota_*$,

whence we obtain the following simplification of (a), (d) and (e):

(i) $\alpha \geqslant \gamma \geqslant \iota \geqslant \tau$ *and* $\tau = \iota$ *or* $\tau = \iota - 1$.

(ii) *If* $\gamma = \iota \geqslant 3$, *then* $\alpha = 4$ *and* $\mathfrak{A}$ *is identical with* $\mathfrak{S}$.

(iii) *If* $\iota \geqslant 4$, *then* $\tau = \iota$.

I will now formulate other conclusions from the homogeneity of $\mathfrak{A}$. Propositions 3.1 (ii), 3.1 (iv) and 1.2 (vii′) imply respectively:

(iv) *Either* $\gamma = \alpha$ *or* $\gamma = \alpha - 1$.

(v) $\iota \geqslant 1$ *and, if* $\alpha \geqslant 3$, *then* $\iota \geqslant 2$.

and

(vi) *If* $\gamma = \alpha = 2$ *then* $\tau \geqslant 2$.

Let us remark incidentally that, in view of 1.2 (viii), proposition (vi) is also true under the hypothesis $\gamma = \alpha = 3$.

I shall now prove that

(vii) *If* $\iota \neq \tau$, *then* $\gamma = \iota \leqslant 3$ *and* $\alpha = \gamma + 1$.

In view of (v), we have $\iota \geqslant 1$ and hence, by definition of ι, there exists a set $I = \{a_1, \ldots, a_\iota\}$ of ι independent elements of A. By hypothesis, we have, on account of (i), $\iota > \tau$, whence there exists a ι-ary non-trivial algebraic operation f. Since I is a set of independent elements, we have

$$f(a_1, \ldots, a_\iota) \notin \{a_1, \ldots, a_\iota\},$$

whence, by 1.2 (i),

$$A = \{a_1, \ldots, a_\iota, f(a_1, \ldots, a_\iota)\}$$

and, consequently, I is a set of generators of A. We thus have $\gamma = \iota$ and $\alpha = \gamma + 1$. Proposition (iii) gives the inequality $\iota \leqslant 3$.

Proposition (vii) is thus proved.

3.3. Examples and conclusions. We can now summarize the discussion of numerical constants α, γ, ι and τ for homogeneous algebras.

(i) *For every homogeneous non-trivial algebra the quadruple* $(\alpha, \gamma, \iota, \tau)$ *is one of the following:*

	α	γ	ι	τ	
1	2	2	2	2	
2	n	n	k	k	where $n > k > 1$
3	n	$n-1$	k	k	where $n-1 > k > 1$
4	n	$n-1$	$n-1$	$n-2$	where $n = 2$, 3 or 4

Let us consider at first the case $\iota=\tau$. In view of 3.2 (iv), we have either $\gamma=\alpha$ or $\gamma=\alpha-1$. If $\gamma=\alpha$, we have either $\gamma=\iota$, whence we obtain, by 3.2 (c), the first row of the table, or $\gamma>\iota$, whence we obtain the second row. It remains to prove in this case that $k>1$. In fact, if $\alpha=2$, we obtain, by 3.2 (vi), $\tau\geqslant 2$, and if $\alpha\geqslant 3$ then, by 3.2 (v), $\iota\geqslant 2$.

If $\alpha>\gamma$, we obtain the third row of the table. It remains to prove in this case that $n-1>k>1$. In fact, the inequality $n-1>k$ follows from 3.2 (b) and the inequality $k>1$ from 3.2 (v) for $\alpha\geqslant 3$ and from 3.2 (vi) for $\alpha=2$.

Passing to the case $\iota\neq\tau$, it is enough to remark that 3.2 (i) and (vii) imply directly the fourth row of the table.

Proposition (i) is thus proved.

Let us consider the following algebras:

$$\mathfrak{L}_{n,m}=(N;\, l_m),\qquad \mathfrak{R}_n=(N;\, r_n),\qquad \mathfrak{R}_{n,m}=(N;\, l_m,\, r_n),$$

where $|N|=n$ and l_m and r_n are operations defined in 1.1. These algebras are, of course, homogeneous.

I shall prove that

(ii) *For every quadruple of numbers in the rows of table* (i) *there exists a homogeneous algebra for which* $(\alpha,\gamma,\iota,\tau)$ *is identical with that quadruple*:

	Algebras	α	γ	ι	τ
1	$\mathfrak{P}_*$, $\mathfrak{P}^*$, $\mathfrak{P}$	1	2	2	2
2	$\mathfrak{L}_{n,k+1}$ *with* $n>k>1$	n	n	k	k
3	$\mathfrak{R}_{n,k+1}$ *with* $n-1>k>1$	n	$n-1$	k	k
4a	$\mathfrak{R}_2$	2	1	1	0
4b	$\mathfrak{R}_3$	3	2	2	1
4c	$\mathfrak{S}$	4	3	3	2

Row 1. See 3.1 (vi).

Rows 2 and 3. On account of the relation $n>k>1$ (supposed in both rows) we have $k+1\geqslant 3$ and $n\geqslant k+1$, whence, by 1.1 (iv), l_{k+1} is non-trivial, and consequently, $\tau\leqslant k$. In order to show that $\tau=k$ it remains to prove that every k-ary algebraic operation in the algebras under consideration is trivial. This follows easily from the fact that if $f_1, f_2, \ldots$ are trivial k-ary operations, then

$$l_{k+1}(f_1,\ldots,f_{k+1})=f_{k+1}$$

and, for $n-1>k$,

$$r_n(f_1,\ldots,f_{n-1})=f_1,$$

on account of definitions of l_{n-1} and r_n.

In order to verify that $\iota=k$ it suffices to remember that $\iota\geqslant\tau$ and remark that every $k+1$ elements $a_1,\ldots,a_{k+1}$ of N are dependent. In fact,

$$l_{k+1}(a_1,\ldots,a_{k+1})=a_1=e_1^{(k+1)}(a_1,\ldots,a_{k+1}),$$

whereas

$$l_{k+1}\neq e_1^{(k+1)}.$$

Since the operation l_{k+1} is quasi-trivial, we have $\gamma=\alpha=n$ for $\mathfrak{L}_{n,k+1}$, and, since r_n is a non-quasi-trivial $(n-1)$-ary operation, we have $\gamma=\alpha-1=n-1$ for $\mathfrak{R}_{n,k+1}$ on account of 3.2 (iv).

Rows 4a, 4b and 4c. All values in these rows are known (see [2], p. 6) and easy to compute.

Propositions (i), (ii) and 3.2 (f) give the final conclusion of our discussion:

(iii) *Table* (i) *consists of all quadruples* $(\alpha,\gamma,\iota,\tau)$ *for homogeneous non-trivial algebras*: *quasi-trivial in rows* 1 *and* 2 *and nonquasi-trivial in rows* 3 *and* 4.

3.4. Appendix. In the preceding paragraphs Świerczkowski's theorem 3.2 (e), or, more precisely, the simplified version of this theorem for homogeneous algebras 3.2 (iii), plays an essential part. Since the proof of 3.2 (e) is based on some earlier results, not easy to prove, it is worth noticing that 3.2 (iii) for homogeneous algebras can be obtained in another way, namely with the aid of the properties of symmetrical operations stated in section 2.

Let us begin by the following lemma:

(i) *If in a homogeneous algebra* $(A;\mathbf{F})$ *with* $|A|=6$ *there is an algebraic symmetrical operation* f *of five variables, then there exists an algebraic quasi-trivial (and consequently non-symmetrical) non-trivial operation* g *of five variables.*

Put $A=\{a_1,\ldots,a_6\}$. The homogeneity and symmetry of f imply

$$(*)\qquad f(a_1,\ldots,a_5)=a_6.$$

Putting

$$g=f(f,\, e_1^{(6)},\, e_2^{(6)},\, e_3^{(6)},\, e_4^{(6)})$$

we obtain, in view of $(*)$,

$$g(a_1,a_2,a_3,a_4,a_5)=f(a_6,a_1,a_2,a_3,a_4),$$

whence, by $(*)$ and the homogeneity of f,

$$(**)\qquad g(a_1,a_2,a_3,a_4,a_5)=a_5.$$

Since, in view of 2.1 (v′),

$$f(a_1, a_1, a_1, a_2, a_5) = a_1,$$

we obtain, on account of 1.2 (i),

$$(\overset{*}{**}) \qquad g(a_1, a_1, a_1, a_2, a_5) = f(a_1, a_1, a_1, a_1, a_2) = a_1 \text{ or } a_2.$$

Formulas (**) and ($\overset{*}{**}$) show that g is non-trivial. It follows from formula (**), the homogeneity of g and proposition 1.2 (i) that g is quasi-trivial (and, by 2.1 (ii), non-symmetrical). Proposition (i) is thus proved.

(ii) *Let us suppose $k \leqslant n \geqslant 6$. If in a homogeneous n-element algebra $\mathfrak{A}$ there exists a k-ary non-trivial algebraic operation, then there exists a non-symmetrical operation with the same properties.*

In view of 2.2 (iii) it suffices to consider the case $n = 6$, $k = 5$. On account of (i) there exists in $\mathfrak{A}$ a non-symmetrical and non-trivial algebraic operation g of five variables, q.e.d.

In order to prove 3.2 (iii) for homogeneous algebras it is sufficient to prove the following proposition:

(iii) *Let $n \geqslant k \geqslant 4$ and $|A| = n$. If the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is homogeneous and if every k-element set in it is a set of independent elements, then every k-ary algebraic operation in $\mathfrak{A}$ is trivial.*

Let us suppose a contrario that there exists in $\mathfrak{A}$ a k-ary non-trivial algebraic operation f. Moreover, let us consider k different elements $a_1, \ldots, a_k$ of A and put $a_{k+1} = f(a_1, \ldots, a_k)$. It follows from the independence of $a_1, \ldots, a_k$ that

$$a_{k+1} \notin \{a_1, \ldots, a_k\}$$

and from 1.2 (i) that

$$(+) \qquad A = \{a_1, \ldots, a_{k+1}\}.$$

Hence $n = k+1$.

I will prove that f is the only algebraic non-trivial k-ary operation. In fact, if g is also an algebraic non-trivial k-ary operation, we have, in view of the independence of $a_1, \ldots, a_k$,

$$g(a_1, \ldots, a_k) \notin \{a_1, \ldots, a_k\},$$

whence, on account of (+),

$$f(a_1, \ldots, a_k) = g(a_1, \ldots, a_k)$$

and, again in view of the independence of $a_1, \ldots, a_k$, we obtain $f = g$.

Let us put, for an arbitrary permutation p of $K = \{1, \ldots, k\}$,

$$f^*(x_1, \ldots, x_k) \equiv f(x_{p_1}, \ldots, x_{p_k}).$$

Operation f^* is thus an algebraic, non-trivial k-ary operation, whence, by the preceding remark, $f = f^*$. Consequently f is symmetrical.

It follows from the existence of symmetrical homogeneous operations in A that $n = 2, 3, 4$ or 6. The case $n = 6$ is impossible on account of (ii). Since $n = k+1$, we obtain $k = 1, 2$ or 3, contrary to our hypotheses.

Proposition (iii) is thus proved and hence we obtain a new proof of 3.2 (iii) for homogeneous algebras.

References

[1] E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45-61.

[2] — *Nombres d'éléments indépendants et nombres d'éléments générateurs dans les algèbres abstraites finies*, Annali di Matematica (4) 59 (1962), pp. 1-9.

[3] — and K. Urbanik, *Algebras in which all elements are independent*, Bull. Acad. Pol. Sc. Série des sc. math., astr. et phys. 8 (1960), pp. 291-293.

[3a] — — *Algebras in which all elements are independent*, Coll. Math. 9 (1962), pp. 199-207.

[4] S. Świerczkowski, *On algebras independently generated by every n elements*, Fund. Math. 49 (1960), pp. 93-104.

[5] — *On two numerical constants associated with finite algebras*, Annali di Matematica (4) 62 (1963), pp. 241-245.

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