Homogeneous operations and homogeneous algebras

by

E. Marczewski (Wrocław)

Introduction. In algebraic systems usually treated in mathematics, such as groups, fields, vector spaces, etc., single elements have some individual properties and their roles in the systems considered are different. Nevertheless, in some research in universal algebra or the general theory of algebraic systems, and especially in the study of the notion of independence (see [1] and all the other papers in the references) there are examples of interesting abstract algebras which are in a certain sense homogeneous, i.e., in which all elements have the same properties. So are some algebras defined by E. L. Post, denoted here by $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ (cf. Marczewski and Urbańska [3] and [3a]), algebra $\mathcal{E}$ of Święcickowski (denoted by $\mathfrak{E}$ by Świerczkowski, see [4], p. 94) and others.

In this paper I adopt the following definitions of homogeneity (see 1.1): an operation $f$ in a set $A$ is homogeneous whenever the formula

$$a_n = f(a_1, ..., a_n)$$

implies

$$h(a_n) = f(h(a_1), ..., h(a_n))$$

for every permutation $h$ of $A$. An algebra is homogeneous if every operation in it is homogeneous (see 3.1).

By replacing permutations in these definitions by all transformations of $A$ into itself, I obtain the definition of strong homogeneity (see 2.3 and 3.1), a notion strictly connected with that of independence (cf. 3.1 (v)); let us remark that strong homogeneity is called homogeneity in [3a]).

This paper contains a study of homogeneous operations and homogeneous algebras.

In the first section I give a representation theorem for homogeneous operations (1.3 (vi)) based on some ideas of Świerczkowski (see [4], especially pp. 95 and 96).

In the second section I consider some classes of homogeneous operations, first of all symmetrical homogeneous operations and strongly homogeneous operations. Symmetry (= commutativity) of an operation may be treated also as a sort of homogeneity, namely as a certain homogeneity of the set of indices (see 2.1 (iii)). It turns out that operations
which are simultaneously homogeneous and symmetrical rarely exist: only in two-, three-, four-, and six-elements sets (2.2 (ii)). Let us remember that strongly homogeneous non-trivial operations are still more exceptional in view of a result of Święckowski [4] such operations exist only in two-element sets (see 2.3 (vii) and (viii)). I gather together all representation and existence theorems in 2.4.

The third section is devoted to homogeneous algebras, especially to their properties connected with the notion of independence. In an earlier paper [2] I considered some numerical constants associated with finite algebras: the minimal number of generators, the maximal number of independent elements, and others. In 3.3 I give the complete discussion of these constants for homogeneous algebras.

Let us remark that in the third section the results concerning symmetrical operations (2.1 and 2.2) are used only in the Appendix (3.4).

**Terminology and notation.** For each finite or denumerable sequence $a_1, a_2, \ldots$ I denote by $(a_1, a_2, \ldots)$ the set of its terms. E.g., $(a, b, a, b, a, b) = (a, b)$.

The cardinal of a set $A$ is denoted by $|A|$. A one-one transformation of $A$ onto $A$ is called a permutation of $A$.

Every mapping $f: A^k \to A$ (where $k = 1, 2, \ldots$) is called an operation of $k$ variables on $A$, or a $k$-ary operation in $A$. For operations in a fixed set $A$, we use the symbol $=\ldots$ in the following sense: $=\ldots$ is satisfied where in $A$. E.g. the formula $f(a_1, a_2) = g(a_1)$ means $f(a_1, a_2) = g(a_1)$ for every pair $(a_1, a_2) \in A^2$.

Operations of the form $\delta^k((a_1, \ldots, a_k) = a_1$, where $k = 1, 2, \ldots$ and $f = 1, 2, \ldots, k$, are called trivial.

An operation $f$ in $A$ is called quasi-trivial if $f(a_1, \ldots, a_k) \in \{a_1, \ldots, a_k\}$ for every $(a_1, \ldots, a_k) \in A^k$.

All terminology and notation concerning abstract algebras, e.g. such notions as algebraic operations, generators, independence, identity of two algebras, etc. agree with my papers [1] and [2].

1. **Homogeneous operations**

**1.1. Definition and examples.** An operation $f: A^k \to A$ is called homogeneous if $h(f(a_1, \ldots, a_k)) = f(h(a_1), \ldots, h(a_k))$ for every permutation $h$ of $A$. It is obvious that

(i) Every trivial operation is homogeneous and quasi-trivial.

(ii) The identity operation $\delta^k(x) = x$ is the only quasi-trivial unary operation.

We will consider the following examples:

$r_n$ (where $n \geq 2$)—an $(n-1)$-ary operation defined in every $n$-element set by the following conditions: if all terms of the sequence $a_1, \ldots, a_{n-1}$ are different, then $r_n(a_1, \ldots, a_{n-1}) \neq \{a_1, \ldots, a_{n-1}\}$, and if there are in it two identical terms, then $r_n(a_1, \ldots, a_{n-1}) = a_1$ (see [2], p. 2).

$l_n$—an $n$-ary operation defined in an arbitrary set as follows: $l_n(a_1, \ldots, a_n) = a_1$ if $a_1, \ldots, a_n$ are all different and $l_n(a_1, \ldots, a_n) = a_n$ in the opposite case (see [2], p. 2).

$p_n$ and $p^*$—two ternary operations defined in every two-element set by the equations

$p_n(x, y, z) = p_n(y, z, x) = p_n(y, z, x) = x$,

$p^*(x, z, y) = p^*(y, z, x) = p^*(y, z, x) = y$.

(Operations defined by E. L. Post. See Mareczewski [2], Mareczewski-Urbanik [3], [3a, b].)

$s$—a ternary operation, defined in any four-element set as follows:

$s(x, y, z) = s(y, z, x)$ if $x, y, z$ are different and $s(x, y, z) = p_n(x, y, z)$ in the opposite case (Święckowski [4], p. 94, cf. Mareczewski [2], p. 2).

Let us remark that in the set $A = \{0, 1\}$ the operations $p_n$ and $p^*$ may be described as follows (see Mareczewski-Urbanik [3], p. 299):

$p_n(x, y, z) = x + y + z (\mod 2)$,

$p^*(x, y, z) = xy + yz + xz (\mod 2)$.

It is obvious that

(iii) Operation $r_n$ for $n \geq 2$ is a homogeneous, non quasi-trivial operation in an $n$-element set; $r_2$ is the transposition in a two-element set.

(iv) Operation $l_n$ for $n \geq 3$ is a homogeneous, quasi-trivial, but non-trivial operation in a set having at least $n$ elements. If the set has less than $n$ elements, then $l_n = \delta^k$, i.e. $l_n$ is the identity operation: $l_n(x) = x$ and $l_n = \delta^k$, i.e. $l_n(x, y) = x$.

(v) Operations $p_n$ and $p^*$ are homogeneous, quasi-trivial but non-trivial.

(vi) Operation $s$ is homogeneous and non quasi-trivial.

1.2. **Properties.** At first we will consider the case where a homogeneous operation is non quasi-trivial. The following simple but fundamental proposition shows that this situation is of a somewhat exceptional character and that it is possible only in finite algebras:

(I) If $f$ is a homogeneous operation in a set $A$ and

\[
b = f(a_1, \ldots, a_k) \in \{a_1, \ldots, a_k\},
\]

then $b$ is the only element of $A$ not belonging to $\{a_1, \ldots, a_k\}$.
Let us denote by \( c \) an arbitrary element of \( A \setminus \{a_1, \ldots, a_k\} \) and by \( h \) a permutation of \( A \) such that

\[
\begin{array}{c}
h(b) = c, \\
h(c) = b, \\
h(x) = x \quad \text{for} \quad x \in A \setminus \{b, c\}.
\end{array}
\]

Hence

\[
h(a_f) = a_f \quad \text{for} \quad f = 1, 2, \ldots, k.
\]

Consequently

\[
\begin{align*}
e &= h(b) = h[f(a_1, \ldots, a_k)] = f[h(a_1), \ldots, h(a_k)] \\
&= f(a_1, \ldots, a_k) = b, \quad \text{q.e.d.}
\end{align*}
\]

Let us note the following corollaries to (i):

(ii) In an infinite set every homogeneous operation is quasi-trivial.

(iii) If (\( \ast \)) for a homogeneous operation \( f \) in a set \( A \) with \( |A| = n \geq 2 \) then \( A \) is finite and there in the sequence \( a_1, \ldots, a_n \) precisely \( n - 1 \) different terms.

(iv) If (\( \ast \)) for a homogeneous operation \( f \) and a sequence \( a_1, \ldots, a_k \) of different elements, then \( |A| = k + 1 \).

The following proposition is obvious:

(v) If \( f \) a homogeneous operation \( f \) in \( A \) and a sequence \( a_1, \ldots, a_k \) of different elements

\[
f(a_1, \ldots, a_k) = a_i \quad \text{(where} \quad 1 \leq i \leq k \text{)}
\]

then

\[
f(b_1, \ldots, b_k) = b_i
\]

for every sequence \( b_1, \ldots, b_k \) of different elements of \( A \).

The preceding propositions imply some conclusions concerning the case of a set having only a few elements. In particular, propositions 1.1 (iii) and 1.2 (iv) and (v) imply

(vii) If \( f \) is a non-trivial homogeneous unary operation in a set \( A \), then \( |A| = 2 \) and \( f = r_A \).

Proposition (v) implies

(viii) In a two element set there are only two non-trivial homogeneous binary operations \( g_1 \) and \( g_2 \):

\[
\begin{align*}
g_1(a_1, a_1) &= a_1 \quad \text{if} \quad a_1 \neq a_2, \\
g_2(a_1, a_2) &= x
\end{align*}
\]

whence

(vii') In a two-element set each quasi-trivial homogeneous binary operation is trivial.

Propositions (iii) and (v) imply

(viii) In a three-element set the operation \( r_A \) is the only non-trivial homogeneous binary operation,

and

(ix) If \( |A| \geq 4 \), then every homogeneous binary operation in \( A \) is trivial.

Finally let us note the following propositions, easy to prove.

(x) Every superposition of homogeneous operations is homogeneous.

(xi) Every superposition of quasi-trivial operations is quasi-trivial.

1.3. Representation theorem. A finite class of sets \( \mathcal{E} = \{E_1, \ldots, E_m\} \) is called a decomposition of \( K \) into \( m \) parts whenever they are different, non void, disjoint and

\[
E = E_1 \cup \cdots \cup E_m.
\]

We denote by \( \mathcal{D}(k, n) \), where \( 1 \leq k < \infty \) and \( 1 \leq n \leq \infty \), the set of all decompositions of \( K = \{1, 2, \ldots, k\} \) into at most \( n \) parts. Obviously, for every \( n \geq k \) (and, in particular if \( n \) is infinite), \( \mathcal{D}(k, n) \) is the class of all decompositions of \( K \).

For every sequence \( a_1, \ldots, a_k \) we denote by \( \mathcal{D}(a_1, \ldots, a_k) \) the decomposition of \( K = \{1, 2, \ldots, k\} \) such that \( i, j \in K \) belong to the same element of \( \mathcal{D}(a_1, \ldots, a_k) \) if \( a_i = a_j \).

The following two propositions are obvious:

(i) \( \mathcal{D}(a_1, \ldots, a_k) \) is a decomposition of \( K \) into \( m \) parts iff the set \( \{a_1, \ldots, a_k\} \) has precisely \( m \) elements.

(ii) Let \( A \) denote an \( n \)-element set and \( \mathcal{D} \) a decomposition of the set \( K = \{1, 2, \ldots, k\} \). There is a sequence \( a_1, \ldots, a_k \in A \) such that \( \mathcal{D} = \mathcal{D}(a_1, \ldots, a_k) \), iff \( \mathcal{D} = \mathcal{D}(b_1, \ldots, b_k) \).

Two sequences \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \) are called similar if \( \mathcal{D}(a_1, \ldots, a_k) = \mathcal{D}(b_1, \ldots, b_k) \). Obviously,

(iii) If \( a_1, \ldots, a_k \in A \), \( h \) is a permutation of \( A \), and \( b_1 = h(a_1), \ldots, b_k = h(a_k) \), then \( \mathcal{D}(a_1, \ldots, a_k) \) and \( \mathcal{D}(b_1, \ldots, b_k) \) are similar.

The following proposition results immediately from the definition of homogeneity and presents a generalization of 1.2 (v):

(iv) If \( f \) is a homogeneous operation in \( A \) and two sequences \( (a_1, \ldots, a_k) \) and \( (b_1, \ldots, b_k) \) are similar, then

\[
\begin{align*}
&\text{if } f(a_1, \ldots, a_k) = a_i & \text{then } f(b_1, \ldots, b_k) = b_i, \\
&\text{if } f(a_1, \ldots, a_k) \in (a_1, \ldots, a_k) & \text{then } f(b_1, \ldots, b_k) \in (b_1, \ldots, b_k).
\end{align*}
\]

We denote by \( \Phi(k, n) \) the class of all functions \( \varphi \) the domain of which is \( \mathcal{D}(k, n) \) and for which either (1) \( \varphi(\emptyset) = \emptyset \) or (2) \( |\varphi| + 1 = n \) and
\( \varphi(\emptyset) = 0 \) (the empty set). In other words, \( \varphi \) is a set-valued function which for every decomposition \( \delta \in \Delta(k, n) \) either distinguishes a set belonging to \( \delta \) or admits for \( \delta \) the value 0, but the second case is possible only if \( k \) is finite and if \( \delta \) is a decomposition into precisely \( n-1 \) parts (cf. Święczkowski [4], p. 96).

For every function \( \varphi \in \Phi(k, n) \) we define a \( k \)-ary operation \( f_\varphi \) in any \( n \)-element set \( A \) by the following conditions: (1) If \( \varphi(\delta(a_1, ..., a_k)) \neq 0 \), then \( f_\varphi(a_1, ..., a_k) = \emptyset \), where \( j \in \varphi(\delta(a_1, ..., a_k)) \); and (2) if \( \varphi(\delta(a_1, ..., a_k)) = 0 \), then \( f_\varphi(a_1, ..., a_k) \in \{ a_1, ..., a_k \} \). The definition is consistent and univocal because (1) if \( \varphi(\delta(a_1, ..., a_k)) \) is non-empty and if \( i \neq j \in \varphi(\delta(a_1, ..., a_k)) \), then \( a_i = a_j \) and (2) if \( \varphi(\delta(a_1, ..., a_k)) = 0 \), then by definition of \( \Phi(k, n) \) \( a \) is finite and \( \delta(a_1, ..., a_k) \) is a decomposition of \( K \) into \( n-1 \) parts or, in other words, \( a_1, ..., a_k \) is a \((n-1)\)-element set, whence \( A \setminus \{ a_1, ..., a_k \} \) is a one-element set.

We can now state the following representation theorem for homogeneous operations:

(v) The correspondence \( \varphi \to f_\varphi \) is a one-one correspondence between the class \( \Phi(k, n) \) and the class of all homogeneous \( k \)-ary operations in an \( n \)-element set \( A \).

Proof. 1. Let us first verify that for \( \varphi \in \Phi(k, n) \) the operation \( f_\varphi \) is homogeneous, or, in other words, that if

\[
\varphi = f_\varphi(a_1, ..., a_k)
\]

then, for every permutation \( h \) of \( A \),

\[
\varphi(h(a)) = f_\varphi(h(a_1), ..., h(a_k))
\]

This is an easy consequence of the preceding definitions, of (iii) and of the remark that if \( (a) = A \setminus \{ a_1, ..., a_k \} \), then \( (h(a)) = A \setminus \{ h(a_1), ..., h(a_k) \} \).

2. We shall prove that for every homogeneous \( k \)-ary operation \( f \) in \( A \) there is a function \( \varphi \in \Phi(k, n) \) such that \( f = f_\varphi \). Namely, for every \( \delta \in \Delta(k, n) \) we consider such a sequence \( a_1, ..., a_k \in A \) that \( \delta(a_1, ..., a_k) = \delta \) (in view of (ii)) and we put

\[
\varphi(\delta) = K \setminus \{ a_1, ..., a_k \}.
\]

In view of (iv) this set depends only on \( \delta \), i.e., it does not depend on the choice of \( a_1, ..., a_k \).

For every \( a_1, ..., a_k \),

\[
\text{If } f(a_1, ..., a_k) \in \{ a_1, ..., a_k \} \text{, then } \varphi(\delta(a_1, ..., a_k)) = 0,
\]

and consequently

\[
f(a_1, ..., a_k) \in \{ a_1, ..., a_k \}.
\]

Since, in view of 1.2 (i), \( A \setminus \{ a_1, ..., a_k \} \) is, in this case, a one-element set, we have here \( f_\varphi(a_1, ..., a_k) = f(a_1, ..., a_k) \).

Hence \( f = f_\varphi \).

3. We shall prove finally that if \( f_\varphi = f_\psi \), then \( \varphi = \psi \). For this purpose, let us consider a decomposition \( \delta \in \Delta(k, n) \).

\[
\text{If } \varphi(\delta(a_1, ..., a_k)) \neq 0 \text{ then } \varphi(\delta(a_1, ..., a_k)) \in \{ a_1, ..., a_k \}
\]

then \( \varphi(\delta(a_1, ..., a_k)) = 0 = \psi(\delta(a_1, ..., a_k)) \).

If \( \delta(\alpha_1, ..., \alpha_k) = f_\varphi(\alpha_1, ..., \alpha_k) \)

then

\[
\text{if } j \in \varphi(\delta(a_1, ..., a_k)) \neq \psi(\delta(a_1, ..., a_k)) \text{, then } \varphi(\delta(a_1, ..., a_k)) = 0.
\]

Consequently

\[
\varphi(\delta(a_1, ..., a_k)) = 0 = \psi(\delta(a_1, ..., a_k))
\]

for every \( a_1, ..., a_k \in A \), whence, by (ii), \( \varphi(\delta) = \psi(\delta) \) for every \( \delta \in \Delta(k, n) \), or, in other words, \( \varphi = \psi \).

Representation theorem (v) is thus proved.

The following easy consequence of the definition of \( f_\varphi \) can be treated as a supplement of (v) for quasi-trivial operations:

(vi) A homogeneous operation \( f_\varphi \) is quasi-trivial iff \( \varphi \) does not vanish (i.e., \( \varphi(\delta) \neq 0 \) for every \( \delta \in \Delta(k, n) \)).

For trivial operations we first formulate the following obvious proposition:

(vii) Let \( \varphi \in \Phi(k, n) \) and \( |A| = n \). In order that \( f_\varphi = \delta^{(n)} \) in \( A \) it is necessary and sufficient that \( \varphi \) be defined as follows: for each decomposition \( \delta = (K_1, ..., K_n) \in \Delta(k, n) \) we have \( \varphi(\delta) = K_p \), where \( j \in K_p \).

For each \( \varphi \in \Phi(k, n) \) we denote by \( P(\varphi) \) the intersection of all sets \( \varphi(\delta) \), where \( \delta \in \Delta(k, n) \).

We shall prove that

(viii) If \( \varphi \in \Phi(k, n) \) and \( |A| = n \), then the following conditions are equivalent:

1. \( f_\varphi \) is trivial in \( A \).
2. \( \varphi \) does not vanish and the class \( \{ \varphi(\delta) : \delta \in \Delta(k, n) \} \) is multiplicative.
3. \( P(\varphi) = 0 \).

It is easy to see that, in the case \( n \geq 2 \), these conditions are equivalent also to

4. \( P(\varphi) \) is a one-element set.

1) \( \Rightarrow \) (ii), then for any two decompositions \( \delta, \gamma \in \Delta(k, n) \) we have, in view of (vii),

\[
\varphi(\delta) \cap \varphi(\gamma) = \emptyset.
\]
Let us denote by $\varepsilon$ any decomposition of $K$ such that $\varphi(\delta) \cap \varphi(\gamma) \neq \varepsilon$.
Then, by (vii),
$$\varphi(\varepsilon) = \varphi(\delta) \cap \varphi(\gamma),$$
q.e.d.

(iii) $+(\delta) (\gamma)$. It follows from (vii) that there is a decomposition $\delta$ such that $\varphi(\delta) = \varphi(\gamma)$, whence (iii).

(vii) $+(\delta) (\gamma)$. If $f(\sigma_1, ..., \sigma_n) = \sigma_i$, or, in other words, $f$ is trivial, q.e.d.

2. Symmetrical homogeneous operations and strongly homogeneous operations

2.1. Symmetrical homogeneous operations. A $k$-ary operation $f$ is called symmetrical or commutative if for each permutation $p: f \to p_f$ of the set $K = \{1, 2, ..., k\}$
$$f(\sigma_1, ..., \sigma_n) = f(\sigma_{p_1}, ..., \sigma_{p_n}).$$

It is obvious that

(i) Operations $p_k, p_{r_k}, \alpha$ and $\sigma$ are symmetrical, whereas (in a set having at least two elements) $\sigma^k$ with $k \geq 2$, $k > 3$ and $\alpha_k$ with $k \geq 4$ are not.

I shall prove that

(ii) If $2 < k \leq n$, then no $k$-ary homogeneous and quasi-trivial operation $f$ in an $n$-element set $A$ is symmetrical.

In fact, if $a_1, ..., a_n$ are different elements of $A$, and if, say,
$$f(a_1, a_2, ...) = a_1,$$
then, by the homogeneity of $f$,
$$f(a_1, a_1, ...) = a_1,$$
whence $f$ is non-symmetrical.

For every permutation $p$ of $K$, every $E \subseteq K$ and every class $F$ of subsets of $K$ we define $p(E)$ and $p(F)$ by the formulas
$$p(E) = \{\sigma: f \in E\}, \quad p(F) = \{p(E): E \in F\}.$$

Let us verify that

(o) $\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n})) = \delta(\sigma_1, ..., \sigma_n)$. 

In fact, $i$ and $j$ belong to the same part of $\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))$ iff $p_i = p_j$ and $p_{i^{-1}}$ belong to the same part of $\delta(\sigma_{p_1}, ..., \sigma_{p_n})$, or, in other words, iff $\sigma_{p_{i^{-1}}} = \sigma_{p_{j^{-1}}}$, which is equivalent to $\sigma_i = \sigma_j$.

A function $\varphi: \Phi(k, n)$ will be called homogeneous whenever $\varphi(p(\delta)) = p(\varphi(\delta))$ for every permutation $p$ of $K$. We may now prove a supplement to the representation theorem which concerns symmetrical operations.

(iii) Homogeneous $k$-ary operation $f_0$ in $A$ (where $\varphi: \Phi(k, n), |A| = n$) is symmetrical iff $\varphi$ is homogeneous.

Let us suppose $\varphi$ homogeneous and let $p$ denote a permutation of $K = \{1, 2, ..., k\}$. We have to prove that

(*) $f_0(\sigma_{p_1}, ..., \sigma_{p_n}) = f_0(\sigma_1, ..., \sigma_n)$

If $\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n})) = 0$, then, in view of (o) and of the homogeneity of $\varphi$,
$$\varphi(\delta(\sigma_1, ..., \sigma_n)) = \varphi\{\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))\} = \varphi(\varphi(\delta(\sigma_1, ..., \sigma_n))) = 0.$$

Hence, in view of 1.2 (i) and of the definition of $f_0$ the element $f_0(\sigma_1, ..., \sigma_k)$, as well as $f_0(\sigma_{p_1}, ..., \sigma_{p_n})$ is the only element of $A$ not belonging to $\{\sigma_1, ..., \sigma_n\}$. Thus we obtain (*) in the case considered.

If $\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n})) \neq 0$, i.e., if there exists a $f$ such that $f \neq \varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))$, or, in other words, $f_0(\sigma_{p_1}, ..., \sigma_{p_n}) = \sigma_{p_1}$, then, in view of (o) and the homogeneity of $\varphi$,
$$p_1 \cdot p(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))) = p(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))) = \varphi(\delta(\sigma_1, ..., \sigma_n)).$$

Hence $f_0(\sigma_{p_1}, ..., \sigma_{p_n}) = \sigma_{p_1} = f_0(\sigma_1, ..., \sigma_n)$.

The formula (*) is thus proved.

Let us suppose now that an operation $f_0$ is symmetrical. We have to prove that
$$\varphi(p(\delta)) = p(\varphi(\delta))$$
for every permutation $p$ of $K$ and every $\delta \in \Delta(k, n)$.

By 1.3 (ii) there exists a sequence $a_1, ..., a_k \in A$ such that $\delta(a_1, ..., a_k) = \delta$.

It is sufficient to prove that

(++) $\varphi(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))) = \varphi(\varphi(\delta(\sigma_1, ..., \sigma_n)))$

or, in view of (o),
$$\varphi(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))) = \varphi(\varphi(\delta(\sigma_1, ..., \sigma_n)))$$
where $q: j \to q(j)$ denotes the inverse of $p$.

The relation $i \in \varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))$ is equivalent consecutively to the following ones:
$$a_m = a_i = f_0(\sigma_1, ..., \sigma_n) = f_0(\sigma_{p_1}, ..., \sigma_{p_n}) = q_i \cdot p(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n}))), \quad i \in p(\varphi(\delta(\sigma_{p_1}, ..., \sigma_{p_n})))$$
whence we obtain (++).
Proposition (iii) is thus proved.

It easily follows from the definition of homogeneity of $\varphi$ that

(4) A function $\varphi \in \Phi(k, n)$ is homogeneous (or, in other words, $f_\varphi$ is symmetrical) iff for each two decompositions of $K = \{1, 2, \ldots, k\}$ into $m \leq n$ parts:

$$\delta = (K_1, \ldots, K_m) \in \Lambda(k, n),$$

$$\delta^* = (K_1^*, \ldots, K_m^*) \in \Lambda(k, n)$$

with $|K_j| = |K_j^*|$ for $j = 1, \ldots, m$, the equation $\varphi(\delta) = K_1$ implies $\varphi(\delta^*) = K_1^*$.

We can now prove

(5) If a function $\varphi \in \Phi(k, n)$ is homogeneous, then for every $\delta \in \Lambda(k, n)$, we have $|E| = |\varphi(\delta)|$ whenever $|\varphi(\delta)| \neq E \neq |\delta|$ (in other words, the set $\varphi(\delta)$ has a different number of elements than any other set belonging to $\delta$).

In fact, if for $\delta = (K_1, \ldots, K_m)$ we have, say, $\varphi(\delta) = K_1$ and $|K_j| = |K_1|$, then, applying (4) to the same decomposition written in two forms:

$$\delta = (K_1, K_2, K_3, \ldots, K_m),$$

$$\delta = (K_1, K_1, K_3, \ldots, K_m),$$

we obtain a contradiction: $\varphi(\delta) = K_1$ and $\varphi(\delta) = K_2$, whereas $K_1 \neq K_2$.

Let us translate (5) into the terms of operations (with the aid of (iii)):

(5') If a homogeneous operation is symmetrical and if

$$f(a_1, \ldots, a_k) = a_1 \neq a_i$$

then

$$|\delta| = |\delta'| = |\delta| \neq |\delta'| = |\delta| \neq |\delta'|.$$  

(6) If $k \leq n$, none symmetrical homogeneous $k$-ary operation in an $n$-element set $A$ is quasi-trivial.

It is sufficient to consider $k$ different elements of $A$ and to apply (5').

2.2. Existence of symmetrical homogeneous operations.

To every decomposition $\delta = (K_1, \ldots, K_m) \in \Lambda(k, n)$ (where all $K_j$ are different) corresponds an arithmetic decomposition of the number $k = |K|$:

$$|K| = |K_1| + \ldots + |K_m|.$$  

We denote by $D(k, n)$ the class of all such decompositions corresponding to decompositions belonging to $\Lambda(k, n)$ or, in other words, the class of all decompositions

(•)

$$k = k_1 + \ldots + k_m$$

where $k_i$ are natural numbers and $m \leq n$. Obviously two decompositions which differ only in the order of summands are treated as identical. By $D^*(k, n)$ we denote the class of all decompositions (•) belonging to $D(k, n)$ such that $m \neq n - 1$.

A decomposition (•) is called good if there is a summand which appears in (•) only once. In the case the decomposition is bad, a decomposition $\delta \in D(k, n)$ is called respectively good or bad if the corresponding arithmetic decomposition belonging to $D(k, n)$ is such.

We shall prove

(i) In an $n$-element set $A$ there exists a homogeneous, symmetrical $k$-ary operation with $k > 1$ iff every decomposition belonging to $D^*(k, n)$ is good.

(ii) In an $n$-element set $A$ there exists a homogeneous symmetrical and quasi-trivial $k$-ary operation with $k > 1$ iff every decomposition belonging to $D(k, n)$ is good.

Let us remark, in view of 2.1 (ii), only the case $k > n$ is essential in (ii).

The necessity in theorems (i) and (ii) follows from 1.3 (v), 1.3 (vi), 2.1 (iii) and 2.1 (v).

In order to prove the sufficiency, let us define $\varphi \in \Phi(k, n)$ as follows.

Let $\delta = (K_1, \ldots, K_m) \in \Lambda(k, n)$.

We may suppose of course that

$$|K_1| \leq |K_2| \leq \ldots \leq |K_m|.$$

If $\delta$ is a bad decomposition, then we put $\varphi(\delta) = 0$. If $\delta$ is good, then there exists a smallest index $s$ such that $|K_s| \neq |K_s|$. We then put $\varphi(\delta) = K_s$.

If every decomposition belonging to $D^*(k, n)$ is good, then $\varphi \in \Phi(k, n)$, whence $f_\varphi$ is homogeneous by 1.3 (v). It is easy to see that $\varphi$ is homogeneous, whence $f_\varphi$ is symmetrical by 2.1 (iii) and theorem (i) is thus proved.

If every decomposition belonging to $D(k, n)$ is good then $f_\varphi$ is a fortiori homogeneous and symmetrical. Moreover, $f_\varphi$ does not vanish, whence, by 1.3 (vi), $f_\varphi$ is quasi-trivial. Theorem (ii) is thus proved.

Theorems (i) and (ii) reduce the problem of the existence of operations simultaneously homogeneous and symmetrical to purely arithmetic questions. I shall prove the following existence theorem for symmetrical homogeneous operations.

(iii) Symmetrical homogeneous operations of more than one variable in an $n$-element set exist iff $n = 2, 3, 4$ or 6. For these values of $n$ there is a symmetrical homogeneous $k$-ary operation (with $k > 1$) iff $k$ and $n$ are relatively prime and $k > n - 1$.

We denote by $S$ the set of all pairs $(k, n)$ (with $k > 1$) such that there exists a $k$-ary homogeneous and symmetrical operation in an $n$-element set or, in other words (by (i)) that every decomposition belonging to $D^*(k, n)$ is good.
Theorem (iii) will be proved by the aid of the following lemmas:

(a) If \((k, n) \not\in S\), then \(k = n - 1\) or \(k > n\).

It is enough to consider the following bad decomposition: \(k = 1 + 1 + ... + 1\).

(b) If \((k, n) \not\in S\), then \((k, m) \not\in S\) for \(m > n + 1\).

This is because a bad decomposition of \(k\) into \(l\) summands where \(l \leq n\) satisfies at the same time the inequalities \(l \leq m\) and \(l \not\equiv m - 1\).

(c) \((k, 2) \not\in S\).

This is because the decomposition \(k = j + j\) is bad.

\((c^*)\) If \((k, n) \not\in S\) then \((k, 4) \not\in S\).

This follows from (c) and (b).

(d) If \(k\) is odd and \(k \geq 3\), then \((k, 3) \not\in S\) and \((k, 4) \not\in S\).

Clearly, each decomposition of an odd number \(k \geq 3\) into two summands is good. Every decomposition into four summands is also good: indeed, if such a decomposition were bad, then every summand would repeat itself two or four times and hence \(k\) would be even.

(e) \((k, 3) \not\in S\) for \(k > 1\) if and only if \(k\) is not divisible by 3.

If \(k\) is divisible by 3, then \(k = j + j + j\) is a bad decomposition. If \(k\) is not divisible by 3, then in every decomposition into three summands at least one summand appears only once.

(f) \((k, 5) \not\in S\) for \(k = 2, 3, ...

In view of (b), (c) and (e), it is sufficient to prove the existence of bad decompositions for \(k \geq 5\) not divisible by 2 and 3, which follows from the equations:

\[7 = 2 + 2 + 1 + 1 + 1,\]
\[9 + 2j = 3 + 3 + 3 + j + j, \quad j = 1, 2, ...

\((f^*)\) \((k, n) \not\in S\) for \(n \geq 7, k = 2, 3, ...

A consequence of (f) and (b).

(g) If \(k\) and 6 are relatively prime and \(k \geq 5\), then \((k, 6) \not\in S\).

There is no bad decomposition of \(k\) into 3, 4 and 5 summands, in view of (d) and (e). In a decomposition of \(k\) into 6 summands the cases

\[k = u + u + v + w + w + w \quad \text{and} \quad k = u + u + v + v + v + v\]

are impossible since \(k\) is not divisible by 2 and by 3, and hence one summand appears precisely once.

The proof of (iii) has been reduced to an easy verification. By (f) and \((f^*)\) if \((k, n) \in S\), then \(n = 2, 3, 4\) or 6. The case \(n = 2\) is treated in lemmas (c) and (d), the case \(n = 3\) in lemma (e), the case \(n = 4\) in \((c^*)\), and (d). Lemma (g) says that if \(k\) and 6 are relatively prime and \(k \geq 5\) then \((k, 6) \not\in S\). It follows from (c), (e), (b) and (a) that for other \(k\)'s we have \((k, 6) \not\in S\).

Theorem (iii) is thus proved.

Passing now to symmetrical, homogeneous and quasi-trivial operations, we will prove a theorem analogous to (iii):

(iv) Symmetrical homogeneous quasi-trivial operations of more than one variable in an \(n\)-element set exist if \(n = 2, 3\) or 4. For these values of \(n\) there exist \(k\)-ary operations having the above mentioned properties if \(k\) is not divisible by any number \(\leq n\).

Let us denote by \(S\) the set of all pairs \((k, n)\) with \(k > 1\) such that there exists a \(k\)-ary homogeneous symmetrical and quasi-trivial operation in an \(n\)-element set, or, in other words, by (ii) that every decomposition belonging to \(D((k, n)\) is good.

Theorem (iv) follows from the following lemmas \((a_1)-(g_5)\):

(a) \((k, n) \in S\), then \(k > n\).

(b) \((k, n) \in S\), then \((k, m) \not\in S\) for \(m > n\).

(c) \((k, n) \in S\), then \((k, n) \not\in S\) for \(n > 2\).

(d) \((k, n) \in S\), then \((k, 2) \not\in S\).

(e) \((k, n) \in S\), then \((k, 3) \not\in S\) for \(n > 3\).

These lemmas are easy to verify.

(f) \((k, n) \not\in S\) for \(n > 5\), \(k > 2\).

This lemma follows from (f), (b), and the relation: \(S \subseteq S\).

(g) \((k, n) \not\in S\), then \((k, 3) \not\in S\) and \((k, 4) \not\in S\).

In fact, under the above assumptions, in each decomposition of \(k\) into two or three summands the equality of all summands is impossible and consequently one at least of them is different from the others. Each decomposition into four summands is also good. Indeed, each bad decomposition into four summands is of the form \(k = u + u + v + v\), which is impossible for odd \(k\).

We finish this paragraph by answering the following question: let \((k, n)\) be a pair of numbers satisfying all conditions of (iii) or (iv); in what manner can one define explicitly a homogeneous symmetrical [and quasi-trivial] \(k\)-ary operation \(f\) in an arbitrary \(n\)-element set? The answer is easy: it is enough to consider the operation \(f_\varphi\) where \(\varphi\) is a function belonging to \(\mathcal{G}(k, n)\) defined in the proof of 2.2 (i) and (ii).

Let us consider the pair \((3, 4)\) for example. According to this rule, a quintenary operation in a four-element set \((u, v, e, d)\) can be defined by the following scheme:

\[5 = 1 + 1 + 1 + 2, \quad 5 = 1 + 2 + 2, \quad 5 = 1 + 1 + 3, \quad 5 = 2 + 3.\]
In each decomposition belonging to \( D(\delta, \eta) \) the number \( |\psi(\delta)| \) is here in bold type. Thus, for example,
\[
f(a, b, c, d, e) = f(d, a, b, c, e) = \ldots = f(d, c, e, b, a) = d,
\]
\[
f(a, b, c, d, e) = f(b, a, d, c, e) = \ldots = f(b, e, d, a, c) = a.
\]

2.3. **Strongly homogeneous operations.** A \( k \)-ary operation in \( A \) is called **strongly homogeneous** if the fundamental equation
\[
\lambda f(x_1, \ldots, x_n) = f(\lambda x_1, \ldots, \lambda x_n)
\]
is satisfied for every transformation \( \lambda \) of \( A \) into itself \(^{(1)}\). It is easy to verify that

(i) Every trivial operation is strongly homogeneous.

(ii) Operations \( x^* \) and \( \lambda x^* \) are strongly homogeneous.

(iii) Every superposition of strongly homogeneous operations is a strongly homogeneous operation.

I shall prove that

(iv) Every strongly homogeneous operation is quasi-trivial.

In fact, if
\[
f(a_1, \ldots, a_n) = a \in (a_1, \ldots, a_n),
\]
then, for the transformation \( \lambda : A \to A \) defined as
\[
\lambda(a) = a,
\]
we have
\[
f(\lambda(a_1), \ldots, \lambda(a_n)) = f(a_1, \ldots, a_n) = a \neq \lambda(a)
\]
and, consequently, (iv) does not hold.

Passing to the representation theorem for strongly homogeneous operations let us define the relation \( \preceq \) in \( D(h, \eta) \): we write \( \delta \preceq \delta' \) if every set belonging to \( \delta \) is contained in a certain set belonging to \( \delta' \).

It is easy to see that

(v) **Let** \( a_j \in A \) **and** \( b_j \in A \) **(j = 1, 2, \ldots, k).** **We have**
\[
\delta(a_1, \ldots, a_k) \preceq \delta(b_1, \ldots, b_k)
\]
**iff** there is a mapping \( \lambda : A \to A \) **such that** \( \lambda(a_j) = b_j \),

A function \( \varphi \in \Phi(k, n) \) is called **monotone** whenever \( \delta \preceq \delta' \) implies \( \varphi(\delta) \subseteq \varphi(\delta') \)(\(^{(2)}\)).

We shall prove the representation theorem for strongly homogeneous operations:

(vi) A strongly homogeneous \( k \)-ary operation \( f_\varphi \) in \( A \) (where \( \varphi \in \Phi(k, n) \), \( |A| = n \) is strongly homogeneous if \( \varphi \) is monotone and not vanishing.

---

\(^{(1)}\) Operations of this kind are called homogeneous in paper [2a] of Marczewski and Urbanik.
\(^{(2)}\) Notions essentially due to Świerczkowski. See [4], p. 96.
As an example of a k-ary strongly homogeneous, non-trivial operation with \( k \geq 3 \) we may consider
\[
f(a_1, \ldots, a_k) = p_4(a_1, a_1, a_1, a_1).
\]

It is known that the class of all strongly homogeneous operations may be described by the common converse of theorems (i), (ii) and (iii) as follows:

(viii) The class of all strongly homogeneous operations is identical with the smallest class of operations in a two-element set, containing trival operations, the operations \( p_k \) and \( p^* \) and closed with respect to superposition. (In other words, the class of all strongly homogeneous operations can be treated as the class of all algebraic operations in the algebra \( \mathcal{A} = (a, b; p_k, p^*) \).)

The proof of (viii) is complicated (see Marczewski-Urbaniak [3a]).

2.4. Table of representation and existence theorems. We list here the main theorems of sections 1 and 2:

**Representation Theorem.** The correspondence \( \varphi \rightarrow f_\varphi \) is one-to-one between \( \Phi(M, n) \) and the class of all \( k \)-ary homogeneous operations in an \( n \)-element set \( M \) (1.3 (v)). In the above correspondence the following are the corresponding properties of \( f_\varphi \) and \( \varphi \):

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( \varphi )</th>
<th>Number of theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>trivial</td>
<td>not vanishing</td>
<td>1.3 (vii)</td>
</tr>
<tr>
<td>quasi-trivial</td>
<td>homogeneous</td>
<td>1.3 (vii)</td>
</tr>
<tr>
<td>symmetrical</td>
<td>not vanishing and nonmonotone</td>
<td>2.1 (iii)</td>
</tr>
<tr>
<td>strongly homogeneous</td>
<td></td>
<td>2.3 (vii)</td>
</tr>
</tbody>
</table>

**Existence Theorem.** The following table gives all pairs \((k, n)\) with \( k \geq 2 \) such that in an \( n \)-element set there exists a non-trivial \( k \)-ary homogeneous operation having the property formulated in the first column.

<table>
<thead>
<tr>
<th>Property of operation</th>
<th>( k )</th>
<th>( n )</th>
<th>Number of theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetrical</td>
<td>2, 3, 4, 6</td>
<td>( k \geq n - 1 )</td>
<td>2.2 (iii)</td>
</tr>
<tr>
<td>symmetrical and quasi-trivial</td>
<td>2, 3, 4</td>
<td>( k ) and a relatively prime ( k ) not divisible by numbers ( \leq n )</td>
<td>2.3 (iv)</td>
</tr>
<tr>
<td>strongly homogeneous and non-trivial</td>
<td>2</td>
<td>( k \geq 3 )</td>
<td>2.3 (vii)</td>
</tr>
</tbody>
</table>

3. Homogeneous algebras

3.1. Definition and properties. An algebra \( (A; F) \) is called homogeneous [strongly homogeneous] if each operation \( f \in F \) is homogeneous [strongly homogeneous].

Proposition 1.2 (a) implies

(i) All algebraic operations in a homogeneous algebra are homogeneous and 1.2 (i) implies

(ii) If \( G \) is a set of generators of a homogeneous algebra \( A \), then either \( G = A \) or \( A \) is finite and \( A \setminus G \) is a one-element set.

In view of (i) we have

(iii) If \( S \) and \( T \) are two equivalent subsets of a homogeneous algebra \( A \) (i.e. \( |S| = |T| \) and \( |A \setminus S| = |A \setminus T| \)), then \( S \) is a set of generators of \( A \) iff \( T \) is so, and \( S \) is a set of independent elements iff \( T \) is so.

Propositions 1.3 (v), (vi), (viii) and (ix) imply

(iv) Every element of a homogeneous algebra \( A \) forms a set of independent elements, and, if \( |A| \geq 3 \), then every pair of elements is a set of independent elements.

It is easy to see (cf. [3a], p. 200) that

(v) An algebra \( A \) is strongly homogeneous iff the whole set \( A \) forms a set of independent elements.

It is easy to prove, by the aid of (v) and proposition 2.2 (vi) of [1], that

(vi) Every trivial algebra is strongly homogeneous, the algebras (defined by Post) \( \mathcal{B}_k = (a, b; p^*) \), \( \mathcal{B}_k = (a, b; p^*) \) and \( \mathcal{B} = (a, b; p_k, p^*) \) are strongly homogeneous and the algebra \( \mathcal{S} = (a, b, c, d; e) \) of Świerkosz is homogeneous but not strongly homogeneous.

Let us add that \( \mathcal{B}_k \), \( \mathcal{B}_k \) and \( \mathcal{B} \) are the only strongly homogeneous non-trivial algebras (cf. [3] and [3a]).

An algebra \( \mathcal{A} = (A; F) \) is called quasi-trivial if each operation \( f \in F \) is quasi-trivial.

(vii) The following conditions for an algebra \( \mathcal{A} = (A; F) \) are equivalent:

(a) \( \mathcal{A} \) is quasi-trivial.
(b) Each algebraic operation in \( \mathcal{A} \) is quasi-trivial.
(c) Each non-void subset of \( A \) is a subalgebra of \( \mathcal{A} \).
(d) \( A \) is the only set of generators of \( \mathcal{A} \).

If \( |A| = n \) is finite, then these conditions are equivalent to the following

(e) Each algebraic \( n \)-ary operation is quasi-trivial.

The implication (a) \( \Rightarrow \) (b) follows from 1.2 (xii).
The implication (b)->(c) and (c)->(d) are obvious. 
If 
\[ a = f(a_1, ..., a_k) \not\in \{a_1, ..., a_k\} \]
for an operation \( f \in F \), then \( A \setminus \{a\} \) is a set of generators of \( \mathbb{A} \). Hence 
(d)->(a).

The implication (b)->(c) is obvious and its converse (c)->(b) follows from the fundamental properties of algebraic operations (see [1], p. 48, (v) and (vii)).

3.2. Numerical properties of finite algebras. For each algebra \( \mathbb{A} = (A; F) \) with \( 2 \leq |A| < \aleph_0 \), we define the following integral numbers (see [2] and [5]):

- \( a = |A| \)
- \( \gamma = \gamma^{*} \) is the smallest number with the following property: every set \( G \subseteq A \) with \( |G| = \gamma^{*} \) is a set of generators.
- \( \gamma = \gamma^{*} \) is the minimal number of generators,
- \( i = i^{*} \) is the maximal number of independent elements,
- \( \iota = \iota^{*} \) is the greatest number with the following property: every set \( I \subseteq A \) with \( |I| = \iota^{*} \) is a set of independent elements,
- \( \tau = \tau^{*} \) is the greatest number with the following property: every algebraic \( \tau \)-ary operation is trivial (obviously this number is defined only for non-trivial algebras; if there are \( \mathbb{H} \) algebraic constants, we put \( \tau = -1 \) by definition, and if there are no algebraic constants, but there is an algebraic non-trivial unary operation, we put \( \tau = 0 \)).

Let us add that if each element of \( A \) is an algebraic constant in \( A \), then the empty set is treated by definition as a set of generators of \( A \).

We have \( \gamma = \gamma^{*} = 0 \).

I will now list some known relations between the numbers defined above. In particular, it is known (2), (ii) and (vii) that

\[(a) \ a \geq \gamma \Rightarrow \gamma \geq i \geq \iota \geq \tau = \iota^{*} \ or \ \tau = \iota \geq -1.
(b) \ \gamma > i \Rightarrow \gamma > i \Rightarrow \gamma > \tau \Rightarrow \gamma > \tau \ (5), \ (vii).
(c) \ \text{If in a non-trivial algebra } a \neq i, \text{ then } \mathbb{A} \text{ is identical with } \mathbb{R}, \mathbb{Q}, \text{ or } \mathbb{K} \text{ and } a = \gamma = i = \iota = \tau = 2 \ (\text{see } \text{Świerczkowski} [4], \text{p. 94,}\n\text{Theorem 1}, \text{Marczewski-Urbanik} [3] \text{and} [3a], \text{Marczewski} [2], \text{p. 6}).
(d) \ \gamma^{*} = \iota^{*} = 3, \text{ then } a = 4 \text{ and } \mathbb{A} \text{ is identical with } \mathbb{S} \text{ (Świerczkowski} [4], \text{p. 94,}\n\text{Theorem 4}).
(e) \ \iota^{*} = 4, \text{ then } \tau = \iota (\text{Świerczkowski} [3]).

Let us also note an easy consequence of 3.1 (vii):

\[(f) \ a = \gamma \iff \text{the algebra is quasi-trivial.} \]

Suppose now that the algebra \( \mathbb{A} = (A; F) \) is homogeneous and that \( 2 \leq |A| < \aleph_0 \). Thus it easily follows from 3.1 (iii) that

\[(o) \ \gamma = \gamma^{*} \text{ and } i = i^{*}, \]

whence we obtain the following simplification of (a), (d) and (e):

\[(i) \ a \geq \gamma \geq i \geq \iota \geq \tau \text{ and } \tau = i \text{ or } i = \tau = -1.
(ii) \ \text{If } \gamma = i = 3, \text{ then } a = 4 \text{ and } \mathbb{A} \text{ is identical with } \mathbb{S}.
(iii) \ \text{If } i = 4, \text{ then } \tau = 1.

I will now formulate other conclusions from the homogeneity of \( \mathbb{A} \).

Propositions 3.1 (ii), 3.1 (iv) and 1.2 (vii) imply respectively:

\[(iv) \ \text{Either } \gamma = a \text{ or } \gamma = a - 1.
(v) \ \text{If } i \geq 1 \text{ and, if } a \geq 3, \text{ then } i \geq 2.
(vi) \ \text{If } \gamma = a = 2 \text{ then } \tau = 2.
\]

Let us remark incidentally that, in view of 1.2 (viii), proposition (vi) is also true under the hypothesis \( \gamma = a = 3 \).

I shall now prove that

\[(vii) \ \text{If } i = \tau = i \geq 3 \text{ and } a = \gamma + 1.
\]

In view of (v), we have \( i \geq 1 \) and, hence, by definition of \( i \), there exists a set \( I = \{a_1, ..., a_k\} \) of \( i \)-independent elements of \( A \). By hypothesis, we have, on account of (i), \( i > \tau \), whence there exists a \( i \)-ary non-trivial algebraic operation \( f \). Since \( I \) is a set of independent elements, we have

\[ f(a_1, ..., a_k) \not\in \{a_1, ..., a_k\}, \]

whence, by 1.2 (i),

\[ A = \{a_1, ..., a_k, f(a_1, ..., a_k)\} \]

and, consequently, \( I \) is a set of generators of \( A \). We thus have \( \gamma = i \) and \( a = \gamma + 1 \). Proposition (iii) gives the inequality \( i \leq 3 \).

Proposition (vii) is thus proved.

3.3. Examples and conclusions. We can now summarize the discussion of numerical constants \( a, \gamma, i, \text{ and } \tau \) for homogeneous algebras.

(i) For every homogeneous non-trivial algebra the quadruple \( (a, \gamma, i, \tau) \) is one of the following:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \gamma )</th>
<th>( i )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

where \( a > k > 1 \) when \( n > k > 1 \)
Let us consider at first the case \( i = \tau \). In view of 3.2 (iv), we have either \( \gamma = a \) or \( \gamma = a - 1 \). If \( \gamma = a \), we have either \( \gamma = n \), whence we obtain, by 3.2 (c), the first row of the table, or \( \gamma > n \), whence we obtain the second row. It remains to prove in this case that \( k > 1 \).

In fact, if \( a = 2 \), we obtain, by 3.2 (vii), \( r \geq 2 \), and if \( a \geq 3 \) then, by 3.2 (v), \( \gamma \geq 2 \).

If \( a > \gamma \), we obtain the third row of the table. It remains to prove in this case that \( n - 1 > k > 1 \). In fact, the inequality \( n - 1 > k \) follows from 3.2 (b) and the inequality \( k > 1 \) from 3.2 (v) for \( a > 3 \) and from 3.2 (vi) for \( a = 2 \).

Passing to the case \( i \neq \tau \), it is enough to remark that 3.2 (i) and (vii) imply directly the fourth row of the table.

Proposition (i) is thus proved.

Let us consider the following algebras:

\[ \mathcal{L}_a = (N; l_a), \quad \mathfrak{R}_a = (N; r_a), \quad \mathfrak{R}_{a,m} = (N; l_a, r_a), \]

where \( |N| = n \) and \( l_a \) and \( r_a \) are operations defined in 1.1. These algebras are, of course, homogeneous.

I shall prove that

(ii) For every quadruple of numbers in the rows of table (i) there exists a homogeneous algebra for which \((a, \gamma, \tau, i)\) is identical with that quadruple:

<table>
<thead>
<tr>
<th>Algebras</th>
<th>( a )</th>
<th>( \gamma )</th>
<th>( \tau )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{L}_a )</td>
<td>( \mathfrak{R}_a )</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \mathcal{L}_{a+1} ) with ( n &gt; k &gt; 1 )</td>
<td>( n &gt; k &gt; 1 )</td>
<td>( n &gt; k &gt; 1 )</td>
<td>( n &gt; k &gt; 1 )</td>
<td>( n &gt; k &gt; 1 )</td>
</tr>
<tr>
<td>( \mathfrak{R}_a )</td>
<td>( \mathfrak{R}_a )</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \mathfrak{R}_a )</td>
<td>( \mathfrak{R}_a )</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \mathfrak{R}_a )</td>
<td>( \mathfrak{R}_a )</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Row 1. See 3.1 (vi).

Rows 2 and 3. On account of the relation \( n > k > 1 \) (supposed in both rows) we have \( k+1 \geq 3 \) and \( n > k+1 \), whence, by 1.1 (iv), \( \mathcal{L}_{a+1} \) is non-trivial, and consequently, \( \tau < k \). In order to show that \( \tau = \gamma = n \) it remains to prove that every \( k \)-ary algebraic operation in the algebras under consideration is trivial. This follows easily from the fact that if \( f_1, f_2, \ldots \) are trivial \( k \)-ary operations, then

\[ \mathfrak{I}_{a+1}(\mathfrak{I}_{a+1}, \ldots, \mathfrak{I}_{a+1}) = f_{a+1} \]

And, for \( n - 1 > k > 1 \),

\[ r_n(f_1, \ldots, f_{n-1}) = f_1, \]

on account of definitions of \( \mathfrak{I}_{a+1} \) and \( r_n \).

In order to verify that \( i = \tau \) it suffices to remember that \( \tau \geq 1 \) and remark that every \( k > 1 \) elements \( a_1, \ldots, a_{k+1} \) of \( N \) are dependent. In fact,

\[ \mathfrak{I}_{a+1}(a_1, \ldots, a_{k+1}) = a_1 \neq \mathfrak{I}_{a+1}(a_1, \ldots, a_{k+1}) \]

whereas

\[ \mathfrak{I}_{a+1} \neq \mathfrak{I}_{a+1} \]

Since the operation \( \mathfrak{I}_{a+1} \) is quasi-trivial, we have \( \gamma = a = n \) for \( \mathfrak{I}_{a+1} \) and, since \( r_n \) is a non-quasi-trivial \( (n-1) \)-ary operation, we have \( \gamma = a - 1 = n - 1 \) for \( \mathfrak{R}_{a, a+1} \) on account of 3.2 (iv).

Rows 4a, 4b and 4c. All values in these rows are known (see [2], p. 6) and easy to compute.

Propositions (i), (ii) and 3.2 (f) give the final conclusion of our discussion.

(iii) Table (i) consists of all quadruples \((a, \gamma, i, \tau)\) for homogeneous non-trivial algebras: quasi-trivial in rows 1 and 2 and non-quasi-trivial in rows 3 and 4.

3.4. Appendix. In the preceding paragraphs Szczerekowski's theorem 3.2 (e), or, more precisely, the simplified version of this theorem for homogeneous algebras 3.2 (iii), plays an essential part. Since the proof of 3.2 (e) is based on some earlier results, not easy to prove, it is worth noticing that 3.2 (iii) for homogeneous algebras can be obtained in another way, namely with the aid of the properties of symmetrical operations stated in section 2.

Let us begin by the following lemma:

(i) If in a homogeneous algebra \((A; \mathcal{F})\) with \(|A| = 6\) there is an algebraic symmetrical operation \( f \) of five variables, then there exists an algebraic quasi-trivial (and consequently non-symmetrical) non-trivial operation \( g \) of five variables.

Put \( A = (a_1, \ldots, a_6) \). The homogeneity and symmetry of \( f \) imply

\[ f(a_1, \ldots, a_6) = a_6. \]

Putting

\[ g = f(f(a_1, a_2, a_3, a_4), a_5, a_6) \]

we obtain, in view of (i),

\[ g(a_1, a_2, a_3, a_4, a_5) = f(a_1, a_2, a_3, a_4), \]

whence, by (i) and the homogeneity of \( f \),

\[ g(a_1, a_2, a_3, a_4, a_5) = a_6 \]
Homogeneous operations and homogeneous algebras

It follows from the existence of symmetrical homogeneous operations in $A$ that $n = 2, 3, 4$ or 6. The case $n = 6$ is impossible on account of (ii). Since $n = k + 1$, we obtain $k = 1, 2$ or 3, contrary to our hypotheses.

Proposition (iii) is thus proved and hence we obtain a new proof of 3.2 (iii) for homogeneous algebras.

References


INSTITUTE OF MATHEMATICS, WROCLAW UNIVERSITY
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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