On quasi-translations in 3-space

by

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The topological translation (') of the plane was characterized by Kerékjártó [1] and Sperner [2] independently by apparently different conditions. To show that their conditions do not characterize the topological translation in 3-space is the purpose of the paper. Our example is naturally constructed from Fox-Artín's pathological one [3] (§ 3).

The notion of quasi-translation is due to Toranaka [4]. His condition is also apparently different from those of Kerékjártó and Sperner, but we shall prove that their three conditions are equivalent to each other for any n-sphere (n ≥ 1) (§ 1).

In § 2 we shall be concerned with locally polyhedral 2-spheres with one singularity in 3-space; this may be of independent interest, even though it appears in the paper only as a preliminary to § 3.

The paper has an appendix, in which we shall prove that if an auto-homeomorphism of a certain kind that includes quasi-translates operates on a manifold, then that manifold must be an n-sphere.

The author of the paper acknowledges with great pleasure his gratitude to Professors R. H. Fox and D. Montgomery for their valuable advices.

§ 1. Let h be an auto-homeomorphism of a compact metric space X. Then h is said to have equi-continuous powers at x ∈ X, if for each ε > 0 there exists δ > 0 such that whenever d(x, y) < δ, d(\(h^n(x), h^n(y)\)) < ε for every integer m. First we prove the following:

* The paper was completed while the author was at the Institute for Advanced Study, being supported by the National Science Foundation of U.S.A. It was announced in the University of Georgia Institute 1961. See Topology of 3-manifolds and related topics, Proe. The Univ. of Georgia Institute, 1961, pp. 229-238, Prentice Hall, Englewood Cliffs, N. J., 1962.

(1) Let g and h be two auto-homeomorphisms of a topological space X. Then g and h are said to be topologically equivalent, if there exists an auto-homeomorphism f of X such that g = f^{-1}h. A topological translation means a transformation that is topologically equivalent to the ordinary translation.
LEMMA 1. Let \( X \) be a compact metric space such that \( X - p \) is connected, where \( p \) is a point of \( X \). Then an auto-homeomorphism \( h \) of \( X \) has equi-continuous powers at every \( x \in X - p \) and does not have them at \( p \), if and only if \( \lim_{m \to \infty} h^m(p) = p \) for every compact subset \( C \) of \( X - p \).

Proof. Suppose that \( h \) has equi-continuous powers at every \( x \in X - p \) and does not have them at \( p \). Then it has been proved that for each \( x \in X \) the sequence \( (h^n(x)) \) converges to \( p \) when \( m \to \infty \) or \( m \to -\infty \) (see Homma and Kinoshita [9] Theorem 4, p. 36). Then \( h \) is almost periodic at \( x \in X - p \) (see [5] Lemmas 1 and 1', p. 30). From this it follows that \( \lim_{m \to \infty} h^m(x) = p \) (see [5] Theorem 1, p. 32).

Further, 
\[
\lim_{m \to \infty} h^m(x) \neq \emptyset \quad \text{and} \quad \lim_{m \to -\infty} h^m(x) \neq \emptyset .
\]

Therefore, 
\[
\lim_{m \to \infty} h^m(x) = p \quad \text{and} \quad \lim_{m \to -\infty} h^m(x) = p .
\]

Now let \( C \) be a compact subset of \( X - p \). Let \( e > 0 \) be given and let \( U(p) = \{(y, d(p, y) < e)\} \). For each \( x \in C \) there exists a natural number \( M_x \) such that \( d(h^n(x), y) < e/2 \) whenever \( m > M_x \). From the assumption that \( h \) has equi-continuous powers at \( x \) it follows that there exists \( \delta_x > 0 \) such that \( d(h^n(x), h^m(y)) < e/2 \) for every \( m \), whenever \( d(x, y) < \delta_x \). Therefore, \( d(h^n(y), y) < e \) whenever \( |m| > M_x \) and \( d(h^n(x), y) < e \). The family of open sets \( U(x) \), where \( U(x) = \{(y, d(x, y) < \delta_x)\} \) covers \( C \), since \( C \) is open, we can choose a finite number of points \( x_1, \ldots, x_n \) such that \( U(x_i) \) covers \( C \). Then, if we put \( M = \max(M_{x_1}, \ldots, M_{x_n}) \), for each \( x \in C \), \( h^m(x) \in U(p) \) whenever \( m > M \). Thus \( \lim_{m \to \infty} h^m(C) = p \).

Conversely assume that \( \lim_{m \to \infty} h^m(C) = p \) for every compact subset \( C \) of \( X - p \). Now we shall prove that \( h \) has equi-continuous powers at every \( x \in X - p \). Let \( e > 0 \) be given and let \( U(p) = \{(y, d(p, y) < e/2)\} \). Let \( U(x) \) be a sufficiently small neighborhood of \( x \). Then \( \lim_{m \to \infty} h^m(U(x)) = p \). Therefore there exists a natural number \( M \) such that \( h^m(U(x)) \subseteq U(p) \) whenever \( m > M \). There exists \( \delta > 0 \) such that if \( d(x, y) < \delta \), then \( y \in U(x) \) and \( d(h^n(x), h^n(y)) < e \) for every \( m < M \). Then, if \( d(x, y) < \delta \), \( d(h^n(x), h^n(y)) < e \) for every integer \( m \). Thus our proof is complete.

Let \( X \) be a compact metric space and \( p \) a point of \( X \). Then an auto-homeomorphism \( h \) of \( X \) is said to satisfy Sperner's condition, if for each compact subset \( C \) of \( X - p \) there exists a natural number \( M \) such that \( C \cap h^m(C) = \emptyset \) whenever \( m > M \).

LEMMA 2. Let \( X \) be a compact metric space and \( p \) a point of \( X \). Then an auto-homeomorphism \( h \) of \( X \) satisfies Sperner's condition if and only if \( \lim_{m \to \infty} h^m(C) = p \) for every compact subset \( C \) of \( X - p \).

Proof. Suppose that \( h \) satisfies Sperner's condition. If \( \lim_{m \to \infty} h^m(u) \neq p \) for some compact subset \( C \) of \( X - p \), then \( h \) does not satisfy Sperner's condition for the compact subset \( C = C \cup U(y) \), where \( U(y) \) is a sufficiently small neighborhood of \( y \). Therefore \( \lim_{m \to \infty} h^m(C) = p \) for every compact subset \( C \) of \( X - p \). The converse is clear.

Topological translation in \( S^n \) was characterized by Kerékkáro [1] as an orientation preserving auto-homeomorphism of \( S^n \) which has equi-continuous powers at every \( x \in S^n - p \) but not at \( p \), where \( p \) is a point of \( S^n \). An orientation preserving auto-homeomorphism of \( S^n \) which satisfies Sperner's condition is also a topological translation, as was proved by Sperner [3]. Terasaka [4] defined a quasi-translation of \( S^n \) as an orientation preserving auto-homeomorphism of \( S^n \) such that for each compact subset \( C \) of \( S^n - p \), \( \lim_{m \to \infty} h^m(C) = p \). Lemmas 1 and 2 imply that Kerékkáro's and Sperner's condition for \( S^n \) and Terasaka's condition for quasi-translations are equivalent to each other for any \( S^n \) (n \( \geq \) 1).

§ 2. A subset \( H \) of a polyhedron \( K \) with a triangulation will be said to be locally polyhedral at \( x \in K \), if there exists a neighborhood \( U \) of \( x \) such that \( U \cap H \) is polyhedral by a subdivision of \( K \).

THEOREM 1. Let \( K \) be a 3-cell with a triangulation. Let \( S^n \) be a 2-sphere in \( K \) such that \( S^n \cap (\partial K) = p \) and that \( S^n \) is locally polyhedral at every point \( x \in S^n - p \). Let \( C \) be the complementary domain of \( S^n \) in \( K \) such that the boundary of \( C \) consists of only \( S^n \). Then \( C \) is a 3-cell.

Proof. Let \( q \) be a point of \( S^n \) which is different from \( p \). First we shall prove the following proposition:

(*) For each \( e > 0 \) there exists a polygonal simple closed curve \( \gamma \) on \( S^n - p \) such that \( e \) does not bound a disk on \( S^n - p \), \( e \) bounds a polyhedral disk \( D \) in \( C \), where \( D \cap S^n = \emptyset \), and that \( D \) is contained in the \( e \)-neighborhood of \( p \).

To prove (*) we may suppose that \( e < d(p, q) \). Let \( A \) be a disk on \( S^n \) such that \( A \) is contained in the \( e \)-neighborhood of \( p \) and that \( A \) contains \( p \) as an inner point. There exists a polyhedral disk \( B \) in \( K \) satisfying the following conditions:

(i) \( B \) is contained in the \( e \)-neighborhood of \( p \),
(ii) \( \partial B \subset \partial K \).
(iii) $\text{Int} B \cap \text{Int} K$.
(iv) $B$ separates $p$ and $q$ in $K$.
(v) $B \cap S^2 = B \cap A$.
(vi) $B \cap A$ consists of at most a finite number of polygonal simple closed curves.

Let $d$ be one of the innermost simple closed curve of $B \cap A$ in the disk $A$. Further, suppose that $d$ bounds a disk $F$ on $A - p$, where $\text{Int} B \cap = \emptyset$. Since $d$ bounds disk $F$ in $B$, we can modify $B$ to $B_1 = (B - F) \cup E$. We may suppose that $B_1$ is small enough so that $B_1$ satisfies the conditions (i)-(v) for $B$ and further, the condition $B_1 \cap A \subset (B \cap A) - d$. Repeating this process, as far as possible, finally we have a disk $B_2$ which satisfies the conditions (i)-(v) for $B$ and in which $B_2 \cap A$ consists of at most a finite number of polygonal simple closed curves that does not bound disks in $A - p$, respectively. The case $B_2 \cap A = \emptyset$ does not occur, since $B_2$ separates $p$ and $q$ in $K$. Now let $e$ be one of the innermost simple closed curve of $B_2 \cap A$ in $B_2$. Then $e$ bounds a disk $D$ in $B_2$, where $\text{Int} D \cap S^2 = \emptyset$. $D$ is contained in the $e$-neighborhood of $p$. It is clear that $e$ does not bound a disk in $S^2 - (p \cup q)$ and that $D$ is contained in $C$. Thus the proof of the proposition (c) is complete.

Now it is easy to prove our theorem. For each $1/n (n = 1, 2, \ldots)$ construct a disk $D_n$ as shown in the proposition (c). We may suppose that $D_m \cap D_n = \emptyset$, if $m \neq n$. Put $\alpha_m = \text{bdry} D_n$. Then $\alpha_1$ bounds a disk $M$ on $S^2$, where $M$ contains $q$, $\alpha_0$ and $\alpha_{n+1}$ bound an annulus $\alpha_n$ on $S^2$. Then

$$M \cup D_1 \cup D_2 \cup D_3 \cup \cdots \cup D_n \cup \alpha_n \cup \alpha_{n+1} \cup \cdots$$

are polygonal 2-spheres which bound closed 3-cells $I_n \subset C$ ($n = 1, 2, \ldots$).

Further, $I_n \cup I_{n+1} = D_{n+1}$ and $\lim_{n \to \infty} I_n = p$. Since $\lim_{n \to \infty} I_n = \emptyset$, it is easy to see that $C$ is a 3-cell. Thus the proof is complete.

A complementary domain $C$ of $S^2$ in $S^2$ will be called trivial, if $C$ is a 3-cell. Harrold and Moise [6] proved that if $S^2$ in $S^2$ is locally polygonal with one singularity, then at least one of the two complementary domains of $S^2$ must be trivial. This theorem will not be used explicitly below, but will be convenient for the understanding of our discussion.

Theorem 2. Let $S^2_1$ and $S^2_2$ be two locally polygonal spheres in $S^2$ with one singularity at $p \in S^2_1 \cap S^2_2$ such that

(i) $S^2_1 \cap S^2_2 = p$,
(ii) $S^2_1 - p$ is contained in the trivial complementary domain of $S^2_2$.

If $S^2_1$ is tamely imbedded in $S^2$, then $S^2_2$ is tamely imbedded in $S^2$.

Proof. Let $C_1$ be the complementary domain of $S^2_1$ which contains a point of $S^2_1 - p$ and let $C_2$ be that of $S^2_2$ which does not contain any point of $S^2_2 - p$. Clearly $C_1 \cap C_2$. Since $C_1$ is a 3-cell, from Theorem 1 (*) it follows that $C_2$ is a closed 3-cell. Since another complementary domain of $S^2_1$ is trivial by our assumption, $S^2_1$ is tamely imbedded in $S^2$.

Theorem 3. Let $S^2_1$ be a locally polygonal 2-sphere in $S^2$ with one singularity at $p \in S^2_1$ and let $S^2_2$ be a locally polygonal 2-sphere in $S^2$ such that $S^2_1 \cap S^2_2 = p$. Suppose that $C$ is the complementary domain of $S^2_1$ such that $S^2_1 - p$ is contained in $C$. Further, let $f$ be a homeomorphism of the 3-cell $C$ into another $S^2_2$ such that $f(S^2_1)$ and $f(S^2_2)$ are two locally polygonal 2-spheres with one singularity at $f(p)$, respectively. If $S^2_1$ is not tamely imbedded in $S^2$, then $f(S^2_1)$ is also not tamely imbedded in $S^2_2$.

Proof. $f(S^2_1)$ is contained in the trivial complementary domain of $f(S^2_2)$. Therefore, if $f(S^2_1)$ is not tamely imbedded in $S^2_2$, then by Theorem 2 $f(S^2_1)$ is not tamely imbedded in $S^2$. If $f(S^2_1)$ is tamely imbedded, then $f$ can be extended to a homeomorphism of $S^2_1$ onto $S^2_2$. Since $S^2_1$ is not tamely imbedded, $f(S^2_1)$ is not tamely imbedded.

Theorem 2 can be generalized to the following:

Theorem 4. Let $S^2_1$ be a tamely imbedded 2-sphere in $S^2$ and $S^2_2$ a locally polygonal 2-sphere in $S^2$ with one singularity at $p$ such that

(i) $S^2_1 \cap S^2_2 = p$,
(ii) $S^2_2$ is contained in the trivial complementary domain of $S^2_1$.

Then $S^2_2$ is tamely imbedded in $S^2$.

Proof. $(S^2_1 \cup S^2_2) - p$ is locally tame in $S^2 - p$. Therefore there exists a triangulation of $S^2 - p$, in which $(S^2_1 \cup S^2_2) - p$ is an open polyhedron (cf. Bing [7]). This means that both $S^2_1$ and $S^2_2$ are two locally polygonal 2-spheres in $S^2$ with one singularity at $p$, respectively. Since $S^2_1$ is tamely imbedded in $S^2$, by Theorem 2, $S^2_2$ is tamely imbedded in $S^2$. Similarly we have the following:

Theorem 5. Let $A$ be an arc in $S^2$ and $S^2$ a locally polygonal 2-sphere with one singularity at $p$ such that

(i) $A \cap S^2 = p$,
(ii) $A - p$ is contained in the trivial complementary domain of $S^2$.

Then $S^2$ is tamely imbedded in $S^2$.

§ 3. Now we construct a quasi-translation $h$ in $S^2$ which is not equivalent to a topological translation.

Put

$$I = \{(x, y, s) | -1 \leq x \leq 1, y^s + s^2 \leq 1\}.$$  

(*) Theorem 1 holds whenever the triangulation of $X - p$ is locally finite.
Let $A$ and $B$ be two cubes in $I$ such that
\[
A = \{(x, y, z) \mid -1 \leq x \leq 1, \ y^3 + z^3 \leq (\frac{1}{2})^3\},
\]
\[
B = \{(x, y, z) \mid 0 \leq x \leq 1, \ y^3 + (x \pm 1)^3 \leq (\frac{1}{2})^3\}
\]
\[
\cup \{(x, y, z) \mid (x-a)^3 + y^3 + (z-b)^3 \leq (\frac{1}{2})^3, \ a^3 + b^3 = (\frac{1}{2})^3, \ -\frac{1}{2} \leq a \leq 0\}.
\]
Put $C = I - (A \cup B)$.

Let $T$ be the dilation of 3-space defined by
\[
T(x, y, z) = (7x, 7y, 7z).
\]
Let us consider $T(B)$. In $T(B)$ we construct three cubes $B_t$ ($t = 1, 2, 3$) such that
\[
B_1 = \{(x, y, z) \mid 0 \leq x \leq 7, \ y^3 + (x \pm 4)^3 \leq (\frac{1}{2})^3\}
\]
\[
\cup \{(x, y, z) \mid (x-a)^3 + y^3 + (z-b)^3 \leq (\frac{1}{2})^3, \ a^3 + b^3 = (4 + \frac{1}{2})^3, \ -4 \leq a \leq 0\},
\]
\[
B_2 = \{(x, y, z) \mid 0 \leq x \leq 7, \ y^3 + (x \pm 7)^3 \leq (\frac{1}{2})^3\}
\]
\[
\cup \{(x, y, z) \mid (x-a)^3 + y^3 + (z-b)^3 \leq (\frac{1}{2})^3, \ a^3 + b^3 = (7 + \frac{1}{2})^3, \ -7 \leq a \leq 0\},
\]
\[
B_3 = \{(x, y, z) \mid 0 \leq x \leq 7, \ y^3 + (x \pm 10)^3 \leq (\frac{1}{2})^3\}
\]
\[
\cup \{(x, y, z) \mid (x-a)^3 + y^3 + (z-b)^3 \leq (\frac{1}{2})^3, \ a^3 + b^3 = (10 + \frac{1}{2})^3, \ -10 \leq a \leq 0\}.
\]
Put
\[
C_t = T(C) \cup \bigcup_{i=1}^{3} (T(B_t) - \frac{1}{2}B_t).
\]

Figure 1

We can repeat this process as shown in Fig. 1 and define $B_{t_{1,0}} \cup C_{n}$, where
\[
C_n = T(C_{n-1}) \cup \bigcup_{i=1}^{3} (T(B_{t_{1,0}}) - \frac{1}{2}B_{t_{1,0}})
\]
for $n \geq 2$.

Now let $J$ be the cube
\[
\{(x, y, z) \mid 1 \leq x \leq 7, \ y^3 + z^3 = 1\}.
\]

In $J$ we construct three cubes $D_t$ ($t = 1, 2, 3$) as shown in Fig. 2. Suppose that
\[
bdry J \cap D_1 = \{(x, y, z) \mid x = 1, \ y^3 + (x - \frac{1}{2})^3 \leq (\frac{1}{2})^3\}
\]
\[
\cup \{(x, y, z) \mid x = 7, \ y^3 + z^3 \leq (\frac{1}{2})^3\},
\]
\[
bdry J \cap D_2 = \{(x, y, z) \mid x = 1, \ y^3 + z^3 \leq (\frac{1}{2})^3\} \text{ or } y^3 + (x - \frac{1}{2})^3 \leq (\frac{1}{2})^3,
\]
\[
bdry J \cap D_3 = \{(x, y, z) \mid x = 7, \ y^3 + z^3 \leq (\frac{1}{2})^3\}.
\]

Put
\[
E_0 = J - (D_1 \cup D_2 \cup D_3).
\]

Figure 2

In each $T(D_t)$ construct three cubes $D_{tj}$ ($j = 1, 2, 3$) as shown in Fig. 3. We may suppose that $B_1, B_2, B_3, D_1, D_2$ and $D_3$ are connected smoothly to $D_{t_{1,0}}, D_{t_{1,1}}, D_{t_{1,2}}$ and $D_{t_{1,0}}$ in a suitable choice as shown in Fig. 3. Put
\[
E_n = T(E_{n-1}) \cup \bigcup_{j=1}^{3} (T(D_{t_{1,0}}) - \frac{1}{2}B_{t_{1,0}}).
\]

Figure 3

We can repeat this process and define $D_{t_{1,0}}$ and $E_n$, where
\[
E_n = T(E_{n-1}) \cup \bigcup_{j=1}^{3} (T(D_{t_{1,0}}) - \frac{1}{2}B_{t_{1,0}}).
\]
Put
\[ F = \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} E_n \cup p, \]
\[ G = A \cup B \cup \bigcup_{n>0} B_{2n-1} \cup \bigcup_{n>0} D_{2n} \cup p, \]
where \( p \) is the point at infinity. Clearly \( G \) is a 3-cell and \( \text{Int} F \) and \( \text{Int} G \) are two complementary domains of the 2-sphere \( H = F \cap G \).

Now let us define a quasi-translation \( h \), which will be seen to be inequivalent to the topological translation. First put
\[ h(p) = p. \]
For each \( q \in T(F) \) we define
\[ h(q) = T^{-1}(q), \]
which is a homeomorphism of \( T(F) \) onto \( F \). \( T(G) \) is a 3-cell and \( G \) is a 3-cell in \( T(G) \). It is easy to see that there is a homeomorphism \( g_0 \) of \( T(G) \) onto \( G \) such that \( g_0(p) = p \), \( g_0(q) = T^{-1}(q) \), whenever \( q \in T(H) \) and that \( s^2 + s^2 + s^2 \geq 1 \), whenever \( (x, y, z) \in g_0(G) \). For each point \( q \in T(G) - G \) we define
\[ h(q) = g_0(q). \]
Since \( G \) is a 3-cell and \( g_0(G) \) is a 3-cell in \( G \), there is a homeomorphism \( g_0 \) of \( G \) onto \( g_0(G) \) such that \( g_0(p) = p \), \( g_0(q) = g_0(q) \) whenever \( q \in H \) and \( s^2 + s^2 + s^2 \geq 2 \), whenever \( (x, y, z) \in g_0(G) \). Put
\[ h(q) = g_0(q) \]
for each \( q \in G - g_0(G) \). Similarly, since \( g_0(G) \) is a 3-cell and \( g_0 g_0(G) \) is a 3-cell in \( g_0(G) \), there exists a homeomorphism \( g_0 \) of \( g_0(G) \) onto \( g_0 g_0(G) \) such that \( g_0(q) = g_0(q) \) whenever \( q \in g_0(H) \) and \( s^2 + s^2 + s^2 \geq 2 \), whenever \( (x, y, z) \in g_0 g_0(G) \). Put
\[ h(q) = g_0(q) \]
for each \( q \in g_0(G) - g_0(g_0(G)) \). We can repeat this process infinitely many times. Then \( h \) is defined as an orientation preserving auto-homeomorphism of \( S^2 \). It is easy to see that \( h \) has equi-continuous powers at \( q \in S^2 - p \) but not at \( p \). Therefore \( h \) is a quasi-translation of \( S^2 \).

It remains to prove that \( h \) is not topologically equivalent to a translation of \( S^2 \).

First we remark that the 2-sphere \( H \) is not tamely imbedded in \( S^2 \). We may suppose that, for each integer \( m \), \( h^m(H) \) is locally polyhedral with one singularity \( p \). One of the simple examples of locally polyhedral 2-spheres \( K \) with one singularity, where \( K \) is not tamely imbedded in 3-space, is reproduced below from Fox-Artin [3].

Let \( k \) be the 2-sphere in question and \( L \) a polyhedral 2-sphere as shown in Fig. 4. Let \( M \) be the closure of the complementary domain of \( L \) which contains \( k - p \). Therefore \( M \) is a 3-cell. It is easy to see that there is a homeomorphism \( f \) of \( M \) onto \( T(G) \) such that \( f(k) = H \). Then, by Theorem 3, \( H \) is not tamely imbedded in \( S^2 \).

![Fig 4](image)

Now let \( k \) be an arbitrary quasi-translation of \( S^2 \). An arc \( \overline{pq} \), where \( q \in S^2 - p \), will be called the positive translation arc of \( h \), if \( h(\overline{pq}) \subset \overline{pq} \).

We shall prove that if \( \overline{pq} \) is a positive translation arc of \( h \), constructed as above, then \( \overline{pq} \) is not tamely imbedded in \( S^2 \). This proves that \( h \) is not topologically equivalent to a translation of \( S^2 \), because if \( h \) were a topological translation, then for each \( q \in S^2 - p \) there would exist a positive translation arc \( \overline{pq} \) tamely imbedded in \( S^2 \).

Let \( q \in S^2 - p \) and \( \overline{pq} \) be a positive translation arc of \( h \). Let \( \overline{h(p)} \) be the subarc of \( \overline{pq} \). It is easy to see that there exists a positive integer \( n \) such that \( \text{Int} h^{-n}(p) \supset \overline{h(p)} \). Then \( \text{Int} h^{-n}(p) \) contains the subarc \( h^n(q) \overline{h^{-n}(p)} \) of \( \overline{pq} \) for each natural number \( n \). Since \( h^n(q) \supset h^{n+1}(q) \) for each integer \( m \), \( \text{Int} h^{-n}(p) \supset \overline{h^{-n}(p)} - p \). Since \( H \) is not tamely imbedded in \( S^2 \), so is \( h^{-n}(H) \), \( h^{-n}(H) \) is the closure of one of the complementary domains of \( h^{-n}(H) \). Further \( h^{-n}(H) \) is a 3-cell and \( h^{-n}(H) \) is locally polyhedral with one singularity \( p \). Then, by Theorem 5, \( \overline{pq} \) is not tamely imbedded in \( S^2 \). Thus our proof is complete.
Appendix. Let $X$ be a compact metric space and $A$ a finite subset of $X$ such that $X-A$ is connected. If $h$ is an auto-homeomorphism of $X$ which has equi-continuous powers at every $x \in X-A$ but not at any $x \in A$, then the number of points of $A$ is at most two (see [5]). Therefore, there are three types of such auto-homeomorphisms.

Let $X$ be compact and $h$ an auto-homeomorphism of $X$ which has equi-continuous powers at every point. Then it seems to be well known that there exists a compact transformation group $G$ of $X$ which contains $h$ as an element and in which $(h^n)$ is dense, where $n$ is an integer. Thus the study of this case will be reduced to that of compact transformation groups.

The purpose of this appendix is to prove that if $X$ is a compact connected closed topological manifold, then other types of these auto-homeomorphisms can operate only on an $n$-sphere.

Theorem 6. Let $X$ be a compact connected closed topological manifold and $h$ an auto-homeomorphism of $X$ which has equi-continuous powers at every $x \in X-p$ but not at $p$. Then $X$ is an $n$-sphere.

Proof. Let $U(p)$ be a Euclidean neighborhood of $p$. Then $X-U(p)$ is compact and contained in $X-p$. Therefore, by Lemma 1, there exists a natural number $N$ such that $h^n(X-U(p)) \subseteq U(p)$. Then $h^{-N}(U(p)) \subseteq X-U(p)$. Therefore $X$ is a sum of two open cells. Thus $X$ is an $n$-sphere by the generalized Schoenflies theorem (see Brown [8]).

Theorem 7. Let $X$ be a compact connected closed topological manifold and $h$ an auto-homeomorphism of $X$ which has equi-continuous powers at every $x \in X-(p \cup q)$ and does not have it at both $p$ and $q$. Then $X$ is an $n$-sphere.

Proof. If $X$ is 1-dimensional, then $X$ is a 1-sphere. Therefore, we may suppose that the dimension of $X$ is equal to or greater than two. Then $X-(p \cup q)$ is connected. Then one of $p$ or $q$, say $p$, is the attractive point of $h$, i.e., for each $x \in X-q$ $(h^n(x))$ converges to $p$ when $n \to \infty$ and the other $q$ is repulsive, i.e., for each $x \in X-p$ $(h^n(x))$ converges to $q$ when $n \to -\infty$ (see [5]). Further, it is known that for each compact subset $C$ of $X-q$ $\lim_{n \to \infty} h^n(C) = p$ and for each compact subset $C$ of $X-p$ $\lim_{n \to -\infty} h^n(C) = q$ (see Homma and Kinoshita [9]).

Let $U(p)$ and $V(q)$ be mutually disjoint Euclidean neighborhoods of $p$ and $q$, respectively. Then there exists a natural number $N$ such that $h^n(X-V(q)) \subseteq U(p)$. Therefore $X-h^{-N}(V(q)) \subseteq U(p)$. This means that $X$ is the sum of two open cells $U(p)$ and $h^{-N}(V(q))$. Thus, again by the generalized Schoenflies theorem, $X$ is an $n$-sphere.

Remark. In the case of Theorem 7 it was known that if the dimension of $X$ is equal to 2 or 3, then $h$ is a topological dilation of $S^n$ or $S^3$, respectively (see [10] and [11]).