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On quasi-translations in 3-space

by

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The topological translation (1) of the plane was characterized by Kerékjártó [1] and Sperner [2] independently by apparently different conditions. To show that their conditions do not characterize the topological translation in 3-space is the purpose of the paper. Our example is naturally constructed from Fox-Artin's pathological one [3] (§ 3).

The notion of quasi-translation is due to Terasaka [4]. His condition is also apparently different from those of Kérekjártó and Sperner, but we shall prove that their three conditions are equivalent to each other for any n-sphere ($n \geqslant 1$) (§ 1).

In § 2 we shall be concerned with locally polyhedral 2-spheres with one singularity in 3-sphere; this may be of independent interest, even though it appears in the paper only as a preliminary to § 3.

The paper has an appendix, in which we shall prove that if an auto-homeomorphism of a certain kind that includes quasi-translations operates on a manifold, then that manifold must be an n-sphere.

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§ 1. Let h be an auto-homeomorphism of a compact metric space X. Then h is said to have *equi-continuous powers* at $x \in X$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, $d\big(h^m(x), h^m(y)\big) < \varepsilon$ for every integer m. First we prove the following:

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(1) Let g and h be two auto-homeomorphisms of a topological space X. Then g and h are said to be *topologically equivalent*, if there exists an auto-homeomorphism f of X such that $g = fhf^{-1}$. A *topological translation* means a transformation that is topologically equivalent to the ordinary translation.

LEMMA 1. *Let X be a compact metric space such that $X-p$ is connected, where p is a point of X. Then an auto-homeomorphism h of X has equi-continuous powers at every $x \in X-p$ and does not have them at p, if and only if $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$ for every compact subset C of $X-p$.*

Proof. Suppose that h has equi-continuous powers at every $x \in X-p$ and does not have them at p. Clearly $h(p) = p$. Then it has been proved that for each $x \in X$ the sequence $\{h^m(x)\}$ converges to p when $m \to \infty$ or $m \to -\infty$ (see Homma and Kinoshita [5] (²) Theorem 4, p. 36). Then h is not almost periodic at $x \in X-p$ (see [5] Lemmas 1 and 1′, p. 30). From this it follows that $\underline{\lim\limits_{m\to\pm\infty}}\, h^m(x) = p$ (see [5] Theorem 1, p. 32). Further,

$$\overline{\lim_{m\to\infty}}\, h^m(x) \neq \emptyset \quad \text{and} \quad \overline{\lim_{m\to-\infty}}\, h^m(x) \neq \emptyset .$$

Therefore,

$$\lim_{m\to\infty} h^m(x) = p \quad \text{and} \quad \lim_{m\to-\infty} h^m(x) = p .$$

Now let C be a compact subset of $X-p$. Let $\varepsilon > 0$ be given and let $U(p) = \{y \mid d(p, y) < \varepsilon\}$. For each $x \in C$ there exists a natural number M_x such that $d(h^m(x), p) < \varepsilon/2$ whenever $|m| > M_x$. From the assumption that h has equi-continuous powers at x it follows that there exists $\delta_x > 0$ such that $d(h^m(x), h^m(y)) < \varepsilon/2$ for every m, whenever $d(x, y) < \delta_x$. Therefore, $d(h^m(y), p) < \varepsilon$ whenever $|m| > M_x$ and $d(x, y) < \delta_x$. The family of open sets $U_\delta(x)$, where $U_\delta(x) = \{y \mid d(x, y) < \delta_x\}$ covers C. Since C is compact, we can choose a finite number of points $x_1, \ldots, x_k$ such that $U_\delta(x_i)$ $(i = 1, \ldots, k)$ covers C. Then, if we put $M = \mathrm{Max}(M_{x_1}, \ldots, M_{x_k})$, for each $x \in C$ $h^m(x) \in U(p)$ whenever $|m| > M$. Thus $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$.

Conversely assume that $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$ for every compact subset C of $X-p$. Now we shall prove that h has equi-continuous powers at every $x \in X-p$. Let $\varepsilon > 0$ be given and let $U(p) = \{y \mid d(p, y) < \varepsilon/2\}$. Further let $U(x)$ be a sufficiently small neighborhood of x. Then $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(\overline{U(x)}) = p$. Therefore there exists a natural number M such that $h^m(\overline{U(x)}) \subset U(p)$ whenever $|m| > M$. There exists $\delta > 0$ such that if $d(x, y) < \delta$, then $y \in U(x)$ and $d(h^m(x), h^m(y)) < \varepsilon$ for every $|m| \leqslant M$. Then, if $d(x, y) < \delta$, $d(h^m(x), h^m(y)) < \varepsilon$ for every integer m. Thus our proof is complete.

Let X be a compact metric space and p a point of X. Then an auto-homeomorphism h of X is said to satisfy *Sperner's condition*, if for each compact subset C of $X-p$ there exists a natural number M such that $C \cap h^m(C) = \emptyset$ whenever $|m| > M$.

(²) Instead of the terminology "*equi-continuous powers*", "*regular*" was used in our papers [5] and [9].

LEMMA 2. *Let X be a compact metric space and p a point of X. Then an auto-homeomorphism h of X satisfies Sperner's condition if and only if $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$ for every compact subset C of $X-p$.*

Proof. Suppose that h satisfies Sperner's condition. If $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) \ni q \neq p$ for some compact subset C of $X-p$, then h does not satisfy Sperner's condition for the compact subset $D = C \cup \overline{U(q)}$, where $U(q)$ is a sufficiently small neighborhood of q. Therefore $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$ for every compact subset C of $X-p$. The converse is clear.

Topological translation in S^2 was characterized by Kerékjártó [1] as an orientation preserving auto-homeomorphism of S^2 which has equi-continuous powers at every $x \in S^2-p$ but not at p, where p is a point of S^2. An orientation preserving auto-homeomorphism of S^2 which satisfies Sperner's condition is also a topological translation, as was proved by Sperner [2]. Terasaka [4] defined a *quasi-translation* of S^n as an orientation preserving auto-homeomorphism of S^n such that for each compact subset C of S^n-p $\overline{\lim\limits_{m\to\pm\infty}}\, h^m(C) = p$. Lemmas 1 and 2 imply that

Kerékjártó's and Sperner's condition for S^n and Tarasaka's condition for quasi-translations are equivalent to each other for any S^n $(n \geqslant 1)$.

§ 2. A subset H of a polyhedron K with a triangulation will be said to be *locally polyhedral* at $x \in H$, if there exists a neighborhood U of x such that $U \cap H$ is polyhedral by a subdivision of K.

THEOREM 1. *Let K be a 3-cell with a triangulation. Let S^2 be a 2-sphere in K such that $S^2 \cap (\mathrm{bdry}\, K) = p$ and that S^2 is locally polyhedral at every point $x \in S^2-p$. Let C be the complementary domain of S^2 in K such that the boundary of C consists of only S^2. Then $\overline{C}$ is a 3-cell.*

Proof. Let q be a point of S^2 which is different from p. First we shall prove the following proposition:

(∗) For each $\varepsilon > 0$ there exists a polygonal simple closed curve c on $S^2-(p \cup q)$ such that c does not bound a disk on $S^2-(p \cup q)$, c bounds a polyhedral disk D in $\overline{C}$, where $D \cap S^2 = c$, and that D is contained in the ε-neighborhood of p.

To prove (∗): We may suppose that $\varepsilon < d(p, q)$. Let A be a disk on S^2 such that A is contained in the ε-neighborhood of p and that A contains p as an inner point. There exists a polyhedral disk B in K satisfying the following conditions:

(i) B is contained in the ε-neighborhood of p,

(ii) $\mathrm{bdry}\, B \subset \mathrm{bdry}\, K$,

(iii) $\operatorname{Int} B \subset \operatorname{Int} K$,

(iv) B separates p and q in K,

(v) $B \cap S^2 = B \cap A$,

(vi) $B \cap A$ consists of at most a finite number of polygonal simple closed curves.

Let d be one of the innermost simple closed curve of $B \cap A$ in the disk A. Further, suppose that d bounds a disk E on $A-p$, where $\operatorname{Int} E \cap B = \emptyset$. Since d bounds disk F in B, we can modify B to $B_0 = (B-F) \cup E$. We may suppose that B_1, the small deformation of B_0, satisfies the conditions (i)-(v) for B and further, the condition $B_1 \cap A \subset (B \cap A) - d$. Repeating this process, as far as possible, finally we have a disk B_2 which satisfies the conditions (i)-(v) for B and in which $B_2 \cap A$ consists of at most a finite number of polygonal simple closed curves that do not bound disks in $A-p$, respectively. The case $B_2 \cap A = \emptyset$ does not occur, since B_2 separates p and q in K. Now let c be one of the innermost simple closed curves of $B_2 \cap A$ in B_2. Then c bounds a disk D in B_2, where $(\operatorname{Int} D) \cap S^2 = \emptyset$. D is contained in the ε-neighborhood of p. It is clear that c does not bound a disk in $S^2 - (p \cup q)$ and that D is contained in $\bar{C}$. Thus the proof of the proposition (*) is complete.

Now it is easy to prove our theorem. For each $1/n$ $(n = 1, 2, \ldots)$ construct a disk D_n as shown in the proposition (*). We may suppose that $D_m \cap D_n = \emptyset$, if $m \neq n$. Put $c_n = \operatorname{bdry} D_n$. Then c_1 bounds a disk M on S^2, where M contains q. c_n and c_{n+1} bound an annulus N_n on S^2. Then

$$M \cup D_1,\ D_1 \cup N_1 \cup D_2,\ \ldots,\ D_n \cup N_n \cup D_{n+1},\ \ldots$$

are polyhedral 2-spheres, which bound closed 3-cells $I_n \subset \bar{C}$ $(n = 1, 2, \ldots)$. Further, $I_n \cap I_{n+1} = D_{n+1}$ and $\lim_{n\to\infty} I_n = p$. Since $\bigcup_{n=1}^{\infty} I_n \cup p = \bar{C}$, it is easy to see that $\bar{C}$ is a 3-cell. Thus the proof is complete.

A complementary domain C of S^2 in S^3 will be called *trivial*, if $\bar{C}$ is a 3-cell. Harrold and Moise [6] proved that if S^2 in S^3 is locally polyhedral with one singularity, then at least one of the two complementary domains of S^2 must be trivial. This theorem will not be used explicitly below, but will be convenient for the understanding of our discussion.

THEOREM 2. *Let S_1^2 and S_2^2 be two locally polyhedral spheres in S^3 with one singularity at $p \in S_1^2 \cap S_2^2$ such that*

(i) $S_1^2 \cap S_2^2 = p$,

(ii) *$S_1^2 - p$ is contained in the trivial complementary domain of S_2^2.*

If S_1^2 is tamely imbedded in S^3, then S_2^2 is tamely imbedded in S^3.

Proof. Let C_1 be the complementary domain of S_1^2 which contains a point of $S_2^2 - p$ and let C_2 be that of S_2^2 which does not contain any point of $S_1^2 - p$. Clearly $C_1 \supset C_2$. Since $\bar{C}_1$ is a 3-cell, from Theorem 1 ([3]) it follows that $\bar{C}_2$ is a closed 3-cell. Since another complementary domain of S_2^2 is trivial by our assumption, S_2^2 is tamely imbedded in S^3.

THEOREM 3. *Let S_1^2 be a locally polyhedral 2-sphere in S^3 with one singularity at $p \in S_1^2$ and let S_2^2 be a polyhedral 2-sphere in S^3 such that $S_1^2 \cap S_2^2 = p$. Suppose that U is the complementary domain of S_2^2 such that $S_1^2 - p$ is contained in U. Further, let f be a homeomorphism of the 3-cell $\bar{U}$ into another S_0^3 such that $f(S_1^2)$ and $f(S_2^2)$ are two locally polyhedral 2-spheres with one singularity at $f(p)$, respectively. If S_1^2 is not tamely imbedded in S^3, then $f(S_1^2)$ is also not tamely imbedded in S_0^3.*

Proof. $f(S_1^2)$ is contained in the trivial complementary domain of $f(S_2^2)$. Therefore, if $f(S_2^2)$ is not tamely imbedded in S_0^3, then by Theorem 2 $f(S_1^2)$ is not tamely imbedded in S_0^3. If $f(S_2^2)$ is tamely imbedded, then f can be extended to a homeomorphism of S^3 onto S_0^3. Since S_1^2 is not tamely imbedded, $f(S_1^2)$ is not tamely imbedded.

Theorem 2 can be generalized to the following:

THEOREM 4. *Let S_1^2 be a tamely imbedded 2-sphere in S^3 and S_2^2 a locally polyhedral 2-sphere in S^3 with one singularity at p such that*

(i) $S_1^2 \cap S_2^2 = p$,

(ii) *S_1^2 is contained in the trivial complementary domain of S_2^2.*

Then S_2^2 is tamely imbedded in S^3.

Proof. $(S_1^2 \cup S_2^2) - p$ is locally tame in $S^3 - p$. Therefore there exists a triangulation of $S^3 - p$, in which $(S_1^2 \cup S_2^2) - p$ is an open polyhedron (cf. Bing [7]). This means that both S_1^2 and S_2^2 are two locally polyhedral 2-spheres in S^3 with one singularity at p, respectively. Since S_1^2 is tamely imbedded in S^3, by Theorem 2, S_2^2 is tamely imbedded in S^3.

Similarly we have the following:

THEOREM 5. *Let A be an arc in S^3 and S^2 a locally polyhedral 2-sphere with one singularity at p such that*

(i) $A \cap S^2 = p$,

(ii) *$A - p$ is contained in the trivial complementary domain of S^2.*

If A is tamely imbedded in S^3, then S^2 is tamely imbedded in S^3.

§ 3. Now we construct a quasi-translation h in S^3 which is not equivalent to a topological translation.

Put

$$I = \{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ y^2 + z^2 \leqslant 1\}.$$

([3]) Theorem 1 holds whenever the triangulation of $K-p$ is locally finite.

Let A and B be two cubes in I such that

$$A = \{(x, y, z) \mid -\tfrac{1}{7} \leqslant x \leqslant 1,\ y^2+z^2 \leqslant (\tfrac{1}{7})^2\},$$

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ y^2+(z\pm\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})^2\}$$
$$\cup \{(x, y, z) \mid (x-a)^2+y^2+(z-b)^2 = (\tfrac{1}{7})^2,\ a^2+b^2 = (\tfrac{4}{7})^2,\ -\tfrac{4}{7} \leqslant a \leqslant 0\}.$$

Put $C = \overline{I-(A\cup B)}$.

Let T be the dilation of 3-space defined by

$$T(x, y, z) = (7x, 7y, 7z).$$

Let us consider $T(B)$. In $T(B)$ we construct three cubes B_i $(i = 1, 2, 3)$ such that

$$B_1 = \{(x, y, z) \mid 0 \leqslant x \leqslant 7,\ y^2+\big(z\pm(4+\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})\big)^2\}$$
$$\cup \{(x, y, z) \mid (x-a)^2+y^2+(z-b)^2 \leqslant (\tfrac{1}{7})^2,\ a^2+b^2 = (4+\tfrac{4}{7})^2,$$
$$-(4+\tfrac{4}{7}) \leqslant a \leqslant 0\},$$

$$B_2 = \{(x, y, z) \mid 0 \leqslant x \leqslant 7,\ y^2+(z\pm 4)^2 \leqslant (\tfrac{1}{7})^2\}$$
$$\cup \{(x, y, z) \mid (x-a)^2+y^2+(z-b)^2 \leqslant (\tfrac{1}{7})^2,\ a^2+b^2 = 4^2,\ -4 \leqslant a \leqslant 0\},$$

$$B_3 = \{(x, y, z) \mid 0 \leqslant x \leqslant 7,\ y^2+\big(z\pm(4-\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})\big)^2\}$$
$$\cup \{(x, y, z) \mid (x-a)^2+y^2+(z-b)^2 \leqslant (\tfrac{1}{7})^2,\ a^2+b^2 = (4-\tfrac{4}{7})^2,$$
$$-(4-\tfrac{4}{7}) \leqslant a \leqslant 0\}.$$

Put

$$C_1 = T(C) \cup \overline{\big(T(B) - \bigcup_{i=1}^{3} B_i\big)}.$$

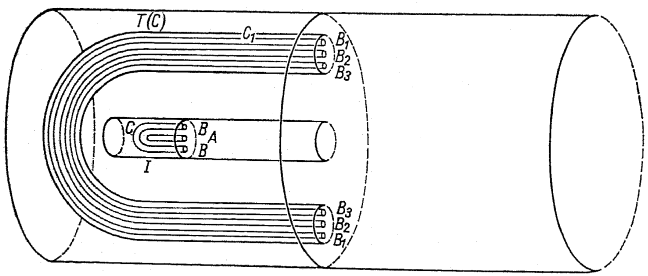

Fig. 1

We can repeat this process as shown in Fig. 1 and define $B_{i_1 i_2 \ldots i_n}$ and C_n, where

$$C_n = T(C_{n-1}) \cup \bigcup_{i_1\ldots i_{n-1}} \overline{\big(T(B_{i_1\ldots i_{n-1}}) - \bigcup_{j=1}^{3} B_{i_1\ldots i_{n-1}j}\big)}$$

for $n \geqslant 2$.

Now let J be the cube

$$\{(x, y, z) \mid 1 \leqslant x \leqslant 7,\ y^2+z^2 = 1\}.$$

In J we construct three cubes D_i $(i = 1, 2, 3)$ as shown in Fig. 2. Suppose that

$$\operatorname{bdry} J \cap D_1 = \{(x, y, z) \mid x = 1,\ y^2+(z-\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})^2\}$$
$$\cup \{(x, y, z) \mid x = 7,\ y^2+z^2 \leqslant (\tfrac{1}{7})^2\},$$

$$\operatorname{bdry} J \cap D_2 = \{(x, y, z) \mid x = 1,\ y^2+z^2 \leqslant (\tfrac{1}{7})^2 \text{ or } y^2+(z+\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})^2\},$$

$$\operatorname{bdry} J \cap D_3 = \{(x, y, z) \mid x = 7,\ y^2+(z\pm\tfrac{4}{7})^2 \leqslant (\tfrac{1}{7})^2\}.$$

Put

$$E_1 = J - \overline{(D_1 \cup D_2 \cup D_3)}.$$

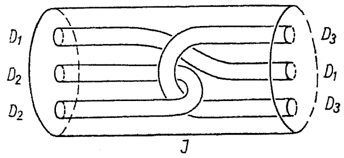

Fig. 2

In each $T(D_i)$ construct three cubes D_{ij} $(j = 1, 2, 3)$ as shown in Fig. 3. We may suppose that B_1, B_2, B_3, D_1 and D_3 are connected smoothly to $D_{11}, D_{12}, D_{21}, D_{22}$ and D_{23} in a suitable choice as shown in Fig. 3. Put

$$E_2 = T(E_1) \cup \bigcup_{i=1}^{3} \overline{\big(T(D_i) - \bigcup_{j=1}^{3} D_{ij}\big)}.$$

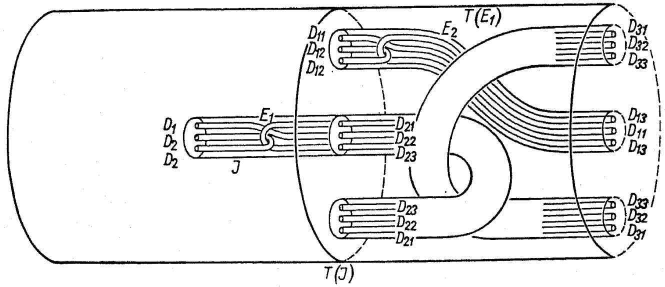

Fig. 3

We can repeat this process and define $D_{i_1 i_2 \ldots i_n}$ and E_n, where

$$E_n = T(E_{n-1}) \cup \bigcup_{i_1\ldots i_{n-1}} \overline{\big(T(D_{i_1\ldots i_{n-1}}) - \bigcup_{j=1}^{3} (B_{i_1\ldots i_{n-1}j})\big)}.$$

Put

$$F = C \cup \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} E_n \cup p ,$$

$$G = A \cup B \cup \bigcup_{i_1 \ldots i_n} B_{i_1 \ldots i_n} \cup \bigcup_{i_1 \ldots i_n} D_{i_1 \ldots i_n} \cup p ,$$

where p is the point at infinity. Clearly G is a 3-cell and $\operatorname{Int} F$ and $\operatorname{Int} G$ are two complementary domains of the 2-sphere $H = F \cap G$.

Now let us define a quasi-translation h, which will be seen to be inequivalent to the topological translation. First put

$$h(p) = p .$$

For each $q \in T(F)$ we define

$$h(q) = T^{-1}(q) ,$$

which is a homeomorphism of $T(F)$ onto F. $T(G)$ is a 3-cell and G is a 3-cell in $T(G)$. It is easy to see that there is a homeomorphism g_0 of $T(G)$ onto G such that $g_0(p) = p$, $g_0(q) = T^{-1}(q)$, whenever $q \in T(H)$ and that $x^2+y^2+z^2 \geqslant 1$, whenever $(x, y, z) \in g_0(G)$. For each point $q \in T(G) - G$ we define

$$h(q) = g_0(q) .$$

Since G is a 3-cell and $g_0(G)$ is a 3-cell in G, there is a homeomorphism g_1 of G onto $g_0(G)$ such that $g_1(p) = p$, $g_1(q) = g_0(q)$ whenever $q \in H$ and $x^2+y^2+z^2 \geqslant 7$ whenever $(x, y, z) \in g_1 g_0(G)$. Put

$$h(q) = g_1(q)$$

for each $q \in G - g_0(G)$. Similarly, since $g_0(G)$ is a 3-cell and $g_1 g_0(G)$ is a 3-cell in $g_0(G)$, there exists a homeomorphism g_2 of $g_0(G)$ onto $g_1 g_0(G)$ such that $g_2(q) = g_1(q)$ whenever $q \in g_0(H)$ and that $x^2+y^2+z^2 \geqslant 7^2$ whenever $(x, y, z) \in g_2 g_1 g_0(G)$. Put

$$h(q) = g_2(q)$$

for each $q \in g_0(G) - g_1 g_0(G)$. We can repeat this process infinitely many times. Then h is defined as an orientation preserving auto-homeomorphism of S^3. It is easy to see that h has equi-continuous powers at $q \in S^3 - p$ but not at p. Therefore h is a quasi-translation of S^3.

It remains to prove that h is not topologically equivalent to a translation of S^3.

First we remark that the 2-sphere H is not tamely imbedded in S^3. We may suppose that, for each integer m, $h^m(H)$ is locally polyhedral with one singularity p. One of the simple examples of locally polyhedral 2-spheres K with one singularity, where K is not tamely imbedded in 3-space, is reproduced below from Fox-Artin [3].

Let K be the 2-sphere in question and L a polyhedral 2-sphere as shown in Fig. 4. Let M be the closure of the complementary domain of L which contains $K - p$. Therefore M is a 3-cell. It is easy to see that there is a homeomorphism f of M onto $T(G)$ such that $f(K) = H$. Then, by Theorem 3, H is not tamely imbedded in S^3.

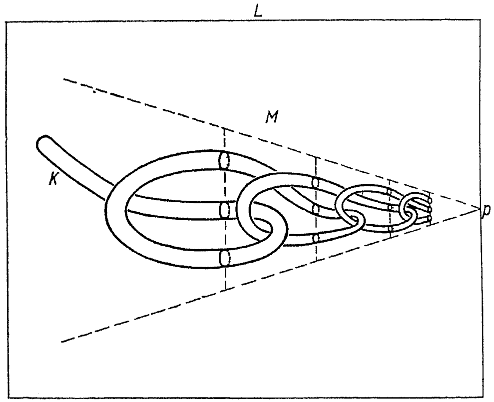

Fig. 4

Now let k be an arbitrary quasi-translation of S^3. An arc $\widehat{qp}$, where $q \in S^3 - p$, will be called the positive translation arc of k, if $k(\widehat{qp}) \subset \widehat{qp}$. We shall prove that if $\widehat{qp}$ is a positive translation arc of h, constructed as above, then $\widehat{qp}$ is not tamely imbedded in S^3. This proves that h is not topologically equivalent to a translation of S^3, because if h were a topological translation, then for each $q \in S^3 - p$ there would exist a positive translation arc $\widehat{qp}$ tamely imbedded in S^3.

Let $q \in S^3 - p$ and $\widehat{qp}$ be a positive translation arc of h. Let $\widehat{qh(q)}$ be the subarc of $\widehat{qp}$. It is easy to see that there exists a positive integer N such that $\operatorname{Int} h^{-N}(G) \supset \widehat{qh(q)}$. Then $\operatorname{Int} h^{-N+n}(G)$ contains the subarc $\widehat{h^n(q) h^{n+1}(q)}$ of $\widehat{qp}$ for each natural number n. Since $h^m(G) \supset h^{m+1}(G)$ for each integer m, $\operatorname{Int} h^{-N}(G) \supset \widehat{qp} - p$. Since H is not tamely imbedded in S^3, so is $h^{-N}(H)$. $h^{-N}(G)$ is the closure of one of the complementary domains of $h^{-N}(H)$. Further $h^{-N}(G)$ is a 3-cell and $h^{-N}(H)$ is locally polyhedral with one singularity p. Then, by Theorem 5, $\widehat{qp}$ is not tamely imbedded in S^3. Thus our proof is complete.

Appendix. Let X be a compact metric space and A a finite subset of X such that $X-A$ is connected. If h is an auto-homeomorphism of X which has equi-continuous powers at every $x \in X-A$ but not at any $x \in A$, then the number of points of A is at most two (see [5]). Therefore, there are three types of such auto-homeomorphisms.

Let X be compact and h an auto-homeomorphism of X which has equi-continuous powers at every point. Then it seems to be well known that there exists a compact transformation group G of X which contains h as an element and in which $\{h^m\}$ is dense, where m is an integer. Thus the study of this case will be reduced to that of compact transformation groups.

The purpose of this appendix is to prove that if X is a compact connected closed topological manifold, then other types of these auto-homeomorphisms can operate only on an n-sphere.

THEOREM 6. *Let X be a compact connected closed topological manifold and h an auto-homeomorphism of X which has equi-continuous powers at every $x \in X-p$ but not at p. Then X is an n-sphere.*

Proof. Let $U(p)$ be a Euclidean neighborhood of p. Then $X-U(p)$ is compact and contained in $X-p$. Therefore, by Lemma 1, there exists a natural number N such that $h^N(X-U(p)) \subset U(p)$. Then $h^{-N}(U(p)) \supset X-U(p)$. Therefore X is a sum of two open cells. Thus X is an n-sphere by the generalized Schoenflies theorem (see Brown [8]).

THEOREM 7. *Let X be a compact connected closed topological manifold and h an auto-homeomorphism of X which has equi-continuous powers at every $x \in X-(p \cup q)$ and does not have it at both p and q. Then X is an n-sphere.*

Proof. If X is 1-dimensional, then X is a 1-sphere. Therefore, we may suppose that the dimension of X is equal to or greater than two. Then $X-(p \cup q)$ is connected. Then one of p or q, say p, is the attractive point of h, i.e., for each $x \in X-q$ $\{h^m(x)\}$ converges to p when $m \to \infty$ and the other q is repulsive, i.e., for each $x \in X-p$ $\{h^m(x)\}$ converges to q when $m \to -\infty$ (see [5]). Further, it is known that for each compact subset C of $X-q$ $\lim_{m\to\infty} h^m(C) = p$ and for each compact subset C of $X-p$ $\lim_{m\to-\infty} h^m(C) = q$ (see Homma and Kinoshita [9]).

Let $U(p)$ and $V(q)$ be mutually disjoint Euclidean neighborhoods of p and q, respectively. Then there exists a natural number N such that $h^N(X-V(q)) \subset U(p)$. Therefore $X-h^N(V(q)) \subset U(p)$. This means that X is the sum of two open cells $U(p)$ and $h^m(V(q))$. Thus, again by the generalized Schoenflies theorem, X is an n-sphere.

Remark. In the case of Theorem 7 it was known that if the dimension of X is equal to 2 or 3, then h is a topological dilation of S^2 or S^3, respectively (see [10] and [11]).

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