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On boundedness in uniform spaces

by

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Introduction. If P is a property defined for sets in uniform spaces, one can raise the question: under what conditions on a uniform space $(S, \mathcal{U})$ does there exist a $V \in \mathcal{U}$ such that $V(X)$ has the property P whenever X has the property P? Such an entourage V might be called P-*conserving*, and such a space P-*conservative*. Depending on P, questions of this kind may range from the trivial (e.g. when P is openness) to the rather deep. Certain potential applications to topological dynamics suggest that questions of this nature may be especially interesting when P is a concept of boundedness. The purpose of this note is to present two simple answers to the above question in the particular case in which P is boundedness as defined by Bourbaki ([2], p. 166).

The following general observation, which is easily proved, will be used later: *If* P *is a hereditary property* (i.e., any subset of a set with property P has property P) *and* $(S, \mathcal{U})$ *is a* P-*conservative space, then the collection of* P-*conserving members of* $\mathcal{U}$ *is a base of the filter* $\mathcal{U}$.

Boundedness. A set X in the uniform space $(S, \mathcal{U})$ is *bounded* if for every $V \in \mathcal{U}$ there holds one of the following equivalent conditions:

(1) $X \subset V^n(x)$ for some $x \in S$ and some positive integer n;

(2) $$X \times X \subset V^n \text{ for some integer } n.$$

When $X \neq \emptyset$, the x in (1) may be assumed to be any point of X. If S is bounded in $(S, \mathcal{U})$, the space itself will be said to be bounded.

When $\mathcal{U}$ is a metric uniformity, boundedness in this sense implies, but is not implied by, the usual metric concept of boundedness (finite diameters). It may be shown, however, that the two concepts of boundedness are equivalent for normed linear spaces and continua.

Elementary facts about boundedness include: it is hereditary; the closure of any bounded set is bounded; the union of two nondisjoint bounded sets is bounded; boundedness is preserved by uniformly continuous maps.

LEMMA 1. *Let $(T, \mathfrak{U}_T)$ be a uniform subspace of the uniform space $(S, \mathfrak{U})$. If $X \subset T$ is bounded in $(T, \mathfrak{U}_T)$, then it is bounded in $(S, \mathfrak{U})$. If also T is dense in $(S, \mathfrak{U})$, then the converse holds.*

Proof. The first assertion is immediate. To prove the second, let T be dense, let $X \subset T$ be bounded in $(S, \mathfrak{U})$, and let $V \in \mathfrak{U}$ be given. Take W to be a symmetric member of $\mathfrak{U}$ such that $W^2 \subset V$. There exists a positive integer n such that $X \times X \subset W^n$. This means that for any $x, y \in X$, there exists a sequence

$$x = x_0, x_1, \ldots, x_n = y,$$

where

$$(x_{k-1}, x_k) \in W \quad (k = 1, 2, \ldots, n).$$

Because T is dense, there exists for each k $(0 < k < n)$ a $\xi_k \in W(x_k) \cap T$. Let $\xi_0 = x$, $\xi_n = y$; then

$$\xi_k \in V(\xi_{k-1}) \cap T \quad (k = 1, 2, \ldots, n).$$

Thus $(x, y) \in \big(V \cap (T \times T)\big)^n$, whence $X \times X \subset \big(V \cap (T \times T)\big)^n$, and this shows that X is bounded in $(T, \mathfrak{U}_T)$.

Simple examples may be constructed to show that when T is not dense, a set $X \subset T$ may be bounded in $(S, \mathfrak{U})$ without being bounded in $(T, \mathfrak{U}_T)$.

LEMMA 2. *Any totally bounded, connected uniform space is bounded.*

Proof. Let $(S, \mathfrak{U})$ be totally bounded and connected, and let $V \in \mathfrak{U}$ be given. Let $W \in \mathfrak{U}$ be open, symmetric, and contained in V. Then by the assumed total boundedness there exists a finite set $A \subset S$ such that $W(A) = S$. If $\{A_1, A_2\}$ is any partition of A $(A_1 \neq \emptyset, A_2 \neq \emptyset)$, it follows from the assumed connectedness that $W(A_1) \cap W(A_2) \neq \emptyset$. This implies that

(3) $$S \times S \subset W^{2n} \subset V^{2n},$$

where n is the number of elements of A. In fact, let $x, y \in S$, and let $x \in W(x_1)$, where $x_1 \in A$. If also $y \in W(x_1)$, let $m = 1$ and stop. If $y \notin W(x_1)$, choose $x_2 \in A$ $(x_2 \neq x_1)$ so that $W(x_1) \cap W(x_2) \neq \emptyset$. Choose $\xi_1 \in W(x_1) \cap W(x_2)$. If $y \in W(x_2)$, set $m = 2$ and stop; if $y \notin W(x_2)$, choose $x_3 \in A$ $(x_3 \neq x_1, x_3 \neq x_2)$ so that $W(\{x_1, x_2\}) \cap W(x_3) \neq \emptyset$, and let ξ_2 be any point in this intersection. Proceeding in this way, one ultimately obtains a sequence

$$x, x_1, \xi_1, x_2, \xi_2, \ldots, x_m, y \quad (m \leqslant n)$$

where each point is W-close to its predecessor. Thus $(x, y) \in W^{2m} \subset W^{2n}$, whence (3).

Examples show that total boundedness alone does not imply boundedness, even for metrizable uniform spaces. In this respect the terminology is unfortunate.

LEMMA 3. *A set in a compact* (1) *uniform space is bounded if and only if it is a subset of some component.*

Proof. Let X be bounded in the compact uniform space $(S, \mathfrak{U})$, and let $x \in X$. For any $V \in \mathfrak{U}$, $X \subset V^n(x)$ for some n. Therefore

$$X \subset A_x = \bigcap \{A_{x,V}\colon V \in \mathfrak{U}\}, \quad \text{where} \quad A_{x,V} = \bigcup \{V^n(x)\colon n = 1, 2, \ldots\}.$$

But according to [2], p. 163, A_x is precisely the component of x in $(S, \mathfrak{U})$; hence X is contained in a component.

On the other hand, any component C of the compact space $(S, \mathfrak{U})$ is bounded in itself, by Lemma 2, and hence in $(S, \mathfrak{U})$, by Lemma 1. Thus any subset of C is bounded in $(S, \mathfrak{U})$.

Conservative uniform spaces. In the rest of this note "boundedness-conservative" will be abbreviated to *conservative*. Let $(S, \mathfrak{U})$ be a conservative uniform space; then $\mathfrak{U}$ has a base $\mathfrak{U}_c$ consisting of (boundedness-) conserving entourages; indeed, $\mathfrak{U}_c$ can be further restricted to open, closed, or symmetric conserving entourages.

THEOREM 1. *If $\mathfrak{U}_c$ contains a maximal element, $(S, \mathfrak{U})$ is either disconnected or bounded.*

Proof. Suppose that V_0 is maximal in $\mathfrak{U}_c$; then in fact V_0 is an upper bound for $\mathfrak{U}_c$ (if there existed a $V \in \mathfrak{U}_c$ such that $V \not\subset V_0$, then $V \cup V_0$ would be a conserving entourage strictly containing V_0). Moreover, V_0 is symmetric. [Proof: $V_s = V_0 \cap V_0^{-1}$ is symmetric, conserving, and (hence) contained in V_0. Thus the same is true of V_s^n for $n = 1, 2, \ldots$, and of $V_1 = \bigcup_n V_s^n$. Let $x \in S$. Since $\{x\}$ is bounded and V_0 is conserving, $V_0(x) \subset V_s^n(x)$ for some n, whence $V_0(x) \subset V_1(x)$. The opposite inclusion follows from the maximality of V_0; thus $V_0(x) = V_1(x)$ for all x, and the symmetry of V_0 follows from that of V_1.] Similarly, $V_0^2 = V_0$. Since also V_0 contains the diagonal $\Delta = \{(x, x)\colon x \in S\}$, V_0 is an equivalence relation on S. There are two cases:

(i) $V_0 = S \times S$. Then for any $x \in S$, $V_0(x) = S$. Since $\{x\}$ is bounded and V_0 is conserving, S is bounded.

(ii) $V_0 \neq S \times S$. Then for any $x \in S$, $V_0(x)$ is a proper closed-open subset of S, so the space is disconnected. This completes the proof of Theorem 1.

(1) Here "compact" means "possessing the Heine-Borel property, not necessarily Hausdorff".

Conservative precompact spaces. The next theorem characterizes those conservative uniform spaces that happen to be precompact, i.e., totally bounded and Hausdorff.

THEOREM 2. *A precompact uniform space is conservative if and only if its (Hausdorff) completion has a finite number of components.*

Proof. Let $(S, \mathcal{U})$ be precompact and let the completion $(\hat{S}, \hat{\mathcal{U}})$ of $(S, \mathcal{U})$ have a finite number of components. As is well known, $(\hat{S}, \hat{\mathcal{U}})$ is compact. Because conservativeness is preserved by uniform isomorphisms, and because (as follows easily from Lemma 1) any dense subspace of a conservative space is conservative, it will suffice to show that $(\hat{S}, \hat{\mathcal{U}})$ is conservative. Let $\mathfrak{C}$ be the (finite) collection of components of $(\hat{S}, \hat{\mathcal{U}})$; then $\mathfrak{C}$ consists of open sets and $V_0 = \bigcup\{C \times C\colon C \in \mathfrak{C}\}$ is a neighborhood of the diagonal and therefore, because of the compactness, belongs to $\hat{\mathcal{U}}$. If X is a bounded set in $(\hat{S}, \hat{\mathcal{U}})$, X is contained in some member C of $\mathfrak{C}$, by Lemma 3. Then $V_0(X) = \emptyset$ or C according as X is or is not empty. In either case, $V_0(X)$ is bounded and V_0 is conserving.

Now we shall show that every completion $(\hat{S}, \hat{\mathcal{U}})$ of a conservative totally bounded space $(S, \mathcal{U})$ has a finite number of components. Again, we observe that $(\hat{S}, \hat{\mathcal{U}})$ is compact, and as is customary we identify $(S, \mathcal{U})$ with that dense subspace of $(\hat{S}, \hat{\mathcal{U}})$ to which it is, by the definition of a completion, uniformly isomorphic. We shall need the following fact: *for any $x \in \hat{S}$, there exists a $V \in \hat{\mathcal{U}}$ such that $V(x)$ is bounded in $(\hat{S}, \hat{\mathcal{U}})$.* In the proof of this fact all entourages mentioned will be taken to be symmetric. First, let $V'_S = V' \cap (S \times S)$, where $V' \in \hat{\mathcal{U}}$, be any conserving member of the relative uniformity $\mathcal{U}$. Let $V \in \hat{\mathcal{U}}$ satisfy $V^3 \subset V'$. For a given $W \in \hat{\mathcal{U}}$, let $W_1 \in \hat{\mathcal{U}}$ satisfy $W_1 \subset V \cap W$. Because S is dense in $(\hat{S}, \hat{\mathcal{U}})$, there exists a point $x_1 \in W_1(x) \cap S$. Because V'_S is conserving and $\{x_1\}$ is bounded in $(S, \mathcal{U})$, there exists a positive integer n such that

$$V'_S(x_1) \times V'_S(x_1) \subset W^n. \tag{4}$$

Now let $y \in V(x)$. Again because S is dense, there exists a $y_1 \in W_1(y) \cap X$. Then

$$y_1 \in W_1(y) \subset V(y) \subset V^2(x) \subset V^2 W_1(x_1) \subset V^3(x_1) \subset V'(x_1).$$

Since $y_1 \in S$ too, it follows from (4) that

$$(x_1, y_1) \in W^n.$$

At the same time, $(x, x_1) \in W_1 \subset W$ and, similarly, $(y_1, y) \in W$. Hence

$$(x, y) \in W^{n+2},$$

from which it is clear that $V(x)$ is bounded in $(\hat{S}, \hat{\mathcal{U}})$. This proves the assertion in italics.

Suppose now that $(\hat{S}, \hat{\mathcal{U}})$ has infinitely many components. Since it is a compact space, at least one component, say C, must be nonopen. Let $x \in C - C^0$, where C^0 is the interior of C, and let $V \in \hat{\mathcal{U}}$ be such that $V(x)$ is bounded, as above. By Lemma 3, $V(x) \subset C$, whence $x \in C^0$. This contradicts the choice of x, so $(\hat{S}, \hat{U})$ must have a finite number of components. The proof of Theorem 2 is complete.

Since every compact Hausdorff space is precompact and is a completion of itself, there is the following immediate consequence of Theorem 2.

COROLLARY. *A compact Hausdorff uniform space is conservative if and only if it has a finite number of components.*

Lemma 2 may also be regarded as a special consequence of Theorem 2.

Conservative nonarchimedean spaces. Another, and quite different type of uniform spaces for which a simple characterization of conservativeness exists is that of the nonarchimedean spaces, introduced by A. F. Monna [3]. A space $(S, \mathcal{U})$ is defined to be nonarchimedean if $\mathcal{U}$ has a base consisting of equivalence relations on S. It has been pointed out by B. Banaschewski [1] that a topological space is zero-dimensional if and only if it admits a nonarchimedean uniform structure.

THEOREM 3. *A nonarchimedean uniform space $(S, \mathcal{U})$ is conservative if and only if $\mathcal{U}$ has a smallest member; i.e., $\bigcap \mathcal{U} \in \mathcal{U}$.*

Proof. Let $(S, \mathcal{U})$ be nonarchimedean. Then according to (2) and the fact that if V is an equivalence relation $V^n = V$ for all n, a set $X \subset S$ is bounded if and only if

$$X \times X \subset V$$

for every equivalence relation $V \in \mathcal{U}$. This means that X is contained in one of the equivalence classes determined by V.

Now let $(S, \mathcal{U})$ be conservative, and let V be a conserving member, which may be taken to be an equivalence relation, of $\mathcal{U}$. Let C be any one of the corresponding equivalence classes and let $x \in C$. Let $W \in \mathcal{U}$ be given and choose W_0 to be an equivalence relation belonging to $\mathcal{U}$ and contained in W. Then

$$C \times C = V(x) \times V(x) \subset W_0 \subset W,$$

because $V(x)$ is bounded. Hence, since V is the union of all such sets $C \times C$, $V \subset W$; that is, V is the smallest member of $\mathcal{U}$.

Conversely, if V is the smallest member of $\mathcal{U}$, it must be an equivalence relation. If X is bounded, it is contained in some equivalence class of V, say C. But then, except in the trivial case $X = \emptyset$, $V(X) = C$, so

$$V(X)\times V(X) = C\times C \subset V.$$

Hence, for any $W \in \mathfrak{U}$, $V(X)\times V(X) \subset W$, $V(X)$ is bounded, and V is conserving.

Corollary. *A Hausdorff nonarchimedean uniform space* $(S, \mathfrak{U})$ *is conservative if and only if* $\mathfrak{U}$ *is discrete, i.e.* $\Delta \in \mathfrak{U}$.

Proof. $(S, \mathfrak{U})$ is Hausdorff if and only if $\bigcap \mathfrak{U} = \Delta$.

References

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The inversion of Peano continua by analytic functions*

by

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1. Introduction. Suppose that f is a function analytic, or even schlicht, in the open disk $|z| < 1$. Suppose that A is an arc which has one end point, p, on the unit circle, but which otherwise lies in the open unit disk. Despite the fact that A itself is locally connected at every point, it may very well happen that the image of $A-p$, $f(A-p)$, will have a closure that is not locally connected. This will occur, for example, for any such arc that leads to a point of $|z| = 1$ which corresponds under a conformal map to a prime end of the fourth kind [4]. If the map is not schlicht, $f(A-p)$ may even be a closed set, but fail to be locally connected. Thus it is *not* true of analytic functions that, given a Peano continuum (1) P in the plane, and a component C of the intersection of P and the open disk, then the closure of $f(C)$ is always a Peano continuum. In this sense, analytic functions are not "Peano-continuum preserving". They do, of course, preserve local connectedness for Peano continua lying entirely in $|z| < 1$, since any continuous map on a Peano continuum preserves this property.

This paper is concerned with the opposite problem: Given a function f into the plane or the extended plane, defined in $|z| < 1$, when is such a function Peano-continuum reversing? By this I mean the following: The map $f(z)$, $|z| < 1$, is *Peano-continuum reversing* provided that if P is any Peano continuum in the extended plane, and C is a component of $f^{-1}(P)$, then the closure of C, $\bar{C}$, is a Peano continuum.

In this paper, I show that bounded analytic or quasiconformal functions, the elliptical modular functions, and some meromorphic functions of bounded characteristic are all Peano-continuum-reversing. These functions are all special cases of the interior light functions of Stoïlow, and, actually, the theorems of this paper follow by purely topological methods from topological hypotheses. Thus the results are quite general, but that was not an aim of the paper. The fact is that I do not know

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(1) A *Peano continuum* is a compact, metric, connected and locally connected space.