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On Egoroff's theorem

by

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I. Although Egoroff's theorem [6] is usually stated for sequences one finds it used in certain instances when the collection of functions involved is non-denumerable ([5], [7]). However, several counter-examples exist in the literature which show that the conclusion of the theorem does not in general follow in this case ([2], [8], [9], [10]). Hahn and Rosenthal [3] must have realized this, although no reference to a counter-example is mentioned, since they state and prove a non-denumerable analogue to Egoroff's theorem, but by placing certain restrictions on the functions not found in the original form of the theorem. Essentially, they prove:

Let m be a measure function on an additive class of sets $\boldsymbol{A}$ of a space X, A an element of $\boldsymbol{A}$ such that $m(A) < +\infty$ and F a real function defined on $A \times (0,1)$ such that for each $x \in A$, $F(x, \cdot)$ is continuous on $(0,1)$ and for each $t \in (0,1)$, $F(\cdot, t)$ is measurable on A. If

$$\lim_{t\to 0} F(x,t) = G(x)$$

a.e. on A, where G is finite a.e. on A, then, for each $\eta > 0$, there exists a set $B \subset A$ such that $m(A-B) < \eta$ and the convergence of $F(\cdot, t)$ to G is uniform on B.

It is the purpose of this note to weaken the hypotheses of the above theorem. In what follows F, m, A, G and $\boldsymbol{A}$ are to have the same significance as above as well as the notation $F(x, \cdot)$ and $F(\cdot, t)$. We obtain our results by replacing the set $(0,1)$ with an infinite set M and varying its nature.

II. We first suppose that M is an infinite subset of a topological space Y which is Hausdorff and second countable while its closure, $\mathrm{cl}\, M$, is countably compact (see Hall and Spencer [4]). This allows us to assume without any loss that if we let M' denote the derived set of M and H a countable subset of M dense in M, then, if $p \in \mathrm{cl}\, M$ but $p \notin H$, then $p \in M' - M$. Let l.s.c. (u.s.c.) denote lower [upper] semi-continuous. If f is a real function defined on a set E and $H \subset E$ then the symbol $\sup f(H)$ means the supremum of the set $\{f(y) \mid y \in H\}$. A corresponding meaning is given to $\inf f(H)$. The main result of this section is the following theorem.

THEOREM 1. *If* (i) $F(\cdot, t)$ *is measurable on* A *for each* $t \in M$, (ii) $F(x, \cdot)$ *is* l.s.c. [u.s.c.] *on* M *for each* $x \in A$, (iii) $\lim_{t\to a} F(x, t)$ *exists and equals* $G(x)$ *a.e. on* A, *and* (iv) $G(x) \leqslant F(x, t)$ $[G(x) \geqslant F(x, t)]$ *for each* $x \in A$ *and for every* t *in some neighborhood* V *of* $a \in M'$, *then for each prescribed* $\eta > 0$ *there exists some set* $B \subset A$ *with* $m(A-B) < \eta$ *on which* $\lim_{t\to a} F(x, t) = G(x)$ *uniformly.*

The theorem cited in the first section turns out to be a corollary of the above theorem. In order to prove the above theorem we need the following lemma.

LEMMA. *If* $F(x, \cdot)$ *is* l.s.c. [u.s.c.] *on* M *for each* $x \in A$ *and* $F(\cdot, t)$ *is measurable on* A *for every* $t \in M$ *and if*

$$\sup F(\cdot, M)(x) = \sup F(x, M),$$
$$\inf F(\cdot, M)(x) = \inf F(x, M),$$
$$\limsup_{t\to a} F(\cdot, t)(x) = \limsup_{t\to a} F(x, t),$$
$$\liminf_{t\to a} F(\cdot, t)(x) = \liminf_{t\to a} F(x, t),$$

then both $\sup F(\cdot, M)$ $[\inf F(\cdot, M)]$ *and* $\limsup_{t\to a} F(\cdot, t)$ $[\liminf_{t\to a} F(\cdot, t)]$ *are measurable on* A, *the latter holding for all* $a \in M'$.

We prove only the l.s.c. part of the lemma, the proof of the u.s.c. part being similar.

Since M is second countable, then it is separable. Let $H = \{r_n\} \subset M$ be a countable set dense in M. Place $f_n(x) = F(x, r_n)$ and $C(x) = \sup_n f_n(x)$. We assert that $C(x) = \sup F(x, M)$. Indeed, all we need show is $C(x) \geqslant \sup F(x, M)$ since $C(x) \leqslant \sup F(x, M)$ follows from $H \subset M$. Therefore, suppose to the contrary that for some $x^* \in A$, $C(x^*) < \sup F(x^*, M)$. Then there exists a $t^* \in M$ such that

$$C(x^*) < F(x^*, t^*) \leqslant \sup F(x^*, M).$$

Hence

$$f_n(x^*) = F(x^*, r_n) \leqslant C(x^*) < F(x^*, t^*) \leqslant \sup F(x^*, M).$$

Since this is true for every n, then $t^* \notin H$ and so $t^* \in M'$. From this follows $t^* \in H'$. Therefore, if we let $N(t^*)$ denote the family of all neighborhoods of t^*, then $U \cap H \neq \emptyset$ for every $U \in N(t^*)$; thus

$$\inf F(x^*, U) \leqslant C(x^*) < F(x^*, t^*).$$

But then

$$\sup\{\inf F(x^*, U),\ U \in N(t_*)\} \leqslant C(x^*) < F(x^*, t^*)$$

which contradicts the hypothesis that $F(x^*, \cdot)$ is l.s.c. on M. Thus our assertion holds and since C is measurable on A, so is $\sup F(\cdot, M)$.

To show the second part of the lemma we make use of a countable basis $\{U_n\}$ at a such that $U_{n+1} \subset U_n$. Define the sequence of functions $\{F_n\}$ by $F_n(x) = \sup F(x, U_n)$. Then by what has just preceded, F_n is measurable on A for every n. Since $\{U_n\}$ is a decreasing sequence of sets, $\{F_n\}$ is a decreasing sequence of functions and thus $\lim_n F_n(x)$ exists for each $x \in A$, say $D(x)$. It follows immediately that $\limsup_{t\to a} F(x, t) \leqslant D(x)$. The opposite inequality also follows. To see this, let V be an arbitrary neighborhood of a. Then there is an integer N such that $U_N \subset V$. This implies $\sup F(x, U_N) \leqslant \sup F(x, V)$. But then $D(x) \leqslant \sup F(x, V)$ and the desired result follows from the arbitrariness of V. Thus $D(x) = \limsup_{t\to a} F(x, t)$. Now D is measurable on A; consequently so is $\limsup_{t\to a} F(\cdot, t)$.

As above we now give a proof of the l.s.c. part of Theorem 1, the proof of the u.s.c. part being similar. By removing from A a set of measure zero we may assume that $F(\cdot, t)$ converges to G everywhere on A and that G is finite on A. Let $\{U_n\}$ and $\{F_n\}$ have the same significance as in the second part of the proof of the Lemma. According to this lemma we know each F_n as well as $\limsup_{t\to a} F(\cdot, t)$ is measurable on A. Since

$$\limsup_{t\to a} F(x, t) = \lim_{t\to a} F(x, t) = G(x) = \lim_n F_n(x),$$

the same must be true of G. Also, in virtue of the fact that G is finite on A and the properties $F_n(x) \geqslant F_{n+1}(x)$, $x \in A$, and (iv) (Theorem 1), for each $\eta > 0$, there exists a measurable subset C of A and an index n^* such that $m(A-C) < 2^{-1}\eta$ and F_n is finite on C for each $n > n^*$. Thus we may apply Egoroff's theorem to the subsequence $\{F_{n^*+k}\}$ $(k = 1, 2, \ldots)$ and assert that there exists a subset B of C such that $m(C-B) < 2^{-1}\eta$ and $\{F_{n^*+k}\}$ converges uniformly on B to G. Consequently, letting $p = n^*+k$, for each $\delta > 0$, there exists an integer N_1 such that

$$|F_p(x) - G(x)| < \delta \quad \text{for each} \quad x \in B \text{ and } p > N_1.$$

According to the definition of F_p, $F_p(x) \geqslant F(x, t)$ for each $x \in A$ and $t \in U_p$. Therefore, if $p > N_1$,

$$F(x, t) - \delta < G(x) \quad \text{for each} \quad x \in B \text{ and } t \in U_p.$$

According to our hypothesis there exists a neighborhood V of a for which $G(x) \leqslant F(x,t)$ for each $x \in A$ and $t \in V$. Let N_2 be an integer such that $p > N_2$ implies $U_p \subset V$. Then, if $p > N_2$,

$$G(x) < F(x,t) + \delta \quad \text{for each} \quad x \in A \text{ and } t \in U_p .$$

Hence if $p > \max(N_1, N_2)$ we have $|F(x,t) - G(x)| < \delta$ for each $t \in U_p$ and $x \in B$, i.e. $F(\cdot, t)$ converges uniformly to G on B at a. Since $m(A-B) < \eta$, our proof is complete.

III. In this section we assume $M \subset Y$ where Y is a space which is a complete chain relative to a partial order relation R. Then every two elements of Y are R-comparable and we may say that $F(x, \cdot)$ is isotone (antitone) on M for each $x \in A$ if $a, b \in M$, aRb imply that $F(x,a) \leqslant F(x,b)$ $(F(x,a) \geqslant F(x,b))$. The space Y can then be considered a topological space by taking as a basis the "intervals" induced by the partial order R. It is known [1] this space is Hausdorff. We assume that M and Y satisfy the further topological conditions stated at the beginning of the second section. The proof of the following theorem is quite similar to that of Theorem 1 and therefore we omit it.

THEOREM 2. *If* (i) $F(\cdot, t)$ *is measurable on* A *for every* $t \in M$, (ii) $F(x, \cdot)$ *is monotonic on* M *for every* $x \in A$ *and* (iii) $\lim_{t \to a} F(x,t) = G(x)$, *a.e. on* A, *then, for each* $\eta > 0$, *there exists a subset* $B \subset A$ *such that* $m(A-B) < \eta$ *and the convergence of* $F(\cdot, t)$ *to* G *is uniform on* B.

IV. Although the conditions imposed on $F(x, \cdot)$ in Theorem 1 are stronger than strict semi-continuity, nevertheless, they do not imply continuity of $F(x, \cdot)$ on M. Aside from the topological conditions imposed on M and Y, the question arises whether or not this theorem is really an extension of the result of Hahn and Rosenthal. To see that the conditions imposed in Theorem 1 are significant, we cite the counter-example given by Vinti [9]. Let $M = A = [0,1]$, $a = 0$, and let $\{E_n\}$ be a sequence of mutually disjoint non-measurable sets whose union is $[0,1]$ and such that any finite union of these sets has inner Lebesgue measure zero. Define F as follows:

$$F(x,t) = \begin{cases} 1 & \text{if} \quad x \in E_n,\ t = x/n,\ x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{t \to 0} F(x,t) = 0$ for each $x \in [0,1]$ and $F(\cdot, t)$ is measurable for every $t \in [0,1]$. But $F(\cdot, t)$ does not converge uniformly on any subset of $[0,1]$ of positive measure. We note that $F(x, \cdot)$ is u.s.c. on $[0,1]$ but no neighborhood of $a = 0$ exists for which $F(x,t) \leqslant 0$ for each $x \in [0,1]$ and every t in this neighborhood.

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