On topological classification of complete linear metric spaces

by

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§ 0. Introduction. In some mathematical theories we deal with objects having two structures: the algebraical and the topological one. Objects of this sort are, for instance, linear topological spaces. For such spaces one may consider three types of classifications: One with respect to both structures (isomorphism classification), another algebraical and the third topological. Only the first of these classifications has been performed completely. It is known that the only algebraic invariant of linear spaces is their algebraic dimension. The present paper is devoted to the third classification of complete linear metric spaces, above

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all of those spaces which are locally convex. One of the reasons of the significance of this subject is that there exist applications of non-linear functional analysis of purely topological character in which the linear structure of underlying spaces is quite immaterial.

Fréchet ([27], p. 95) and Banach ([3], p. 242) asked if every infinite-dimensional separable Banach space was homeomorphic to the Hilbert space. Recent results due to Kadeč, Klee, Besaga and Pelczynski suggest an affirmative answer to this problem. From the hypothesis that all separable infinite-dimensional Banach spaces are homeomorphic it can be inferred that all separable infinite-dimensional $F$-spaces, except perhaps $\mathbb{R}^n$, $n$ denoting the real line, are homeomorphic. Concerning unseparable $F$-spaces we know only that some of them (for instance certain spaces of continuous functions) are homeomorphic to Hilbert spaces of a suitable density character. No proof of topological non-equivalence of two complete linear metric spaces having the same density character is known. Hence at the present stage the study of topological classification of complete linear metric spaces aims establishing that some spaces or some classes of spaces are homeomorphic to Hilbert spaces.

A natural problem which arises is to characterize the topological structure of Hilbert spaces in purely topological terms. One of the possible approaches to this problem is the following [(4)]. Let us call a Keller retract any complete metric space which is an absolute retract relative to metric spaces and is topologically homogeneous. The following conjecture may be posed:

**Conjecture.** Any Keller retract which is not locally compact is homeomorphic to a Hilbert space of a suitable density character. Any locally compact Keller retract is homeomorphic to a locally compact convex subset of $l_1$ (all such subsets are topologically described in Klee [49]).

No results contradicting this conjecture are known to the author. However, to prove its truth seems extremely difficult.

There are several methods of establishing a homeomorphism between linear topological spaces:

1. **Direct method** ([Mazur, [63]]) Consider a square integrable function $x = x(t)$ defined on $[0, 1]$. Let

$$y(t) = |x(t)|^{\infty} \text{sgn} x(t).$$

It is easily seen that the mapping $y$ which sends $x$ to $y$ is one-to-one from $l_1$ onto $L_p$. It is also easy to check that $h$ is a homeomorphism. Similarly any space $l_2$ is homeomorphic with $l_1$. Hence all the spaces $l_p$ and $L_p$ are homeomorphic with the Hilbert space. This is historically the first example, due to Mazur in 1929, of non-isomorphic Banach spaces which are homeomorphic.

For the applications of the direct method in case of Orlicz spaces and "$L_p$-spaces" of vector-valued functions, see Kaczmarz [35], Stone [81], Bourbaki [16], § 6 Exercice 18. Another application: Proposition 7.2 of this paper.

2. **Coordinate methods.** The discovery of those methods by Kadeč in 1953 and 1956 marked essential progress towards solving the problem of Fréchet and Banach. The first of these methods is applicable to spaces $X$ possessing an increasing system $(X_n)$ of finite-dimensional subspaces with the so-called "Bernstein property". To any vector $x$ of such a space a sequence $a_nx$ of its "coordinates" may be attached in such a way that $\{a_nx : x \in X\}$ coincides with the class of all real sequences whose absolute values monotonically tend to zero, and that the mapping $h$ between two such spaces, with $a_nhx = a_nx (n = 0, 1, ...)$, is a homeomorphism. The coordinates are inclinations of the vector from subspaces $X_n (n = 0, 1, ...)$.

By the use of this method Kadeč [36] proved the non-trivial fact that $\gamma = 1$ (cf. 7.4). Klee and Long [56] proved that all infinite-dimensional normed linear spaces are homeomorphic. Besaga [6] extended this result to some more general classes of linear metric spaces. The special case of the first coordinate method, concerning spaces with an unconditional basis, is discussed in § 7.

The second method (Kadeč [37], [38], [39], Klee [51]) is in a certain sense dual to the first: coordinates are inclinations from subspaces having finite deficiencies. The most important result obtained by this method is that all infinite-dimensional separable conjugate Banach spaces are homeomorphic to the Hilbert space (Kadeč [39] and Klee [51]).

Another coordinate method was employed recently by Kadeč [40] for the proof that in every infinite-dimensional separable Banach space with an unconditional basis, the positive cone, with respect to this basis, is homeomorphic to the positive cone in $l_1$.

3. **Decomposition method** (Besaga-Pelczynski [8], [9]). Homeomorphisms are constructed by means of decompositions of spaces under consideration into Cartesian products. The general idea of this method is a simple abstract algebraic scheme: Propositions 8.3 and 12.1, cf. also Pelczynski [70], Proposition 4; an essential role is also played by the Barlow-Graves theorem on cross-section for linear operators. Similar methods were employed previously by Borůvka [16], Pełczyński [69], Kadeč-Lewin [41].

By the use of the decomposition method we can show that if $Y$ is an $F$-space homeomorphic with the Hilbert space $l_2(\mathbb{R})$, then every
I. Preliminaries

§ 1. Notation and terminology. The symbols $\cup$, $\cap$, $\ldots$ will be used for set union, intersection and difference, $+$, $-$, $\ldots$ being reserved for algebraic operations: both for vectors and sets in linear spaces. For instance, if $t$ is a number, $y$ is a vector in a linear space $X$ and $A$, $B \subset X$, then $tA = \{tx : x \in A\}$, $A + B = \{a + b : a \in A, b \in B\}$, $A - y = \{a - y : a \in A\}$, etc. The empty set will be denoted by $\emptyset$ and $0$ will represent the number zero and the neutral element of the linear space under consideration. card $A = \text{cardinality of the set } A$. If $A$ is a finite set then $\text{card } A = n$, where $n$ is the cardinality of the set $A$.

The purpose of this paper is to present the more important known results concerning topology and classification of complete linear metric spaces with special emphasis on the decomposition method. There are, however, some exceptions to this: A large part of the considerations of Chapter II is valid for incomplete spaces, too. In §10 using the decomposition method, we establish homeomorphisms between some non-linear spaces. In §11 the classification of $F$-spaces considered as uniform spaces is studied.

A more comprehensive discussion of the subjects omitted from this paper can be found in Klee’s expository paper [54] and in the papers listed in the reference section. For the classification of compact and locally compact convex subsets in $F$-spaces, see Keller [45] and Klee [49]. The classification of closed convex bodies can be found in Klee [44], Corson-Klee [19] and Bessaga-Klee [7], [82]. Cones are studied in Klee-Klee-Klee [18] and Kadeč [40]. An essential progress in the field of uniform structure of Banach spaces and related classification has been made by Lindenstrauss; see his recent paper [62].

In the present paper nineteen open problems are mentioned. These problems were partially published by Klee [53], Poljak-Polczyński [72], Bessaga-Polczyński [10]; some of them were posed in the report On Keller’s retract and topological classification of linear metric spaces during the Congress of Soviet Mathematicians in Leningrad in 1961.

I would like to express my gratitude to Professor Victor Klee; his seminars, his papers and discussion with him greatly influenced the present paper. I want also to express my thanks to Dr. A. Poljak-Polczyński (who is a co-author of a great part of the results of this paper) for his valuable remarks during the preparation of the paper. I am very much indebted to Professor M. I. Kadeč, who communicated to me the unpublished part of the proof of the theorem on topological equivalence of spaces $\mathfrak{a}$ and $I$ (the proof of the lemma $(*)$, p. 269) and gave his consent to my publishing it here.
spaces $X$ and $Y$ are homeomorphic; $Y|X$ if there exists a topological space $Z$, such that $X \approx Y \times Z$, $X$ is then said to be divisible by $Y$.

Unless otherwise explicitly stated, all the linear spaces considered in this paper are over the field of reals. However, all the proofs can be automatically transferred to the complex case.

A real-valued function $||_e$ defined on a linear space $X$ is called a norm if it satisfies the following conditions: (1) $0 = 0$, (2) $||x + y|| \leq ||x|| + ||y||$ for $x, y \in X$, (3) $||\lambda x|| = ||\lambda|| ||x||$ for $x \in X$, $\lambda \in \mathbb{R}$, (4) $||x|| = 0$ implies $x = 0$; $||_e$ is called a distance function (or $F$-norm), or a pseudonorm if it satisfies the conditions (1), (2), (4) or the conditions (1), (3), (3), respectively. A topological space which is simultaneously a linear space with linear operations continuous in the topology is called a linear topological space (l.t.s.) By a subspace of an l.t.s. $X$ we mean a closed linear manifold in $X$. Two l.t.s.'s $X$ and $Y$ are called isomorphic (written $X = X'$) if there is a linear homeomorphism (isomorphism) between these spaces. An $F$-norm $||_e$ in particular a norm defined on an l.t.s. $X$, will be called admissible, iff $f(x, y) = ||x|| - ||y||$ is a metric compatible with the topology of $X$; $f(x, y)$ is called complete iff the metric $f$ is complete. A sequence of pseudonorms $(||_e||)$ in $X$ will be called admissible iff $||_e|| \leq ||_f|| \leq \cdots$ for every $x \in X$ and the $F$-norm $||_e||$

$|a| = \sum_{k=1}^{\infty} 2^{-m} \langle a\rangle_{(1 + |a|)}$

is admissible. An l.t.s. $X$ with a fixed admissible $F$-norm $||_e||$, in particular with a fixed norm $||_e||$, is called a linear metric space (l.m.s.), in particular a normed space. If this fixed $F$-norm $||_e||$ or norm $||_e||$ is complete, $X$ is said to be a complete linear metric space or a Banach space (shortly B-space), respectively. An l.t.s. $X$ in which no admissible norm can be defined is said to be non-normable. An l.t.s. $X$ with a fixed admissible sequence of pseudonorms $(||_e||)$ such that the $F$-norm $||_e||$ is complete is called an $F$-space (or $E_{\infty}$-space in the terminology of Marczewski [66]). Obviously, every $F$-space is a B-space, with all the pseudonorms coinciding with the norm. Any convex set with a non-empty interior in an l.t.s. $X$ is called a convex body.

By a linear operator (linear functional) we mean a continuous linear vector-valued (scalar-valued) function defined on a linear topological space. If any mapping is denoted by a capital letter, it will be assumed to be a linear operator or functional. If $X$ is an l.t.s., $X^*$ will denote its conjugate space, i.e. the space of all linear functionals defined on $X$, equipped with its strong topology (Bourbaki [1], IV, 3.1). The conjugate of a normed space $X$ is a Banach space under the (admissible) norm $\|F\| = \sup \{ Fx : \|x\| = 1 \}$. $X$ is called reflexive iff every linear functional $\phi$ defined on $X^*$ is of the form $\phi x = Fx_0$, $x_0 \in X$. If $X$ is an l.t.s., $F : X^* \to X$ is called hyperplanes in $X$.

A series $\sum a_n x_n$ with $a_n$ in an l.t.s. $X$ is said convergent to $x$ iff $\lim_{n \to \infty} \sum a_n x_n = x$; it is said to be unconditionally convergent iff for every permutation $(\varphi_n)$ of indices the series $\sum a_n x_n$ is convergent. A sequence $(x_n)$ is called a basis of $X$ iff every $x \in X$ is uniquely representable in a form $x = \sum_{n=1}^{\infty} a_n x_n$; a system $(x_1, x_2, \ldots)$ is said to be an unconditional basis iff every $x \in X$ has a unique (up to permutation of terms) representation $x = \sum_{n=1}^{\infty} a_n x_n$, the series being unconditionally convergent and $A_2 \subset A_1$ with $card A_2 \approx k_2$. A basis $(x_n)$ will be called primitive iff $\sum a_n x_n$ converges for every real sequence $(a_n)$.

§ 2. List of special spaces. The spaces $L_p(s)$ ($p > 0$). Elements of the space $L_p(s)$ are real functions $x = (\xi_i)$ defined on a abstract set $A$, with card $A = k$, such that card $\{ \xi_i : \xi_i \neq 0 \leq k \}$ and $\sum |\xi_i|^p < \infty$, the norm being defined by the formula $|x| = \left( \sum |\xi_i|^p \right)^{1/p}$, where $\gamma = \min(1, 1/p)$.

The space $C_0(s)$ consists of real functions $x = (\xi_i)$ defined on a set $A$, with card $A = k$, such that card $\{ \xi_i : \xi_i > 0 \leq k \}$ for every $\delta > 0$. Here $|x| = \sup |\xi_i|$.

The spaces $L_1(s)$ and $C_0(s)$ are shortly denoted by $L_1$ and $C_0$. Instead of $l_1(s)$ and $l_\infty$ we also write $l(s)$ and $l$.

The spaces $L_\infty(s)$ are measurable functions $x = x(t)$ defined on $A$, such that $\int |x(t)|^p d\gamma = < + \infty$.

The norm, $|x| = \left( \int |x(t)|^p \right)^{1/p}$, where $\gamma = \min(1, 1/p)$.

The space $s$, which is composed of all real sequences $x = (s_n)$, under the topology given by the pseudonorms $|x| = \sup |s_n|$

The Köthe spaces ($\ell$). Let $(a_{m,n})_{m,n=1}^{\infty}$ be an infinite matrix of positive numbers such that

$s_{a_1} \leq s_{a_2} \leq \cdots \leq s_{a_{m,n+1}} \leq \infty \quad (n = 1, 2, \ldots)$

By $M(s_{a})$ we shall denote the space of all real sequences $x = (s_n)$, such that $|s_n| = \sup \{a_{m,n}|s_n| \leq \infty$.

(*) The class of Köthe spaces coincides with that of nuclear "Stufenräume" — Köthe [57] and [58] § 30.9 (see Grothendieck [52], Chapter II, p. 59, Proposition 8).

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The space \( \mathcal{M}(A) \) of all real bounded functions \( x \in (\mathcal{A}) \) defined on an abstract set \( \mathcal{A} \); \( \|x\| = \sup \|x_i\| \).

The spaces \( L^p(p) \) and \( L^p(p, n) \) (for \( p \leq 1 \)) are complete linear metric spaces without being F-spaces. \( M(\mathcal{A}) \) and \( \mathcal{X} \) are non-normable F-spaces. All the other spaces listed above are Banach spaces.

\section{3. Auxiliary theorems on linear functional analysis}

3.1. Suppose that \( X \) is a complete l.m.s. and \( \mathcal{U} \subset X \times X \) \((n = 0, 1, \ldots)\) are linear operators, with \( \lim_{n \to \infty} \mathcal{U}_n \mathcal{X} = \mathcal{U}_0 \mathcal{X}\). Then the convergence is uniform on any compact subset of \( X \).

3.2. Let \( X \) and \( Y \) be complete linear metric spaces. If \( T \) is a linear operator from \( X \) onto \( Y \), then \( T \) is isomorphic to the quotient space \( X / \ker T \), where \( \ker T = T^{-1} \).

This is an immediate consequence of the well-known closed graph theorem.

3.3 (Schauder). Every Banach space \( X \), with \( \mathcal{M}(X) \subset \mathcal{X} \), is a linear image of the space \( \mathcal{I}(\mathcal{A}) \).

Let \( \{x_i\} \) be an arbitrary system of points of \( X \) dense in the unit sphere of \( X \). The required linear operator \( \mathcal{T} \) from \( \mathcal{I}(\mathcal{A}) \) onto \( X \) can be defined by the formula \( \mathcal{T}([x]) = \sum_i \lambda_i x_i \) (see Banach-Mazur [3], p. 111 and Klee [31], Prop. 2.1).

3.4. Every F-space \( X \) is isomorphic to a subspace of the F-space \( \mathcal{Y} = \int_0^1 X_1 \), where \( X_1 \) is the completion of the quotient space \( Z = X / \{x \in X : x = 0\} \) considered as a normed space under the norm \( \|\cdot\| \).

\section{Icmc}

3.7. If \( X \) is a Banach space such that \( X \times \mathcal{X} \) is reflexive, then \( X \) is reflexive.

This follows directly from the fact that every \( F \in (X \times \mathcal{X}) \) has the unique representation \( F(x, y) = F(x, y) + F(y, x) \). For \( F(x, y), F(y, x) \in \mathcal{X} \).

3.8. If \( X \) is a B-space whose conjugate \( X^* \) contains a reflexive subspace \( Y \), then there exists a linear operator \( U \) from \( X \) onto \( Y \).

\textbf{Proof.} Define \( (Ux)F = Fx \) for \( F \in \mathcal{Y} \). Since the adjoint operator \( U^* \) with \( (U^*F)F = Fx \) is the embedding operator and has, evidently, the continuous inverse, by Banach [2], p. 146, Theorem 1, \( U \) is onto \( Y \).

3.9. If \( X \) is a non-reflexive B-space with an unconditional basis, then \( X \) contains either a subspace isomorphic with \( c_0 \) or a subspace isomorphic with 1 (James [34]).

3.10. For every infinite-dimensional F-space \( X \) the condition (a) \( \dim_\mathcal{X} \exists x \in \mathcal{X} \) \( \exists x \in \mathcal{X} \) \( \|x\| = 0 \) is either 0 or \( \infty \), for sufficiently large \( \alpha \), implies the following one:

(b) \( X \) is isomorphic to no Cartesian product \( \mathcal{Y} \times \mathcal{Z} \), \( Y \) being a B-space.

If \( X \) is non-normable, then (b) is equivalent to the condition (c) \( X \) contains a subspace isomorphic to a K"othe space.

\textbf{Proof.} Non (b) implies non (a). Let \( x = (y, (\{x_i\}) \in \mathcal{Y} \times \mathcal{X} \). The special system of admissible pseudonorms \( \|x\| = \|y\| + \max \|x_i\| \) does not satisfy (a). But from 3.5 it follows that no admissible system of pseudonorms satisfies (a).

For the proof of the second assertion, see Bessaga-Pelczyński-Rolewicz [13].

3.11. If \( \{x_n\} \) is a basis in an F-space \( X \), then every subsequence \( \{x_{n_k}\} \) forms a basis in its linear hull. If \( \{x_n\} \) is an unconditional basis, then \( \{x_{n_k}\} \) is also unconditional.

3.12. Let \( X \) be an F-space with a non-primitive basis \( \{x_n\} \). Then there is a subsequence \( \{x_{n_k}\} \) whose closed linear hull satisfies condition (a) of 3.10.

\textbf{Proof.} (cf. Bessaga-Pelczyński-Rolewicz [14], proof of Theorem 1).

Among the pseudonorms \( \|\cdot\| \) of the space \( X \) there is one \( \|\cdot\| \) such that \( \|x_n\| = 0 \) for infinitely many \( n \), say \( p_1, p_2, \ldots \), because otherwise the series \( \sum_k p_n \|x_n\| \) would be convergent for any real sequence \( \{p_n\} \). Put \( Y = \text{closed linear hull of} \{x_{n_k}\} \).

By 3.11, every \( y \in Y \) can be represented in a form \( y = \sum_{n=1}^\infty a_n x_n \). The formulas

\[ \left| \sum_{n=1}^\infty a_n x_n \right| = \sup_{\sum_{n=1}^\infty |a_n|} \left| \sum_{n=1}^\infty a_n x_n \right| \]
define an admissible system of pseudonorms on $Y$. Since all these pseudonorms are norms, condition (a) is satisfied.

3.13. Every $F$-space with a primitive basis $(x_\mu)$ is isomorphic to the space $s$.

Proof. The required isomorphism is given by the formula

$$T \sum_{\mu \in A} \lambda \mu x_\mu = (x_\mu).$$

Let us notice that, conversely, every basis in $s$ is primitive (Banach [22], Chapter III, Theorem 13 and Besaga-Pelczynski-Roelcke [14], Theorem 5).

II. Infinite products of spaces and their mappings

§ 4. Coordinate spaces and coordinate products. Through this section $A$ is assumed to be a fixed abstract set. $\mathfrak{B} = (\mathfrak{B}_\mu)_\mu(A)$, $\mu \in A$, with $\mathfrak{B}_\mu = 0$ for $\mu \neq \mu$ and $\mathfrak{B}_\mu = 1$ for $\mu = \lambda$.

Definition 1. An l.m.s. $E$ whose elements are real-valued functions $x = (x_\mu)$ defined on $A$, satisfying the following conditions:

(a) If $x = (x_\mu) \in E$, $\gamma = (\gamma_\mu) \in w(A)$, then $\gamma x = (\gamma_\mu x_\mu) \subseteq \gamma \gamma x = |x| \gamma x$,

(b) Card supp $x < \kappa_1$ for every $x \in E$, where supp $x = \{ \mu : (x_\mu) \neq 0 \}$.

(c) Every $x \in E$ is a sum of the unconditionally convergent series $\sum \mathfrak{B}_\mu x_\mu$, where $\mu$ runs over supp $x$.

(d) For every $x \in E$, we have $\gamma \mathfrak{B}_\mu x_\mu$,

will be called a coordinate space. If $A = \{1, 2, \ldots\}$, the space $E$ will be called a coordinate coordinate space. We shall use also terms: coordinate $F$-space, coordinate normed space, etc. For any coordinate space $E$, by $E^*$ we shall denote the positive cone in $E$, i.e. the set of those elements in $E$ all coordinates of which are non-negative.

The class of coordinate $F$-spaces coincides, up to isomorphisms, with the class of $F$-spaces having an unconditional basis. Let us also observe that

4.1. Every coordinate space $E$ can be embedded in a complete coordinate space $E$ consisting of all such functions $y = (y_\mu)$ defined on $A$ for which card supp $y < \kappa_1$, and the series $\sum \mathfrak{B}_\mu y_\mu$ satisfies the Cauchy conditions as a series of elements of $E$.

The spaces $L_p(\mathfrak{B}_\mu)$, $Q_0(\mathfrak{B}_\mu)$, $M(\mathfrak{B}_\mu)$ are defined in section I.2 are complete coordinate spaces. As an example of an incomplete coordinate space we can consider the subset of $A$ consisting of all finite-supported sequences, or that consisting of all absolute summable sequences, etc.

Definition 2. Suppose that $E$ is a coordinate space and $(X_\mu)_\mu<\kappa_1$ is a family of normed spaces with norms $\| \cdot \|_\mu$, respectively. Denote by

$$\sum_{\mu < \kappa_1} X_\mu$$

the l.m.s. consisting of all systems $x = (x_\mu)$, with $x_\mu \in X_\mu$, such that $\|x_\mu\|_\mu \leq F_\mu$, with the $F$-norm $\|x_\mu\| = \|x_\mu\|_\mu$. In the case where all $X_\mu$ coincide with a fixed space $X$, we shall write $\sum_{\mu} X_\mu$ instead of $\sum_{\mu < \kappa_1} X_\mu$.

The space $\sum_{\mu} X_\mu$ will be called a coordinate product of the spaces $X_\mu$ in the sense of $E$ (f); $X_\mu$ and $E$ will be called the factors of the product and its generating space, respectively. Products corresponding to countable-coordinate generating spaces are said to be countable.

If $E$ is an $F$-space and $X_\mu$ are $F$-spaces, then $Y = \sum_{\mu} X_\mu$ is an $F$-space, the pseudonorms $\|x_\mu\|_\mu = \|x_\mu\|_\mu$, are admissible for $Y$.

Let us mention that for $X$ which is a $B$-space the spaces $\sum_{\mu} X_\mu$, $\sum_{\mu} X_\mu$, $\sum_{\mu} X_\mu$, $\sum_{\mu} X_\mu$, $\sum_{\mu} X_\mu$ are isomorphic with the tensor products: $X \otimes I$, $X \otimes I$, $X \otimes I$, $X \otimes I$, $X \otimes I$, respectively (for the definitions of tensor products and related notions, see Grothendieck [26]).

Let $Y = \sum_{\mu} X_\mu$. We shall employ the following notation: if $y = (y_\mu)$ is in $Y$, then $m y = (m y_\mu)_\mu E^*$, $s y = (s y_\mu)_\mu E^*$, where $m = s = 0$ if $s \neq 0$ and $s = 0$ if $s \neq 0$. For an arbitrary system $y = (x_\mu)$ with $x_\mu \in X_\mu$ (not necessarily belonging to $Y$) we write $F y = m y$. In the case where $A = \{1, 2, \ldots\}$, we shall use also the following notation: $S y = \sum_{\mu} S y_\mu$, $S^* y = \sum_{\mu} S^* y_\mu$, $S^* y = -S^* y$. It is easy to show that

$$S y = \sum_{\mu} S y_\mu,$$

is the function $S^* y = S^* y_\mu$, $S^* y = -S^* y$. For every $y \in \sum_{\mu} E_\mu$ (E being the completion of $E$ defined in 4.1) we have $\lim_{\mu \to \infty} S^* y = y$.

Since every coordinate space is a coordinate product with $X_\mu = E_\mu$, all the notations and the last proposition are valid for coordinate spaces.

§ 5. Direct sums

Definition 3. A complete l.m.s. $X$ will be called a direct sum (d.s.) of its subspaces $X_n$ ($n = 1, 2, \ldots$) if there exist continuous linear projections $T_n$ from $X$ onto $X_n$ such that

$$T_n T_m = 0 \quad \text{for} \quad n \neq m; \quad x = \sum_{\mu} T_n x_\mu, \quad \text{for any} \ x \in X.$$

(1) Coordinate products with special generating spaces were considered in Banach [22], p. 243, and denoted by $(X_1, X_2, \ldots)$; see also Day [36], Chapter 2, § 2.

(2) This notion of direct sum has been introduced by Grothendieck [26]. It was studied by S. Mazur in his seminar in 1955 and independently by W. McArthur and his students.
X will be called an unconditional direct sum (u.d.s.) of a family \((X_i)_{i \in I}\) of its subspaces if there are continuous linear projections \((T_i)\) from \(X\) onto \(X_i\), respectively, such that

\[(6) \quad T_i T_j = 0 \quad \text{for} \quad i \neq j; \quad \text{the cardinality of the set} \quad \text{supp} x = \{i : T_i x \neq 0\} \quad \text{is} \quad \leq k_x \quad \text{for every} \quad x \in X; \quad \sum \text{supp} x, \quad \text{the series being unconditionally convergent.}

Let us remark that the class of direct sums (unconditional direct sums) of one-dimensional subspaces coincides with the class of spaces having bases (unconditional bases).

5.1. Theorem. If \(X\) is an u.d.s. of its subspaces \(X_i (i \in A)\), \(Y\) is an l.d.s., \(h : X_i \to Y\) are continuous and such that \(h_0 = 0\) for every \(i\), and for every \(x\) in \(X\) the series \(\sum_{i} h_i x = h x\) is unconditionally convergent, then the mapping \(h\) is continuous.

The proof makes use of the following proposition,

5.2. Let \(X\) be a d.s. of its subspaces \(X_n\) and let \(h : X_n \to Y\) be continuous and such that \(h_n = 0\) for all \(n\) and \(x = \sum \text{supp} x\) is convergent for every \(x\) in \(X\). Then \(h\) is continuous.

Proof of 5.2. Put

\[g_n x = \sum_{i=1}^{n} h_i T_i x.\]

The functions \(g_n\) are continuous as finite sums of continuous functions.

We are going to show that they are equiregular at \(0\). Suppose otherwise; then one can choose a neighbourhood \(U\) of zero in \(Y\), vectors \(x_n \in X\) and increasing sequence \((k_n)\) of indices, such that

(a) \[\left| \sum_{i=1}^{k_n} T_i x_n \right| < 2^{-n} \quad \text{for} \quad p < k_n,
\]

(b) \[g_{k_n} x_n \in V; \quad (g_{k_n} x_n + V) \cap V = \emptyset.\]

Put

\[c_n = \sum_{i=k_n+1}^{\infty} T_i x_n.\]

Since \(X\) is complete, condition (a) implies that the series \(\sum c_n - c\) is convergent. Since according to the hypothesis the series \(\sum_{n} h_n T_n x\) converges, we get

\[y = \sum_{n=k}^{\infty} h_n T_n x \to 0.\]

But

\[\sum_{n=k}^{\infty} h_i T_i x = \sum_{n=k}^{\infty} h_i T_i c_n = \sum_{n=k}^{\infty} h_i T_i c_n = g_{k_n} x_n - g_{k_n+1} x_n = 0,
\]

and by (b), \(y_n \in V\), which is contradictory to \(y_n \to 0\).

We have just proved that for every neighbourhood \(V\) of zero in \(Y\) there is a number \(\delta(V) > 0\) such that

(c) \[\text{if} \quad x \in X, \quad |x| < \delta(V), \quad \text{then} \quad y_n x \in V.
\]

Suppose \(x_n \to 0\) in \(X\). Let \(V\) be a neighbourhood of \(0\) in \(Y\). Put

\[S^2 = \sum_{i=1}^{n} T_i x_n, \quad S^2 x = x - S^2 x.
\]

By Definition 3, \(S^2 x \to 0\), by 3.1, this convergence is uniform on the compact set \(\{x_n, x_{n+1}, \ldots\}\); therefore there exists an index \(k_n\) such that

(d) \[|S^2 - S^2 x| < \delta(V) \quad \text{for} \quad k = 0, 1, \ldots
\]

By (c) and (d), we have

\[h_{k_n} x_n = h_{k_n} x_n - h_{k_n} x_n + (g_{k_n} x_n - g_{k_n} x_n) \in V + V + (g_{k_n} x_n - g_{k_n} x_n).
\]

Since \(g_{k_n}\) is continuous, for sufficiently large \(k\) we have \(g_{k_n} x_n - g_{k_n} x_n \in V\); thus, for such \(k\), \(h_{k_n} x_n \in h_{k_n} x_n + V + V\), which means that \(h_{k_n}\) is continuous.

Proof of the Theorem. Let \(Y\) be a separable subspace of \(X\). Let \(A_n = \bigcup \text{supp} x_i\). From the separability of \(Y\) it follows that the set \(A_n\) at most countable, say \(A_n = \{a_1, a_2, \ldots\}\). Thus \(Y\) is contained in a d.s. of spaces \(X_{a_1}, X_{a_2}, \ldots\) and by 5.2, \(h_i\) is continuous on \(Y\), but a function which is continuous on every separable subspace must be continuous on the whole space.

Remark 1. An u.d.s. \(X\) of its subspaces which is an F-space can be defined as a module over the space \(w(A)\) such that: the operation of multiplication by elements of \(w(A)\) is bilinear over \(w(A) \times X\), if \(A_n\) are disjoint subspaces of \(A\), and \(x_{a_n}\) are its characteristic functions, then \(x_n = \sum x_{a_n}\) (in norm convergence) for any \(x \in X\). Here we have \(T_i x = x_{a_i} x_n T_i x\).

Such an approach suggests an idea of generalization of the concepts considered to the case in which the role of \(w(A)\) would be played by a space of all bounded measurable functions defined on a space with a measure.

§ 6. Coordinate and cylindrical mappings. In this section we shall discuss the question what sorts of continuous mappings (resp. homeomorphisms) of generating spaces and of factors determine continuous mappings (resp. homeomorphisms) of coordinate products.
By 5.1, we have:
6.1. Suppose that $E_i$ and $E_4$ are coordinate spaces with common $A$, $X_i, Y_i$ ($i \leq A$) are Banach spaces, and $X = \sum x_iX_i$, $Y = \sum x_iY_i$.
If $h$: $X_i \rightarrow Y_i$ are continuous and such that $h(0, 0) = 0$ and $(h, F_E) \in \Gamma$ for any $x \in X$, then the mapping $k$: $X \rightarrow Y$ with $k = (h, F_E)$ is continuous.
If, moreover, $h_i$ are homeomorphisms and $h$ is onto $Y$, then $h$ is a homeomorphism from $X$ onto $Y$.

Definition 4. Suppose that $Y_i = \sum x_iX_i$ ($i = 1, 2$) are coordinate products. Any mapping $g$: $Y_1 \rightarrow Y_2$ (in particular $g$: $E_i = \sum x_iX_i \rightarrow E_j = \sum x_jX_j$, or $g$: $E_1 \rightarrow E_j$) fulfilling the conditions:
(We) superpositions $F_kg$ are continuous and such that $F_ky = 0$ implies $F_kg = 0$ for any $k \leq A$;
(W2) $F_kg = F_kg$ for every $y \in Y_i$, $x \in X_k$,
is called a coordinate mapping.

If $Y_i$ are countable-coordinate products, then any mapping $g$: $Y_1 \rightarrow Y_2$, or $g$: $E_1 \rightarrow E_2$ satisfying condition (We) and the following one:
(We) there exists a homeomorphism $g$ from $R^+$ onto itself such that $[s_g, g] < \infty$, $y \in Y_i$, $x = \sum x_iX_i$,
will be called a cylindrical mapping.

A homeomorphism $h$ from $X_1$ onto $Y_2$ such that both $h$ and $h^{-1}$ are coordinate (resp. cylindrical) mappings will be called a coordinate (resp. cylindrical) homeomorphism.

6.2. Suppose that $E_1$ and $E_2$ are coordinate spaces with a common $A$, and $X_i$ ($i \leq A$) are normed spaces. If $h$: $E_1 \rightarrow E_2$ is a coordinate (cylindrical) mapping, then $k$, with $hy = s_{g(x)} - h(m(x))$, is a coordinate (resp. cylindrical) mapping from $Y_1$ onto $Y_2$. If $h$ is one-to-one and onto $E_1$, then $h$ is one-to-one and onto $Y_1$.

Proof. (i) We have $F_kh = (F_kg)(y)(F_kh \mod y)$.

If $h$ satisfies (We), from (a) and 4.2, we see that $F_kh$ is continuous at all points $y$, for which $F_ky \neq 0$. But if $F_ky = 0$ and $y \rightarrow y$, we have again $F_kh = (F_kg)(y)(F_kh \mod y) = F_kh = 0$, because the sequence of the first factors is bounded and the second factors tend to zero.

(ii) Suppose that $h$ satisfies (W2). Then $F_kh = (s_{g(x)} - h(m(x)) \mod y) = (s_{g(x)} - h(m(x)) \mod y) = hhy$.

(iii) Suppose that $h$ satisfies (We). By 4.2, we have $[s_yh_k] = [s_yh_k] = \sup [s_g, g_y, k]$.

The second statement of the proposition is an obvious consequence of the first.

6.3. Suppose that $E_i$ ($i = 1, 2$) are complete coordinate spaces, $X_i$ ($i \leq A$) are Banach spaces, $Y = \sum x_iX_i$. Then every coordinate mapping $h$: $X_1 \rightarrow X_2$ or $h$: $E_1 \rightarrow E_2$ is continuous.

Proof. The continuity of $h$ in the case $h$: $Y_1 \rightarrow Y_2$ follows directly from 6.1. Assume $h$: $E_1 \rightarrow E_2$; then the mapping $k$: $\sum x_iR \rightarrow \sum x_iE$ is by 6.2 a coordinate mapping again, and therefore it is continuous. But it is easy to see that in this case $k$ is the restriction of $h$ to $E_i$, whence it is also continuous.

6.4. Let $Y_i = \sum x_iX_i$ ($i = 1, 2$) be two countable-coordinate products.

Any cylindrical mapping $h$: $Y_1 \rightarrow Y_2$ is continuous.

Lemma 1. Suppose $Y = \sum x_iX_i$ and $y_k \in Y$ ($k = 0, 1, ...$). Then $y_k \rightarrow y_k$, iff the following conditions are satisfied:
(a) $F_ky_1 \rightarrow E_ky_2$ ($k = 1, 2, ...$),
(b) $\lim \sup [s_h, y_k] = 0$.

Proof of Lemma 1. The necessity of (a) is obvious. Since $\lim s_ky = 0$, for every $y$ belonging to the complete l.m.s. $\sum x_iX_i$, the tilda denotes the completion defined in 4.1), by 3.1, this convergence is uniform on $\langle y_k, y \rangle$. But this means that condition (b) is necessary.

The sufficiency. Given $\varepsilon > 0$. By (b) there exists an $N$, such that $[y_k - y] < \sum x_iy_i + [s_ky] + [s_ky] < \varepsilon + \varepsilon + \sum x_ky_k - x_ky_k < 2\varepsilon$, for sufficiently large $k$.

Proof of 6.4. If $y_k \rightarrow y$, then by Lemma 1, $F_ky_1 \rightarrow E_ky_2$ and $\sup [s_h, y_k] \rightarrow 0$; but by (We) and (W2), we have $F_kh_2y_2 \rightarrow E_kh_2y_2$ and $\sup [s_h, y_k] \rightarrow 0$, whence by Lemma 1, $h_ky_k \rightarrow h_y$.

6.5. Suppose $X_i = \sum x_iX_i$, $Y_i = \sum x_iZ_i$, $E$ being normed spaces.

If $h$: $X_i \rightarrow Z_i$ are continuous and such that $[s_h, a] \leq A[\|x\|]$, for some constant $A$, for all $x \in X_i$ and all $i \leq A$, then the mapping $h$: $Y_i \rightarrow Y_i$, with $h(x) = (y, a)$ is continuous. If $h_i$ are homeomorphisms from $Y_i$ onto $Z_i$ such that $[\|x\|] \leq \sum x_iX_i \leq A[\|x\|]$, then $h_i$ is an homeomorphism.

Proof. Begin with the case where $Y_i$ and $Y_i$ are countable coordinate products. It is obvious that $h$ fulfills condition (We). On the other
hand, by the definition of $h$, by condition (d) of Definition 1 and by
the homogeneity of norm in $E$, we have

$$[h_{a}h_{b}] = [h_{a} \mod y_{a}] \leq \Delta[y_{a}] \mod y_{a} = \Delta[y_{a}y_{a}];$$

thus $h$ fulfills also (V3), and by 6.4, $h$ must be continuous.

The general case of products of arbitrary power can be reduced
to the case of countable products as follows: Every sequence $(y_{a})$ in $Y$
may be embedded in a countable coordinate product corresponding to
those indices $a$ for which $\sum_{a} 1 \leq 2^{\|y_{a}\|} \neq 0$; therefore, using the countable
version of the proposition, we conclude that the convergence of $(y_{a})$
implies that of $(h_{a})$.

The second statement of this proposition follows directly from the
first one.

In the case of coordinate products with complete factors the
assumption that $E$ is a normed space can be omitted in the formulation of
3.5. Indeed, the mappings $h_{a}$, with $[h_{a}a] \leq A[\|a\|]$ determine a mapping
$h: E_{\mathbb{N}} \rightarrow E_{\mathbb{N}}$ between complete spaces which, according to 3.1, is con-
tinuous. Hence the mapping $h$ which is a restriction of $h$ to $E_{\mathbb{N}}$ is also
continuous.

In particular, we have:

6.6. Suppose that $X$ and $Z$ are Banach spaces, and $E$ is a coordinate
space. If there exists a homeomorphism $h$ from $X$ onto $Z$, with the property
$|h| = |a|$, then the formula $h(a): h(a)$ defines a homeomorphism from
$E_{\mathbb{N}} X$ onto $E_{\mathbb{N}} Z$.

From 3.6 follows:

6.7. Let $X$, $Z$ be Banach spaces, and let $E$ be a coordinate
space. If $X \approx Z$, then $\exists \exists E_{\mathbb{N}} X \approx E_{\mathbb{N}} Z$.

Proof. According to Klee [48], the unit sphere of an arbitrary
infinite-dimensional Banach space is homeomorphic with the hyperplane
(subspace of deficiency one) of this space. Hence $X \approx Z$ implies that
the unit sphere in the spaces $E \times X$ and $E \times Z$ are homeomorphic. Let
$x$ be a homeomorphism from the unit sphere of the space $E \times X$ onto
that of $E \times Z$; then the formula $h(a) = [a] \cdot \varphi(a)$ defines a homeomorphism from
$E_{\mathbb{N}} X \times E_{\mathbb{N}} Z$ onto $E_{\mathbb{N}} Y$. Thus by 6.6,

$$\Sigma E_{\mathbb{N}} (X \times R) \approx \Sigma E_{\mathbb{N}} (Y \times R), \text{ i.e. } E_{\mathbb{N}} X \approx E_{\mathbb{N}} Z.$$

§ 7. Applications.

7.1. The mapping $h(x_{n}) = \Sigma \xi_{n} \cdot \xi_{n}^{*}$ is a coordinate homeo-
morphism from $E_{\mathbb{N}}(X)$ onto $E_{\mathbb{N}}(Y)$ ($p > 0$). (Marczewski [65].)

This is an immediate consequence of 6.3.

7.2. For every Köthe space $M(a_{n})$ there is a coordinate homeo-
morphism from $c_{0}$ onto $M(a_{n})$. (Bessaga-Pelczynski [9], Lemma 3; cf.
Rog Tclis [76].)

Proof. Without loss of generality we may assume that the matrix
$a_{n}$ satisfies the additional condition

$$x^{*} \lim_{n \rightarrow \infty} a_{n} (x) = \infty \text{ for } n = 1, 2, \ldots.$$

(for every matrix satisfying (x) of section 5.2, we can make
(x) by multiplying the rows of the matrix by some positive constants;
obviously such an operation does not change the topology in $M(a_{n})$).

From (x) it follows that there are homeomorphism $h_{a}$ from $R$ onto $R$
such that $h_{a}(1/a) = 1/a_{n}$ for $a = 1, 2, \ldots$, and $h(x)$ is defined by
the limit of $h_{a}(x)$ as $a$ approaches $x$. Condition (x) implies that
the space $M(a_{n})$ can be characterized as a class of sequences
$(y_{a})$ for which $|y_{a}| < 1/a_{n}$, for $a = 1, 2, \ldots$ and $n > N_{a}$. Hence
$(y_{a})$ is in $M(a_{n})$ if $\sum |y_{a}| < 1/a_{n}$, for $a = 1, 2, \ldots$ and $n > N_{a}$, i.e.
$(y_{a}) \in M(a_{n})$ iff $h_{a}^{-1}(y_{a}) \rightarrow 0$. The last condition means that the mapping
$h((a, b)) = (b, h_{b}(a))$ is the required coordinate homeomorphism from $c_{0}$ onto
$M(a_{n})$.

Let us remark that there is no coordinate homeomorphism from $c_{0}$
onto $l$. In fact: take $\exists \exists X \in c_{0}$, $X_{n} > 0$. Obviously, for every $N_{0}$,
$s_{n}, t_{n}, \ldots, t_{n} \in c_{0}$. Thus if there existed a coordinate homeomorphism $h((a, b))$
from $c_{0}$ onto $l$, we would have $\sum_{c_{0}} |h_{c_{0}}(a) X_{n} | = \infty$ and there would
exist an increasing sequence $(p_{n})$, with $p_{n} = 1$, of positive integers, such
that $\sum_{c_{0}} |h_{c_{0}}(a) X_{n} | > 1$. Take $(y_{n})$ in $c_{0}$, with $y_{n} = y_{n}$ for $p_{n} = 1$
and $y_{n} = y_{n}$ for $p_{n} > 1$; we now get $\sum_{c_{0}} |h_{c_{0}}(a) y_{n} | = \infty$, which leads to a contradiction.

Now we shall describe a method of construction of cylindrical
cylindrical homeo-
morphisms due to Kaděk. Let $E$ be a countable-coordinate $B$-space.
Consider its positive cone $E^{+}$. Put

$$d_{x}(x) = \|h_{a}x\|, \text{ for } x \in E^{+} (k = 0, 1, \ldots),$$

i.e. $d_{x}(x)$ denotes the distance between $x$ and the subspace spanned upon
the first $k$ unit vectors of $E$. Obviously $d_{x}(x)$ is non-increasing and tends
to zero (written $d_{x}(x) \rightarrow 0$) for every $x$ in $E^{+}$. One can prove (see Bern-
stein [5] and Kaděk [30]) that:

For every real sequence $(h_{a})$ with $h_{a} \rightarrow 0$, there exists an $x$ in $E^{+}$ such that

$$d_{x}(x) = 0 \quad (k = 0, 1, \ldots).$$

The space $E$ is said to have the Bernstein property if for every $(h_{a})$,
with $h_{a} \rightarrow 0$, there is in $E^{+}$ at most one point $x$ (i.e. exactly one point)
satisfying (a).
7.3. If spaces $E_1$ and $E_2$ have the Bernstein property, then the mapping $h$, which sends any $x \in E_1$ to the $y$ in $E_2$ such that $d_0x = dy$ ($k = 0, 1, ...$), is a cylindrical homeomorphism.

Proof (cf. Kadets [36], Klee-Long [56]). The Bernstein property ensures that $h$ is one-to-one and onto $E_2$. It is obvious that $\text{sgn} \, h = \text{sgn} \, e$ for $e \in E_2$, and that

$\|S_n h e\| = \|S_n e\|$ for every $e \in E_2$ ($n = 0, 1, ...$).

This means that $h$ satisfies condition (V$_n$).

Take $x \to x_0$ in $E_1$. By (b), $\sup_k \|h e\| < \infty$, and by Lemma 1,

$\sup_k \|h e\|, S^k e_0 \leq \sup_k \|h e\| = \|h e_0\| = 0$.

Hence the set $A = \langle h e_0, h e_1, ... \rangle$ is bounded and can be approximated by finite-dimensional linear sets $S^k e_0$; therefore it is totally bounded (precompact). Let $y$ be a cluster point of this set. Since $d_0 h e_0 = d_0 h e_1$ and $d_0$ are continuous, $d_0 y = d_0 e_0$ ($n = 0, 1, ...$), and, by the Bernstein property, $y = h e_0$ is the only cluster point of the set $A$. Thus $h$ is continuous and using 4.2, we infer that $h$ satisfies condition (V$_n$). The assumptions concerning $h$ and $h^{-1}$ being symmetrical, we infer that $h^{-1}$ is also continuous.

7.4 (Kadets [36]). There is a cylindrical homeomorphism between $E_1$ and $E_2$.

Proof. The space $I$ has the Bernstein property; if $x = \langle t e_0 \rangle$, then $x = (x_k) \in I^*$, then $t = t_0 = t_0 = t_0$. Thus, to complete the proof, it is enough to find in $c_0$

an admissible norm $\|\|$, under which this space has the Bernstein property. Let $\varphi(t) = \lim_n t e_0/t$ for $0 < t < 2$, and $(a_n)$ and $(c_n)$ be sequences of positive numbers such that

\[ \sum_{n=1}^{\infty} a_n < \infty, \quad \lim_{n \to \infty} 2^n \varphi(2^{-n} e_0) \leq \sum_{n=1}^{\infty} a_n = 0, \quad \sum_{n=1}^{\infty} c_n = 1. \]

Now for any $x = \langle t e_0 \rangle$ we put $\|x\| = \inf \{ t : 0 < t : \sum_{n=1}^{\infty} a_n \varphi(t e_0/n) \leq 1 \}$.

Since $\varphi(1) = 1$ and $\lim t e_0/t = \infty$, we get $\|x\| = \|x\|$ for every $x$ in $c_0$, i.e. the new norm $\|\|$ is admissible. The Bernstein property for the space $c_0$ equipped with this norm is, equivalent to the following statements:

(a) For an arbitrary sequence of positive numbers $a_n$, 0, the system of equations

$\sum_{n=m}^{\infty} a_n \varphi(t e_0/n) = 1 \quad m = 0, 1, ...$

has at most one solution $x = \langle t e_0 \rangle$ in $c_0$.

Proof of (a). Take a fixed $n$ and write the first $m$ equations in the reverse order:

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

(b)

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

(c)

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

(d)

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1,$

(e)

$\sum_{k=n}^{\infty} a_k \varphi(t e_0/k) = 1.$

Let $Z_n$ denote the set of those $x = \langle t e_0 \rangle$ which satisfy the equations (b). For any real-valued function $f$ defined on $c_0$ we shall write

$\Delta_n f = \sup \{ |f(x) - f(y)| : x, y \in Z_n \}.$

Since $a_n > 0$, the first of equations (b) gives us

$0 < \xi_k < 2^n a_{n+1}$

for $k > n$.

Thus, since the function $\varphi$ is increasing and the sequence $(a_n)$ does not increase, we have

$\Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k) \leq \sup \{ \sum_{k=0}^{\infty} a_k \varphi(t e_0/k) : (t e_0) \in Z_n \} \leq \sum_{k=0}^{\infty} a_k \varphi(2 a_{n+1}/a_n),

$\Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k) \leq \varphi(2 a_{n+1}/a_n) \sum_{k=0}^{\infty} a_k$ for $m - n = 2$.

Since $a_i < a_i$ for $j > i$, we have

$\Delta_n \varphi(t e_0/k) \leq \Delta_n \varphi(t e_0/k).$

This combined with the fact that the function $\varphi$ is convex gives

$\Delta_n \varphi(t e_0/k) \leq \Delta_n \varphi(t e_0/k)$ for $j > i$, $m = 1, 2, ...$

We are going to estimate $\Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k)$, for $m = n - 1$.

By the m-th of the equations (b), we have

$\Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k) \leq \Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k) + ... + \Delta_n \sum_{k=0}^{\infty} a_k \varphi(t e_0/k).$
Thus, by (d) and (e),
\[(t) \quad \lambda_n \varphi(\xi_n \delta_n) \leq \sum_{k=1}^{n-1} a_{k-1} \varphi(\xi_{k-1} \delta_{k-1}) + \varphi(2\delta_n \delta_n) \cdot \sum_{k=n}^{\infty} a_k.
\]
This inequality, for \(m = n - 1\), gives
\[\lambda_n \varphi(\xi_n \delta_n) \leq \sum_{k=1}^{n-1} (2^{k-1} \delta_n \delta_n) \cdot \sum_{k=n}^{\infty} a_k.
\]
Now, by induction, we obtain
\[(g) \quad \lambda_n \varphi(\xi_n \delta_n) \leq 2^{m-n} \varphi(2 \delta_n \delta_n) \cdot \sum_{k=n}^{\infty} a_k \quad (m < n - 1).
\]
Since \(\delta_n \to 0\), the product \(\prod_{n=1}^{\infty} (1 - \delta_n \delta_n)\) is divergent, whence
\[\sum_{n=1}^{\infty} (1 - \delta_n \delta_n) = \infty.
\]
Now, from the conditions (a), it follows that
\[1 - \delta_n \delta_n \geq \delta_n \]
for some increasing sequence \((n)\) of indices, and that
\[\lim_{n \to \infty} \varphi(2 \delta_n \delta_n) \cdot \sum_{k=n}^{\infty} a_k = 0 \quad (m = 1, 2, ...).
\]
The estimation (g) now gives
\[\lim_{n \to \infty} \lambda_n \varphi(\xi_n \delta_n) = 0 \quad (m = 1, 2, ...).
\]
Since \(\varphi(t) = \tan t/\lambda \geq t/2\), we get
\[\lim_{n \to \infty} \lambda_n \xi_n = 0 \quad \text{for} \quad k = 1, 2, ...
\]
But this just means that the system of equations (a) has no more than one solution.

The consequences of results of §6 and §7 can be summarized as follows:

7.8. THEOREM. Assume that \(X_1, X_2, \lambda \in A\), are B-spaces which are pairwise homeomorphic under homeomorphisms \(\lambda_i: X_1 \to Y_i\), such that
\[\lambda^{-1} \cdot |\omega| \leq |h| \leq A \cdot |\omega|, A \text{ being a constant independent on } \lambda, \text{ and let } E = E(\lambda, A), E' = E(p, \lambda, A, p, q).
\] Then \(\Sigma X_1 \approx \Sigma Y_1\). Moreover, if \(\text{card } A = n\), then for \(E\) and \(E'\) being any two of the spaces \(c_0, c_0^p, c_0^p\), the assertion \(\Sigma X_1 \approx \Sigma Y_1\) holds.

III. Spaces homeomorphic with a Hilbert space

§8. General theorems. We recall that the symbol \(Y_{,X}\) denotes that \(X\) is homeomorphic with a Cartesian product \(Y \times Z\) for some topological space \(Z\).

8.1. THEOREM. Let \(X\) and \(Y\) be F-spaces. If either \(X\) is a subspace of \(Y\) or \(Y\) is an image of \(X\) under a linear operator, then \(Y \approx X\).

8.2. THEOREM. Let \(X\) be an F-space, with \(nX = X\). If \(l(s)\), then \(X \approx l(s)\) (cf. Besaga-Pelczyński [9], Theorem 1).

The first theorem is a corollary to the Bartle-Graves [4] result, which states that if \(Y\) is a subspace of \(X\) (\(Y\) and \(X\) being F-spaces) then there exists a continuous mapping \(\varphi: Y \to X\) sending any coset \([x]\) of the quotient space \(X/Y\) to a vector belonging to this coset. This mapping induces the homeomorphism \(f = ([x], x - ax)\) between \(X\) and \(Y \times Y\). To prove the sufficiency of the second assumption of 8.1, we use 3.2.

The proof of the second theorem is based on three auxiliary propositions:

8.3. Let \(X\) and \(Y\) be topological spaces and \(Y \approx Y^m\). Then the condition \(X/Y\) implies \(X \approx X \times Y\); the condition \(Y/X\) implies \(X \approx X \times Y\).

Hence the both conditions imply \(X \approx X\).

Proof. For the case \(X/Y\), there exists a \(W\) such that \(Y \approx X \times W \approx Y \times X^m \approx W \times Y^m \approx X \times X^m \times Y^m \approx Y^m \times X^m \times Y\), and there exists a \(Z\) such that \(Z \approx Z \times Z \approx Z \times Y^m \approx Z \times X^m \times Y \approx Z \times Y^m \times X\).

8.4. If \(X\) is an F-space, with \(nX = X\), then \(X \approx l(s)\).

Proof. By 3.5 and 8.1, \(X \approx X\) \(\prod_{q=1}^{\infty} X_q\), where \(X_q\) are B-spaces, with \(nX_q \leq s\). By 3.4 and 8.1, \(X_q \approx l(s)\) (\(q = 1, 2, ...\)). Hence
\[\left(\prod_{q=1}^{\infty} X_q\right) \approx l(s)\] and \(l(s) \approx l(s)^m\).

8.5. Let \(X\) be a B-space. If \(X = X\) \(\Sigma X\), then \(X \approx X\). In particular \(l(s) \approx l(s)^m\).

To prove this proposition we shall need the following:

LEMMA 2. Suppose that \(X\) is a B-space and \(E = M(s)\) is a Köthe space. Then there exists a linear operator \(T\) from \(Y = \Sigma X\) unto \(\Sigma X\), namely
\[T(s) = \sum_{k=1}^{m} a_k \xi_k\]
Proof of Lemma 2 (i). It is obvious that \( T \) is additive; moreover,

\[
|T(\{a_n\})| \leq \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{\infty} a_k a_{k+1} \right) \sup_{k \in \mathbb{N}}(a_k a_{k+1} a_{k+2}) \cdot \\
\leq \left( \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{\infty} a_k a_{k+1} a_{k+2} \right) \right) \sup_{n \in \mathbb{N}}(a_n)_{k+1}. 
\]

Thus \( T \) is continuous. It remains to prove that \( T \) is onto \( \Sigma X \). For any \( x \in X \) and \( \epsilon = \{a_n\} \in E \), we shall write

\[
e \otimes x = \{a_n x\} \in \Sigma X.
\]

Obviously

\[
|e \otimes x| = |e| \cdot |x|.
\]

Let \( x = \{a_n\} \) be an arbitrary element in \( \Sigma X \). We are going to find any in \( \Sigma X \) such that \( T x = x \). Put \( y_n = (0, \ldots, 0, 1, 0, \ldots, 0) \in \epsilon \), and let \( U \) be the linear operator from \( E \) onto \( \epsilon \) defined in 3.6. Since, obviously, \( \|y_n\| = 0 \) for all \( n \), we form the propositions 3.2 and 3.3, it follows from the existence of a such that \( inf \{ \|e_i\| \} \) for \( n > a' \). Thus we can choose vectors \( v_0 \) (\( k = 1, 2, \ldots, n \)) in \( E \) such that

\[
U x_k = y_k = |a_k| = 2 \cdot a_k \quad \text{for} \quad k > a'.
\]

Let \( y_0 \) \( = \sum_{k=1}^{n} a_k |a_k| \) if \( a_k \neq 0 \), and \( y_0 = 0 \) if \( a_k = 0 \). By (a) and (b), the series

\[
y = \sum_{k=1}^{\infty} y_k \otimes x_k
\]

is convergent and

\[
Ty = \sum_{k=1}^{\infty} (T y_k) \otimes x_k = \sum_{k=1}^{\infty} y_k \otimes x_k = x. \quad \text{Q.E.D.}
\]

Proof of Proposition 8.5. By Theorem 7.5, we have \( X = \Sigma X \approx \Sigma \Sigma_0 X \). But according to Lemma 2 and 8.4, \( \Sigma_0 X \approx \Sigma_0 \Sigma X \), whence \( \Sigma_0 X \approx \Sigma X \). But, obviously \( \Sigma X \approx \Sigma X \); therefore \( \Sigma X \approx \Sigma_0 X \). Now taking \( Y = \Sigma X \), and applying 8.3, we get \( X \approx \Sigma X \).

The second assertion follows from the obvious fact that 1(\( x \) \( \approx \) \( \Sigma 1(0) \).

Proof of Theorem 8.3. According to 8.4 and 8.5, from the hypothesis of the theorem it follows that the spaces \( X \) and \( Y = I(\epsilon) \) fulfill the assumption of 8.3, whence \( X \approx Y \).

\section{F-spaces.}

9.1 Theorem. Let \( X \) be an infinite-dimensional separable F-space. Each of conditions (i)-(xviii), listed below, is sufficient in order that \( X \) be homeomorphic with 1.

\footnote{(i) This proposition can be derived from 3.6 and some of Grothendieck’s results on tensor products (cf. Bessaga-Pelczynski [9], proof of Lemma 1). The present proof, however, does not make any use of the apparatus of tensor products.}

(i) there exists a B-space \( Y \) such that \( X \approx Y \),

(ii) \( X \) is isomorphic neither to \( B \)-space nor to any product \( x \times \Sigma X \), \( Y \) being a B-space,

(iii) \( X \approx I(\epsilon) \),

(iv) \( X \) is non-normable, but there exists a continuous norm defined on \( X \).

(v) \( X \) is reflexive (in particular, nuclear) and non-isomorphic with \( s \),

(vi) \( X \) is a B-space and \( X \) contains a reflexive infinite-dimensional subspace \( Y \),

(vii) \( X \) has a non-primitive unconditional basis,

(viii) \( X \) is non-normable and has a basis no subsequence of which is primitive,

(ix) there exists a semi-compact metric space \( Q \) having at least one cluster point and an F-space \( Y \) such that \( X \approx Y \),

(x) \( X \) is a linear image of a space \( R^\infty \), \( Q \) being a compact space,

(xi) \( X \) is a space of complex-valued functions on a compact space \( Q \), under the norm-sup, and has the following property

\[\text{(G1) For every } \epsilon > 0 \text{ and every } \alpha \in E^* \text{ there is an } \epsilon \epsilon X \text{ such that } \sup_{n \in Q} |\epsilon(n)| - \alpha(x) < \epsilon.\]

(xii) \( X \) is a Dirichlet algebra (in the sense of Grothendieck [1]),

(xiii) \( X = \Sigma X, \) where \( E \) is a countable coordinate space and \( X \) a (\( n = 1, 2, \ldots, m \)) inductive-dimensional B-space,

(xiv) \( X \approx R^\infty \), \( Y \) being a infinite-dimensional F-space non-isomorphic with \( s \).

(xv) \( X \neq s \), \( X \approx R^\infty \), where \( T \) is an arbitrary topological space,

(xvi) \( X \) is a space of B-space with an unconditional basis,

(xvii) \( X \) is a subspace of \( s \),

(xviii) \( X \) is a subspace of a space \( R^\infty \), \( Q \) being a countable compact space.

Proof. (i) The sufficiency of this condition has been proved by Kadeč [38] and Klee [51].

(ii) If this condition is satisfied, then, according to 3.10, \( X \) contains a subspace \( Y \) isomorphic to a Köthe space \( M(a_m) \) whence, by 8.1, \( M(a_m) \approx \Sigma X \), and by 7.2 and 7.4, \( I(X) \approx X \). Using 8.2, we get \( X \approx I \).

(iii) Since \( I(1 \times x) = s \), and \( l(1 \times x) \), Theorem 8.5 gives \( I(1 \times x) = l \).

(iv) Since the space \( s \) admits no continuous norm (no convex body in \( s \) is linearly bounded), condition (iv) implies (ii).

(v) If \( X \) is a B-space, condition (v) implies (i) and therefore is sufficient. So it is enough to restrict our attention to the case where
$X$ is non-normable and (ii) is not fulfilled, i.e. $X \cong s \times Y$, where $Y$ is a $k$-space. But, by (i), $Y$ must be reflexive, whence, by (i), $Y \cong I$, and therefore $X \cong I$. Now, by (ii), $X \cong I$.

(vi) Under this condition, by 3.8 and 8.1, $Y' = X$, but by (ii), $Y' \cong I$, i.e. $lI$, Now, by 8.2, $X \cong I$.

(vii) If $X$ is a $k$-space, to get the statement, we apply in turn: (v), 3.9, 7.4, 8.1 and 8.2. If $X$ is non-normable, according to 3.11, 3.12 and 3.10, there is a subspace $Y$ of $X$, with an unconditional basis, non-isomorphic to any product of a space $Z$. We have $Y \cong I$, by (ii), or by what we have just proved. Hence, by 8.1 and 8.2, $X \cong I$.

(viii) Under this assumption, according to Bessaga-Pelczynski [12], Lemma 4, no subspace of $X$ is isomorphic to $s$, whence, by (ii), $X \cong I$.

(ix) Begin with the case $Y = R$. Let $\{q_n\} \subset Q$, with $q_n \to q$. From Tietze’s extension theorem it follows that the formula $T_x = [q(q_n) \rightarrow x(q_n)]$ defines a linear operator from $R^n$ onto $q_n$. Hence, by 8.1 and 7.4, $lI$, and, by 8.2, $X \cong I$. The statement in the general case follows from the fact that $R^n \cong R$ and from 8.2.

(x) If $X$ contains a subspace isomorphic to $s$, the assertion follows from 7.4, 8.1, 8.2. Otherwise, by Pelczynski [16], Theorem 5, and Eberlein [23], $X$ is reflexive. Therefore, by (v), $X \cong I$.

(xi) Since $X$ is infinite-dimensional, $Q$ is infinite and therefore there is a sequence of non-empty disjoint open subsets $\{G_n\}$ in $Q$. According to Urysohn’s Lemma and condition (G1), one can find a sequence $\{a_n\}$ in $X$ such that

$$1 \geq \sup_{t \in S} |a_n(t)| \geq 1 - 2^{-n-1}, \quad \sup_{t \in S} |a_n(t)| \leq 2^{-n-1} \quad (n = 1, 2, ...).$$

Let $Y$ be the closed linear hull of $a_n$. It is easy to see that $\sum_{n=1}^{\infty} a_n(s)$ is norm-convergent in $\ell^1$ if $\lim n \to 0$; moreover

$$\frac{1}{2} \sum_{n=1}^{\infty} |a_n| \leq \frac{1}{2} \sup_{t \in S} |a_n(t)| \leq 2 \sum_{n=1}^{\infty} |a_n|,$$

i.e. $X$ is isomorphic to $s$. Now, by 7.4, 8.1, 8.2, $X \cong I$.

(xii) It is easily seen that every Dirichlet algebra has the property (G1) (cf. Glicksberg [29]).

(xiii) If the generating space $E$ does not consist of all real sequences, the unit vectors constitute a non-principal basis for $E$, whence, by (vii), $E \cong I$. But since $E$ is isomorphic to a subspace of $X$, from 8.1 and 8.2 follows $X \cong I$. In the other case the formulas $\|a_n\| = \max_{t \in S} |a_n(t)| = \{\sum_{n=1}^{\infty} |a_n| = 0\}$, $X$ being defined in 3.10. Hence, by 3.10 and (ii), $X \cong I$.

(xiv) If $Y$ is a Banach space, the assertion follows from (xii). If $Y$ satisfies (ii), then $Y \cong \mathbb{R}$, and, by 8.1, 8.2, $X \cong I$. If $Y \cong R^\infty$, where $Z$ is an infinite-dimensional $k$-space, then $Z' = X$, but $Z' \cong I$, and by 8.2, we get again $X \cong I$.

(xv) $X \cong T^n \cong T^n = X$, i.e. condition (xv) is satisfied.

(xvi) Under this assumption, according to Bessaga-Pelczynski [11], Theorem 5, $X$ contains an infinite-dimensional subspace $Y$ with an unconditional basis. Hence, by (vii), 8.1 and 8.2, $X \cong I$.

(xvii) By Kadet–Pelczynski [42], any subspace of $L$ is either reflexive or contains a subspace isomorphic to $s$. In both cases, by (v), 8.1, 8.2, $X \cong I$.

(xviii) Under this condition, according to Pelczynski-Semadeni [74], p. 214, $X$ contains a subspace isomorphic to $s$. Now, by 7.4, 8.1 and 8.2, $X \cong I$.

Observe that, in view of Theorems 8.1 and 8.2, to show that a separable $k$-space $X$ is homeomorphic with the Hilbert space (or with $l$), it is sufficient to find a subspace of $X$ satisfying one of the conditions of 9.1. As we have mentioned in the Introduction, for any concrete known Banach space it is enough to use condition (v). The classification of $k$-spaces is closely related to that of Banach spaces; this is due to the following proposition (Bessaga-Pelczynski [9], Remark 1).

9.3. Under the conjecture that all separable infinite-dimensional $k$-spaces are homeomorphic with $l$, every separable infinite-dimensional $k$-space is homeomorphic with $l$.

From the conjecture that there is an $k$-space $Y$ with $Y \neq s$ and $X \cong s$, it follows that all non-normable separable $k$-spaces are homeomorphic with $l$.

Proof. The first statement follows directly from 9.1 (ii), (iii). The assumption that $Y$ is non-isomorphic with $s$ and $Y \cong s$ gives $s \cong s'^{\infty} \cong Y'^{\infty}$, whence, by (xiv), $s \cong l$. This implies that every non-normable $k$-space which does not fulfill (ii) is homeomorphic with $l$.

We see that if it is true that all separable infinite-dimensional Banach spaces are homeomorphic, then all the topological types among infinite-dimensional separable $k$-spaces are represented either by two spaces, $l$ and $s$, or only by $l$. If the first situation occurs, the class of $k$-spaces homeomorphic with $s$ consists of one members. Hence it would be worthwhile to solve:

**Problem 1.** Is the space $s$ homeomorphic with $l$ in Banach’s monograph [2], p. 233, it is mentioned that $s \neq l$, but no proof of this fact is known.
It is interesting to know whether or not an analogical method can be applied to separable complete linear metric spaces. This is connected with the following three questions:

**Problem 3.** Suppose that $X$ is a complete linear metric space and $Y$ is a subspace of $X$. Is the formula $X = Y 	imes Y$ valid?

**Problem 4.** Is it true that every separable infinite-dimensional complete linear metric space with an unconditional basis is homeomorphic with $[0, 1]$?

We know that the answer to Problem 2 is positive if $Y$ is an $F$-space (Michael [67]). Perhaps a good approach to Problem 4 is to modify Kadoh's [40] method in a suitable way.

Also the following question is still open.

**Problem 5.** Is the space $S$ of all measurable real functions on $I$, with the topology of measure convergence, homeomorphic to $[0, 1]$?

Let us come back to $F$-spaces. In the non-separable case all that we know can be summarised in the following theorem.

**9.3. Theorem.** Let $X$ be an infinite-dimensional $F$-space, with density character $\kappa$. Each of the conditions (xii)-(xxiv) listed below is sufficient in order that $X$ be homeomorphic with $I(\kappa)$.

(xix) $X$ contains a reflexive subspace homeomorphic with $l(\kappa)$.

(xx) $X$ is an abstract $L$-space (Kakutani [43]).

(xxi) $X = B^0$, where $B$ is a compact space which admits a sequence $(\mu_n)$ of Baire measures such that measure algebras $\mathfrak{B}(\mu_n, B)$ are homogeneous in the sense of Maharam [63] and super $\mathfrak{B}(\mu_n, B)$ is $\kappa$- (cf. Bessaga, Pelewicz [9], Corollary 6).

(xxii) $X = B^0$, where $G$ is a compact topological group.

(xxiii) $X = B^0$, $Q$ being a Stone-Cech compactification of a discrete set

(xxiv) $X = B^0$, where $Q$ is a semicompact space containing a closed subspace $Q_1$, with $B^0 \approx l(\kappa)$.

The proof of the sufficiency of condition (xix) is the same as that of (xv). To prove (xx), we represent the space $X$ as a space $L(\mu)$ (according to Kakutani's [43] representation theorem) and then we use Mazur's [65] homeomorphism which sends any function $x$ belonging to $L(\mu)$ to the function $|x|^\mu \text{sgn } x$ in $L(\mu)$.

The proof of (xii), (xxi) and (xxiii) is given in Pelewicz [71]. The sufficiency of (xxiv) follows from the fact that, according to Urysohn-Tietze extension theorem, the linear operator of restriction of functions defined on $Q$ to the subset $Q_1$ is onto $B^0$, and from Theorems 8.1, 8.2.

In particular, the following problems remain open:

**Problem 6.** Is the space $c_0(\kappa)$ homeomorphic with $I(\kappa)$, for $\kappa > \kappa_0$?

**Problem 7.** Is it true that every reflexive Banach space of density character $\kappa$ $(\kappa > \kappa_0)$ is homeomorphic with $l(\kappa)$?

**Problem 8.** Is it true that every Banach space with an unconditional basis of density character $\kappa$ $(\kappa > \kappa_0)$ is homeomorphic with $l(\kappa)$?

**§ 10. Spaces $l(\kappa) \times W$ and $W^\infty$.** From 6.3 and 8.5 it follows that

10.1. For an arbitrary topological space $W$ the conditions $W \times I \approx I$ and $W[I] \approx I$ are equivalent.

Hence the problem for which $W, W \times I \approx I$ is reduced to the study of divisors of $I$. We have

10.2. If $W[I]$, then $W^\infty[I]$ for any metric semi-compact space $Q$. If $W[I] \approx I$ then $W[I] \approx I$.

**Proof.** The condition $W[I]$ implies $W^\infty[I]$, but $W^\infty[I]$ is a separable $F$-space, with $W^\infty[I]$ and by 8.2, $W^\infty[I] \approx I$.

The condition $W[I] \approx I$ gives $W[I] \approx I$. Now Proposition 8.5 gives our second assertion.

10.3. Any closed convex body $W$ in an arbitrary separable $F$-space $X$ is a divisor of $I$.

**Proof.** If $X$ is one-dimensional, there are only three, up to homeomorphisms, convex bodies in $X$, namely: $J = B^0$, $R^+ = \mathbb{R}^+$, $R = \mathbb{R}$. We have

10.4. Any locally compact closed convex subset of $W$ in an arbitrary $F$-space $X$ is a divisor of $I$.

**Proof.** By Klee [48], $W$ is representable as $R^p \times (R^+)^q \times R^p$, with $p < r, q < r, r < \kappa_0$. Now, by 10.5, $W[I]$. Let us consider another example of a divisor of $I$.

10.5. Let $\Gamma = ([u, v]) \in R \times R$: $0 < v < g(u)$, $|u| < 1$, with $g(t) = 0$ for $t < 1$, $g(t) = 1$ for $t \geq 1$. Then $\Gamma[\infty] \approx \mathfrak{F}$, whence in particular $\Gamma[I]$.

**Proof.** Put

\[
A = \{[u] \in \mathbb{R}^p : 0 < u \leq 1, u \not\in \mathbb{R} \}
\]

\[
B = \{[u] \in \mathbb{R}^q : 0 < u \leq 1, u \not\in \mathbb{R} \}
\]

Our proposition is obviously equivalent to the statement $A \approx B$.
§ 11. Special homeomorphisms. Begin with the following:

DEFINITION 5. Let $Y$ and $X$ be linear topological spaces. A mapping $\varphi : X \rightarrow Y$ will be called:

(a) Lipschitzian,
(b) Lipschitzian for small distances,
(c) Lipschitzian for large distances,
(d) uniformly continuous

iff for every neighbourhood $U$ of zero in $X$ there is a neighbourhood $V$ of $X$ such that $x_1 - x_2 \in U$ implies $\varphi x_1 - \varphi x_2 \in IV$ (a) for all real $t$, (b) for all $t \leq 1$, (c) for all $t \geq 1$, (d) for $t = 1$;

(e) $\varphi$ is called locally uniformly continuous iff every $x$ in $X$ has a neighbourhood in which $\varphi$ is uniformly continuous.

A homeomorphism $h$ from $X$ onto $Y$ will be called uniform, Lipschitzian, Lipschitzian for small distances, etc., iff both $h$ and $h^{-1}$ are of the corresponding type.

From Definition 5 it follows that

11.1. (1) A mapping $\varphi$ is Lipschitzian iff $\varphi$ is Lipschitzian for both small and large distances.

(2) If $\varphi$ is Lipschitzian for either small or large distances, then $\varphi$ is uniformly continuous.

(3) If $\varphi$ is uniformly continuous and homogeneous $\Leftrightarrow \varphi(\lambda x) = \lambda \varphi(x)$ then $\varphi$ is Lipschitzian (cf. Corson-Klee [19], Corollary 5.5).

(4) If $X$ is a locally convex linear topological space, then every uniformly continuous mapping $\varphi$ from a linear topological space $X$ into $Y$ is Lipschitzian for large distances. In particular, the notion of a Lipschitzian mapping and of a Lipschitzian mapping for small distance, for locally convex spaces, coincide (cf. Corson-Klee [19], Proposition 5.3).

The statements (1)-(3) are obvious. To prove (4), we shall need:

LEMMA 3. Let $U$ be a neighbourhood of zero in an l.t.s. $X$. Write $U^n = \{x \in X : x = x_1 + \ldots + x_n, x_1, \ldots, x_n \in X, x_2 = 0, x_n = 2, x_n = 0, x_1 = -x_{n-1} \epsilon U\}$. Then we have $U^n \supset nU$; moreover, if $U$ is convex, then $U^n = nU$.

Proof of Lemma 3. We obviously have

\[ U^n = \underbrace{U + \ldots + U}_n. \]

This immediately gives the first statement: $U^n \supset nU$. Now suppose that $U$ is convex. If $y = x_1 + \ldots + x_n$ with $x_i \in U$ $(i = 1, \ldots, n)$ is an arbitrary vector in $nU$, then

\[ y = \frac{1}{n} \left( x_1 + \ldots + x_n \right), \]
and \( \left( \frac{1}{n} x_1 + \ldots + \frac{1}{n} x_n \right) \) is in \( U \) as a convex linear combination of vectors \( x_1, \ldots, x_n \). This means that \( y \in nU \), which completes the proof of the lemma.

Now assume that \( \phi: X \to Y \) is uniformly continuous, \( Y \) is a locally convex l.t.s., and \( X \) is an arbitrary l.t.s. Let \( V \) be a neighbourhood of zero in \( Y \). Without loss of generality we may assume that \( V \) is convex. Take a neighbourhood \( U_1 \) in \( X \) such that \( x_1 - x_2 \in U_1 \) implies \( \phi x_1 - \phi x_2 \in V \), and another neighbourhood \( U_2 \) with \( \tau \in U_1 \) for any \( 0 \leq \tau \leq 2 \). Let \( u < t < u+1 \), where \( u \) is a positive integer. We have \( tU \subseteq tU_1 \subseteq U_1 \), for any \( 0 \leq \tau \leq 2 \), whence \( tU \subseteq U_1 \subseteq U_2 \). Now from the definition of \( U_1 \) it follows that the condition \( \phi x_1 - \phi x_2 \in tU \) implies \( \phi x_1 - \phi x_2 \in U_1 \), which implies \( \phi x_1 - \phi x_2 \in V \). Whence, by the lemma, \( \phi U_1 \subseteq \phi U_2 \subseteq V \).

The notion of a Lipschitzian mapping defined above generalizes that of a mapping fulfilling the Lipschitz condition in the sense of norm, for normed spaces. In general, for linear metric spaces these two notions do not coincide. To observe this, consider the one-dimensional space \( R \) equipped with two \( F \)-norms: the absolute value \( |\cdot| \) and the following one \( ||\cdot|| = |\cdot|/(1+|\cdot|) \). The identity operator \( I: \langle X, |\cdot| \rangle \to \langle X, ||\cdot|| \rangle \) obviously fulfills condition (a) of Definition 5 but does not satisfy the triangular Lipschitz condition. The reason we use Definition 5 is that we want any linear operator to be Lipschitzian.

The above consideration suggest an abstract approach to the definition of a “Lipschitz structure” as a uniform space with an operation of multiplying neighbourhoods of the diagonal by real numbers, subject to some axioms. Another axiomatic definition of “Lipschitz structure” can be found in Sandberg [17]. The reader interested in abstract problems of Lipschitzian mappings are also referred to Katetov [44] and Efremov [24].

Let us observe that if spaces \( X \) and \( Y \) are equivalent under a homeomorphism \( f \) such that \( f \) and \( f \) are differentiable, then obviously they are linearly isomorphic. The differential \( [df]_{x_0} \) is a required linear operator (isomorphism) from \( X \) onto \( Y \). Thus there is no reason to study in this paper the differentiable homeomorphisms, \( C^\omega \)-isomorphisms, analytical isomorphisms, etc.

With each of the types of homeomorphisms considered above a classification of linear topological spaces is connected. We shall discuss in more detail only the case of uniform homeomorphisms and the related classification. To begin with, we shall recall the terminology on uniform spaces [cf. Bourbaki [17] and Kelley [46]].

By a uniform space \( \langle X, U \rangle \) (shortly \( X \)) we mean a pair consisting of a topological space \( X \) and a filter \( U \) of subsets of \( X \times X \) satisfying the following axioms:

(i) every \( U \in U \) contains the diagonal \( \{(x, x) : x \in X \} \),

(ii) if \( U \in U \), then \( \{(y, x) : (x, y) \in U \} \in U \),

(iii) for every \( U \in U \) there exists \( V \in U \) with \( U \wedge V \in U \), the operation \( \wedge \) being defined as follows: \( U \wedge V = \{(x, y) : \exists z \in X \text{ with } (x, z) \in U, (z, y) \in V \} \),

(iv) for every \( x \in X \) the family of all the sets \( U_\alpha(a) = \{(x, y) : (x, y) \in U_\alpha \} \), with \( U_\alpha \in U \), is a base of neighbourhoods of the point \( x \) in the topology of \( X \).

The filter \( U \) is called the uniformity of \( X \) and its members are called the neighbourhoods of the diagonal or entourages.

A mapping \( \phi \) between two uniform spaces \( \langle X, U \rangle \) and \( \langle Y, \Omega \rangle \) is said to be uniformly continuous iff for every \( U \in \Omega \) we have \( \phi(U \times X) \in \Omega \). A one-to-one mapping \( h \) from \( X \) onto \( Y \) is called a uniform homeomorphism iff both \( h \) and \( h^{-1} \) are uniformly continuous.

In the sequel we shall write \( U_n = U_1 \circ U_2 \circ \ldots \circ U_n \).

A uniform space \( \langle X, U \rangle \) is called uniformly bounded iff for every \( U \in U \) there is a positive integer \( n \), with \( U_n = X \times X \). \( X \) is said to be locally uniformly bounded iff there exists an open set \( V \) such that for every \( U \in U \) we have \( V \times V \subseteq U_n \) for some \( n \).

There exist uniformly bounded linear metric spaces; for instance the space \( S \) of all measurable functions defined on \( J \). No locally convex l.t.s. is uniformly bounded. As we shall see later, among the locally convex spaces only the normed spaces are locally uniformly bounded.

Of course, uniform boundedness (local uniform boundedness) is an invariant under uniform homeomorphisms; moreover, it is easy to show that.

11.2. If \( X \) is uniformly bounded (locally uniformly bounded), then the image of \( X \) under an arbitrary uniformly continuous (uniformly continuous and open) mapping is uniformly bounded (locally uniformly bounded).

DEFINITION 6. Let \( \langle X, U \rangle \) be a uniform space and let \( U \subseteq U \). Write

\[ \mathcal{F}(U, \mathcal{U}, \varepsilon) = \sup \{ n : \text{for every } x \in X \text{ there are } x_1, \ldots, x_n \in U^0(a) \text{ such that } (x_i, x_j) \in U^\varepsilon \text{ for } i \neq j \} . \]

Let \( \mathcal{F}(U, \mathcal{U}) \) be the class of all non-negative functions \( \mathcal{F}(\cdot) \), defined for \( \varepsilon > 0 \), such that there is an \( \varepsilon_0 > 0 \), with \( \varepsilon(\varepsilon' > 0) \) for \( \varepsilon < \varepsilon_0 \).

Write

\[ \mathcal{F}(X) = \bigcup_{U \subseteq U} \mathcal{F}(U, \mathcal{U}) . \]

\( \mathcal{F}(X) \) will be called the approximative dimension of the uniform space \( X \).
Observe that the conditions $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $\mathcal{V}_1 \subseteq \mathcal{V}_2$ imply $M(\mathcal{U}_1, \mathcal{V}_1) \subseteq M(\mathcal{U}_2, \mathcal{V}_2)$. Hence

11.3. If $\mathcal{B}$ is any basis for the uniformity $\mathcal{U}$ of a uniform space $X$, then $\Phi(X) = \bigcup_{\mathcal{B} \subseteq \mathcal{U}} M(\mathcal{U}, \mathcal{V})$.

We can also easily check that

11.4. If $Y$ is uniformly homeomorphic with $X$, then $\hat{\Phi}(Y) = \hat{\Phi}(X)$.

Now assume that $X$ is a locally convex linear topological space. The uniformity $\mathcal{U}$ of $X$ is then determined by the class $\mathcal{C}$ of all convex symmetric neighbourhoods of zero. Namely, the family $\{U: U \in \mathcal{C}\}$, with $\hat{U} = \{(x, y) \in X \times X: \pi_1 \in U\}$, is a basis of the filter $\mathcal{U}$. From Lemma 3 it follows that

$$\hat{U} = \hat{U}^n (n = 1, 2, ...).$$

This means that $X$ is locally uniformly bounded if $X$ is locally bounded as a linear topological space, i.e., according to Kolmogorov [60], $i$ if $X$ is isomorphic to a normed space. Hence, by 11.2.

11.5. If $X$ is a normed space and $Y$ is a non-normable $F$-space, then $Y$ is not uniformly homeomorphic with $X$.

Assume that $U$ and $V$ are in $\mathcal{C}$. Put

$M(U, V, \varepsilon) = \{x_1, ..., x_k \in U, \text{ with } x_i - x_j \in V \ (i \neq j)\}.$

Now define $M(U, V)$ in the same way as in Definition 6, with $M(\ldots)$ replaced by $M(\ldots)$ and $\mathcal{C}$ replaced by $\mathcal{U}$.

Since $1 < 2, \left\langle\frac{1}{2^n}\right\rangle \leq \frac{1}{2^n}$ for $0 < \varepsilon < 1$,

from (*), it follows that

$M(U, 4V, \varepsilon) \subseteq M(U, V, \varepsilon) \subseteq M(U, V, \varepsilon)$.  

Hence, by 11.3, we get

11.6. If $X$ is a locally convex linear topological space, then $\hat{\Phi}(X) = \Phi(X)$.

The functor $\Phi(X)$ is known invariant under isomorphisms of linear topological spaces, called the Kolmogorov approximate dimension. It has been studied and computed for concrete spaces, in Pełczyński [73], Kolmogorov [53], Mitjaig [63], Rolewicz [75]. According to 11.4 and 11.6 the Kolmogorov approximate dimension is also an invariant under uniform homeomorphisms. We know many examples of separable infinite-dimensional $F$-spaces distinguishable by means of $\Phi(\cdot)$. For instance, if $H^f$ denotes the space of all entire functions of $n$ complex variables, then $\Phi(H^f) = \Phi(H^f)$, for $n \neq m$ (cf. Rolewicz [75]). It has been shown also that such properties of $F$-spaces as being a Schwartz space or being nuclear can be characterized by means of approximative dimension.

The above method cannot be applied to infinite-dimensional $B$-spaces, because $\Phi(\cdot)$ does not distinguish between such spaces. The uniform structures of Banach spaces have been studied in a recent paper [62] by Lindenstrauss. Examples of non-uniformly homeomorphic separable infinite-dimensional $B$-spaces are given in it. All this implies that classification with respect to uniform homeomorphisms does not coincide with the topological classification as well as in the case of Banach spaces as in the case of $F$-spaces. The following problem is still open.

**Problem 17.** Do there exist two non-isomorphic $F$-spaces, which are uniformly homeomorphic?

§ 12. Abstract decomposition scheme, invariant infinite powers, radial homeomorphism. It is easily seen that proposition 8.3 can be generalized as follows:

12.1. Let $\mathcal{B}$ be an algebra consisting of objects $X, Y, ...$ with two operations $X \times Y$ and $X^\mathcal{B}$ and one relation $\sim$. Assume that the following axioms are satisfied:

(a) $X \sim X$; $X \sim Y$ and $Y \sim Z$ imply $Z \sim X$.
(b) $X \sim X$ and $Y \sim Y$ implies $X \times Y \sim X \times Y$.
(c) $X \sim Y$ implies $X^\mathcal{B} \sim X^\mathcal{B}$.
(d) $(X \times Y) \times Z \sim X \times (Y \times Z)$; $(X \times Y)^\mathcal{B} \sim X^\mathcal{B} \times Y^\mathcal{B}$.

If $X$ and $Y$ are elements of $\mathcal{B}$ such that $X \sim X \times Y$, $X \sim X \times W$ for some $Z$, $W \in \mathcal{B}$, and $Y \sim X \times Y \times Y$, then $X \sim Y$.

Pełczyński has showed in his paper [70] used a concrete model of scheme 12.1, in which $\mathcal{B}$ is the class of all Banach spaces, with $\sim$ and $X \times Y$, $X^\mathcal{B}$ being interpreted as the relation of isomorphism and Cartesian product and $\Sigma_\mathcal{B} X$, respectively. For the purposes of topological classification, the symbol $\sim$ should denote the relation of being homeomorphic; the natural interpretation of $X \times Y$ is as a Cartesian product. The infinite power $X^\mathcal{B}$ can be interpreted in different ways. For instance, as in 8.3, we may assume $X^\mathcal{B} = X^\mathcal{B}$; obviously $X^\mathcal{B}$ is invariant (i.e., fulfills axiom (a)) in the class of all topological spaces. In Bessaga-Pełczyński [7] we assumed $X = \Sigma_\mathcal{B} X$; this is, of course, invariant in the class of all Banach spaces. Similarly, from Theorem 7.5 it follows that $X^\mathcal{B} = \Sigma_\mathcal{B} X$ is also invariant in the class of all $B$-spaces.

It is natural to ask if every coordinate product is invariant; in other words:

**Problem 18.** Suppose that $E$ is a coordinate space, and $X$ and $Y$ are $B$-spaces. Does the condition $X \sim Y$ imply $\Sigma_\mathcal{B} X \sim \Sigma_\mathcal{B} Y$?
Let us mention another similar question:

**Problem 19.** Let $X, Y, X_1, Y_1$ be $B$-spaces. Do the conditions $X \equiv X_1, Y \equiv Y_1$ imply that the tensor products $X \otimes Y$ and $X_1 \otimes Y_1$ or $X \otimes Y$ and $X_1 \otimes Y_1$ are homeomorphic?

According to Proposition 5.6, if $X$ and $Y$ are homeomorphic by means of a norm-preserving homeomorphism, then $\Sigma_\infty X \equiv \Sigma_\infty Y$ holds.

We have

12.2. Let $X$ and $Y$ be normed spaces. The following conditions are equivalent:

(a) There is a non-preserving homeomorphism between $X$ and $Y$.

(b) The unit spheres in $X$ and $Y$ are homeomorphic.

(c) If $X$ and $Y$ are hyperplanes in $X$ and in $Y$, then $X_i \equiv Y_i$.

(d) There is a homeomorphism $h$ from $X$ onto $Y$ such that $h_{\lambda x} = \lambda h x$, $\|h_x\| = \|x\|$ for every real $\lambda$ and every $x \in X$.

Proof. The implication (a) $\rightarrow$ (b) is obvious.

(b) $\rightarrow$ (d). This follows from Klee's [48] result stating that hyperplanes of infinite dimensional normed spaces are homeomorphic with unit spheres.

(d) $\rightarrow$ (a) is obvious.

**Definition 7.** Two normed spaces $X$ and $Y$ satisfying any of the equivalent conditions (a)-(d) of 12.2 are called **radially homeomorphic**.

To solve Problem 18 it would be sufficient to show that any two homeomorphic $B$-spaces are radially homeomorphic.

§ 13. Conjectures weaker than that of topological equivalence of all infinite-dimensional Banach spaces. Consider the following sentences:

(I) Every separable infinite-dimensional $B$-space is homeomorphic with $l$.

(2) Every separable infinite-dimensional $B$-space is radially homeomorphic with $l$.

(2') Every separable infinite-dimensional $B$-space is divisible by $s$.

(2'') Every separable infinite-dimensional $B$-space is homeomorphic with a certain non-normal $F$-space.

(3) In every infinite-dimensional $B$-space all the closed convex bodies are homeomorphic.

(3') In every infinite-dimensional $F$-space all the closed convex bodies are homeomorphic.

(4) For every infinite-dimensional $B$-space $X$, we have $X \times R \equiv X$.

(4') Every two homeomorphic $B$-spaces are radially homeomorphic.

(6) There are only finitely many topologically different separable infinite-dimensional $B$-spaces.

(7) For every $B$-space $X$, the condition $X \times R \equiv X$ implies $X \equiv l$.

(7') Every $B$-space homeomorphic with $l$ is radially homeomorphic with $l$.

(8) In every $B$-space homeomorphic with $l$, all the closed convex bodies are homeomorphic with $l$.

None of the above statements has been proved. All the known implications between them can be expressed by the following diagram:

![Diagram](image-url)

Proof. (I) $\rightarrow$ (I'). This follows from 12.2.

(1) $\rightarrow$ (2). To prove this, it is sufficient to note that by 9.1 (ii), $s(l)$. (2) $\rightarrow$ (2'). This follows from 9.1 (ii) and (iii).

(2) $\rightarrow$ (3). Let $W$ be a convex body in $X$. By Corson-Klee [19], $W \equiv (Z/Y) \times X \times (\mathbb{R}^+)^2$, where $p, r$ are non-negative integers and $Z$ is a subspace of $X$ of the deficiency $p+r$. Let $Y$ be a separable infinite-dimensional subspace of $Z$. By 8.1, $W \equiv (Z/Y) \times X \times (\mathbb{R}^+)^2$. Now assuming (2), we get $W \equiv (Z/Y) \times X \times (\mathbb{R}^+)^2 \times s$. But, by Bessaga-Klee [1], $s \equiv (\mathbb{R}^+)^2 \times X \times (\mathbb{R}^+)^2 \times s \approx X$, whence $W \equiv X \times (\mathbb{R}^+)^2 \equiv X$.

(3) $\rightarrow$ (3'). According to Bessaga-Klee [7], if $X$ is a non-normal $F$-space, then all the closed convex bodies in $X$ are homeomorphic with $X$.

(3') $\rightarrow$ (3). This follows from 12.2.

(7) $\rightarrow$ (7). Assume $X \times R \equiv X$. Put $X_0 = X \times R$, $X_1 = X$, $X_{n+1} = X_0 \times X_{n+1}$, a subspace of $X_0$ of deficiency one ($n = 1, 2, \ldots$). Under hypothesis (8) there are $p, q \in [0, q]$, with $p = q$. But this implies that $X_{n+1} \equiv X_0 \times X^2 \approx X_0 \times X \equiv X$, since $X$ is separable and $X_{n} \equiv X_{n+1}$, by 8.2, we get $X \equiv l$.

All the other arrowed implications are trivial.

**References**


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On Egoroff's theorem

by

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I. Although Egoroff's theorem [6] is usually stated for sequences instances when the collection of functions involved is non-degenerate (F3) [7]. However, several counter-examples exist in the literature which show that the conclusion of the theorem does not in general follow in this case (F2), (F3), (F9), (F10)). Hahn and Rosenthal [3] must have realized this, although no reference to a counter-example is mentioned, since they state and prove a non-degenerate analogue of Egoroff's theorem, but by placing certain restrictions on the functions not found in the original form of the theorem. Essentially, they prove:

Let \( m \) be a measure function on an additive class of sets \( A \) of a space \( X \), \( A \) an element of \( A \) such that \( m(A) < +\infty \) and \( E \) a real function defined on \( A \times (0, 1) \) such that for each \( x \in A \), \( F(x, \cdot) \) is continuous on \( (0, 1) \) and for each \( t \in (0, 1) \), \( F(\cdot, t) \) is measurable on \( A \). If

\[
\lim_{t \to 0} F(x, t) = G(x)
\]

a.e. on \( A \), where \( G \) is finite a.e. on \( A \), then, for each \( \eta > 0 \), there exists a set \( B \subset A \) such that \( m(A - B) < \eta \) and the convergence of \( F(\cdot, t) \) to \( G \) is uniform on \( B \).

It is the purpose of this note to weaken the hypotheses of the above theorem. In what follows \( F, m, A, G, A \) and \( A \) to have the same significance as above as well as the notation \( E(\cdot, \cdot) \) and \( F(\cdot, \cdot) \). We obtain our results by replacing the set \( (0, 1) \) with an infinite set \( M \) and varying its nature.

II. We first suppose that \( M \) is an infinite subset of a topological space \( Y \) which is Hausdorff and second countable while its closure, \( clM \), is countably compact (see Halm and Spencer [4]). This allows us to assume without any loss that if we let \( M' \) denote the derived set of \( M \) and \( H \) a countable subset of \( M \) dense in \( M \), then, if \( p \in clM \) but \( p \notin H \), then \( p \in M' - M \). Let \( l.s.c. \) (u.s.c.) denote lower [upper] semi-continuous. If \( f \) is a real function defined on a set \( E \) and \( H \subset E \) then the