On topological classification of complete linear metric spaces *

by

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§ 0. Introduction. In some mathematical theories we deal with objects having two structures: the algebraical and the topological one. Objects of this sort are, for instance, linear topological spaces. For such spaces one may consider three types of classifications: One with respect to both structures (isomorphical classification), another algebraical and the third topological. Only the first of these classifications has been performed completely. It is known that the only algebraic invariant of linear spaces is their algebraic dimension. The present paper is devoted to the third classification of complete linear metric spaces, above

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all of those spaces which are locally convex. One of the reasons of the significance of this subject is that there exist applications of non-linear functional analysis of purely topological character in which the linear structure of underlying spaces is quite immaterial.

Fréchet ([27], p. 95) and Banach ([2], p. 242) asked if every infinite-dimensional separable Banach space was homeomorphic to the Hilbert space. Recent results due to Kadeč, Klee, Bessaga and Pełczyński suggest an affirmative answer to this problem. From the hypothesis that all separable infinite-dimensional Banach spaces are homeomorphic it can be inferred that all separable inifinite-dimensional F-spaces, except perhaps $R^{\aleph_0}$, R denoting the real line, are homeomorphic. Concerning unseparable F-spaces we know only that some of them (for instance certain spaces of continuous functions) are homeomorphic to Hilbert spaces of a suitable density character. No proof of topological non-equivalence of two complete linear metric spaces having the same density character is known. Hence at the present stage the study of topological classification of complete linear metric spaces aims establishing that some spaces or some classes of spaces are homeomorphic to Hilbert spaces. A natural problem which arises is to characterize the topological structure of Hilbert spaces in purely topological terms. One of the possible approaches to this problem is the following (1). Let us call a *Keller retract* any complete metric space which is an absolute retract relative to metric spaces and is topologically homogeneous. The following conjecture may be posed:

CONJECTURE. *Any Keller retract which is not locally compact is homeomorphic to a Hilbert space of a suitable density character. Any locally compact Keller retract is homeomorphic to a locally compact convex subset of* l_2 (all such subsets are topologically described in Klee [49]).

No results contradicting this conjecture are known to the author. However, to prove its truth seems extremely difficult.

There are several methods of establishing a homeomorphism between linear topological spaces:

1. *Direct method* (Mazur, [65]). Consider a square integrable function $x = x(t)$ defined on $[0,1]$. Let

$$y(t) = |x(t)|^{p/2}\operatorname{sgn} x(t)\,.$$

It is easily seen that the mapping h which sends x to y is one-to-one from L_2 onto L_p. It is also easy to check that h is a homeomorphism. Similarly any space l_p is homeomorphic with l_2. Hence all the spaces L_p and l_p are homeomorphic with the Hilbert space. This is historically the first example, due to Mazur in 1929, of non-isomorphic Banach spaces which are homeomorphic.

(1) This idea was presented during the Congress of Soviet Mathematicians (Leningrad 1961) in the report *On Keller retracts and topological classification of linear metric spaces* by Pełczyński and the author.

For the applications of the direct method in case of Orlicz spaces and "L_p-spaces" of vector-valued functions, see Kaczmarz [35], Stone [81], Bourbaki ([18], § 6 Exercise 10). Another application: Proposition 7.2 of this paper.

2. *Coordinate methods.* The discovery of those methods by Kadeč in 1953 and 1955 marked essential progress towards solving the problem of Fréchet and Banach. The first of these methods is applicable to spaces X possessing an increasing system $\{X_n\}$ of finite-dimensional subspaces with the so-called "Bernstein property". To any vector x of such a space a sequence $d_n x$ of its "coordinates" may be attached in such a way that $\{d_n x\colon x \in X\}$ coincides with the class of all real sequences whose absolute values monotonically tend to zero, and that the mapping h between two such spaces, with $d_n hx = d_n x$ $(n = 0, 1, \ldots)$, is a homeomorphism. The coordinates are inclinations of the vector from subspaces X_n $(n = 0, 1, \ldots)$.

By the use of this method Kadeč [36] proved the non-trivial fact that $c_0 \approx l$ (cf. 7.4). Klee and Long [56] proved that all $\aleph_0$-dimensional normed linear spaces are homeomorphic. Bessaga [6] extended this result to some more general classes of linear metric spaces. The special case of the first coordinate method, concerning spaces with an unconditional basis, is discussed in § 7.

The second method (Kadeč [37], [38], [39], Klee [51]) is in a certain sense dual to the first: coordinates are inclinations from subspaces having finite deficiencies. The most important result obtained by this method is that all infinite-dimensional separable conjugate Banach spaces are homeomorphic to the Hilbert space (Kadeč [39] and Klee [51]).

Another coordinate method was employed recently by Kadeč [40] for the proof that in every infinite-dimensional separable Banach space with an unconditional basis, the positive cone, with respect to this basis, is homeomorphic to the positive cone in l_2.

3. *Decomposition method* (Bessaga-Pełczyński [8], [9]). Homeomorphisms are constructed by means of decompositions of spaces under consideration into Cartesian products. The general idea of this method is a simple abstract algebraic scheme: Propositions 8.3 and 12.1, cf. also Pełczyński [70], Proposition 4; an essential role is also played by the Bartle-Graves theorem on cross-section for linear operators. Similar methods were employed previously by Borsuk [15], Pełczyński [69], Kadeč-Lewin [41].

By the use of the decomposition method we can show that if Y is an F-space homeomorphic with the Hilbert space $l_2(\aleph)$, then every

F-space X which contains Y and has the density character equal to $\aleph$ is homeomorphic with $l_2(\aleph)$. Another result states that every separable infinite-dimensional Banach space with an unconditional basis is homeomorphic with l_2. These two facts make it possible to show that every "known" separable infinite-dimensional Banach space is homeomorphic with l_2; it is so because no infinite-dimensional Banach space without an infinite-dimensional subspace with an unconditional basis is known.

The main results of this paper are Theorem 8.2 and its corollaries, 9.1 and 9.3, contained in Chapter III. These results were previously published in Bessaga-Pełczyński [9]. The proof of Theorem 8.2 (the main theorem in [9]) is based on some considerations of Chapter II (propositions: 6.2, 6.3, 6.4 and 7.4) which were omitted in the short sketch of the proof of proposition 2 in [9].

The purpose of this paper is to present all the more important known results concerning topological classification of complete linear metric spaces with special emphasis on the decomposition method. There are, however, some exceptions to that: A large part of the considerations of Chapter II is valid for incomplete spaces, too. In § 10 using the decomposition method, we establish homeomorphisms between some non-linear spaces. In § 11 the classification of F-spaces considered as uniform spaces is studied.

A more comprehensive discussion of the subjects omitted from this paper can be found in Klee's expository paper [54] and in the papers listed in the references. For the classification of compact and locally compact convex subsets in F-spaces, see Keller [45] and Klee [49]. The classification of closed convex bodies can be found in Klee [48], Corson-Klee [19] and Bessaga-Klee [7], [82]. Cones are studied in Corson-Klee [19] and Kadeč [40]. An essential progress in the field of uniform structure of Banach spaces and related classification has been made by Lindenstrauss; see his recent paper [62].

In the present paper nineteen open problems are mentioned. These problems were partially published by Klee [55], Pełczyński [72], Bessaga-Pełczyński [10]; some of them were posed in the report *On Keller retracts and topological classification of linear metric spaces* during the Congress of Soviet Mathematicians in Leningrad in 1961.

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I. Preliminaries

§ 1. Notation and terminology. The symbols $\cup$, $\cap$, $\setminus$ will be used for set union, intersection and difference, $+$, $\cdot$, $-$ being reserved for algebraic operations: both for vectors and sets in linear spaces. For instance, if t is a number, y is a vector in a linear space X and $A, B\subset X$, then $tA = \{tx\colon x\in A\}$, $A+B = \{x_1+x_2\colon x_1\in A,\ x_2\in B\}$, $A-y = \{x-y\colon x\in A\}$, etc. The *empty set* will be denoted by $\emptyset$ and 0 will represent the number zero and the neutral element of the linear space under consideration. $\operatorname{card} A$ = *cardinality* of the set A. *If and only if* will be rendered by *iff*. The symbols $\{p_n\}$ or $\{p_1, p_2, \ldots\}$, $\{p_1, \ldots, p_k\}$, $\{p_\lambda\}_{\lambda\in\Lambda}$ (or shortly $\{p_\lambda\}$, if the set Λ is fixed in considerations) will denote a sequence, a k-tuple and an indexed system of points, respectively. $\langle p\rangle$ represents a one-point set and $\langle p_1, p_2, \ldots\rangle$ is the set of elements of the sequence $\{p_n\}$. $[x]$ will denote the *coset* determined by the vector x in a quotient space under consideration. The only type of parenthes used for denoting the values of functions is $(\ldots)$, but usually we omit them, associating symbols written consecutively to the right, whether they denote functions or numerical multipliers, e.g. $fgx = f(g(x))$.

For any topological space Q, wQ will denote the *density character* (*weight*) of Q, i.e. the infimum of cardinalities of bases of neighbourhoods for Q; if $A\subset Q$, then $\bar{A}$, $\operatorname{Int} A$ will denote the closure, the interior of A. By *compact space* we always mean a Hausdorff topological compact (= bicompact in Russian terminology) space. Q is said to be a *semi-compact space* iff there exists a sequence $\{Q_n\}$ of compact subspaces of Q such that

$$(*)\qquad Q = \bigcup Q_n,\qquad Q_n\subset \operatorname{Int} Q_{n+1}\quad \text{for}\quad n = 1, 2, \ldots$$

Symbols $X\times Y$ and $\prod_{\lambda\in\Lambda} X_\lambda$ denote *topological product* of spaces; $X^{\aleph_0} = \prod_{\lambda\in\Lambda} X_\lambda$, with all $X_\lambda = X$, $\operatorname{card}\Lambda = \aleph_0$. If Q is a compact [semicompact] space, and W is a metric space, then by W^Q (cf. Kuratowski [61], § 88, VI) we shall denote the space of all W-valued continuous functions: $f, g, \ldots$ defined on Q with the topology given by the metric

$$\varrho(f, g) = \sup_{p\in Q}\varrho(fp, gp)$$

$$\Big[\,\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n}\sup_{p\in Q_n}\varrho(fp, gp)/\big(1+\sup_{p\in Q_n}\varrho(fp, gp)\big),$$

where $Q = \bigcup Q_n$ is an arbitrary decomposition (*) for Q (the topology does not depend on the choice of concrete decomposition)$\Big]$. By R, R^+, J we shall denote the set of all real numbers and intervals $\{t\colon t\geqslant 0\}$, $\{t\colon 0\leqslant t\leqslant 1\}$, respectively. We shall write $X\approx Y$, if the topological spaces X and Y are homeomorphic; $Y|X$ if there exists a topological space Z, such that $X \approx Y \times Z$; X is then said to be divisible by Y.

Unless otherwise explicitly stated, all the linear spaces considered in this paper are over the field of reals. However, all the proofs can be automatically transferred to the complex case.

A real-valued function $\|\cdot\|$ defined on a linear space X is called a *norm* iff it satisfies the following conditions: (1) $\|0\| = 0$, (2) $\|x+y\| \leqslant \|x\|+\|y\|$ $(x, y \in X)$, (3) $\|tx\| = |t|\,\|x\|$ $(x \in X,\ t \in R)$, (4) $\|x\| = 0$ implies $x = 0$; $\|\cdot\|$ is called a *distance function* (or *F-norm*), or a *pseudonorm* iff it satisfies the conditions (1), (2), (4) or the conditions (1), (2), (3), respectively. A topological space which is simultaneously a linear space with linear operations continuous in the topology is called a *linear topological space* (l.t.s.). By a subspace of an l.t.s. X we mean a closed linear manifold in X. Two l.t.s.'s X and Y are called *isomorphic* (written $X = Y$) iff there is a linear homeomorphism (isomorphism) between these spaces. An F-norm $|\cdot|$, in particular a norm defined on an l.t.s. X, will be called *admissible*, iff $\varrho(x, y) = |x-y|$ is a metric compatible with the topology of X; $|\cdot|$ is called *complete* iff the metric ϱ is complete. A sequence of pseudonorms $\{|\cdot|_a\}$ in X will be called admissible iff $|x|_1 \leqslant |x|_2 \leqslant \ldots$ for every x in X and the F-norm

$$(\alpha) \qquad |x| = \sum_{a=1}^{\infty} 2^{-a}|x|_a/(1+|x|_a)$$

is admissible. An l.t.s. X with a fixed admissible F-norm $|\cdot|$, in particular with a fixed norm $\|\cdot\|$, is called a *linear metric space* (l.m.s.), in particular a *normed space*. If this fixed F-norm $|\cdot|$ or norm $\|\cdot\|$ is complete, X is said to be a *complete linear metric space* or a *Banach space* (shortly *B-space*), respectively. An l.t.s. X in which no admissible norm can be defined is said to be *non-normable*. An l.t.s. X with a fixed admissible sequence of pseudonorms $\{|\cdot|_a\}$ such that the F-norm (α) is complete is called an *F-space* ($= B_0$*-space* in the terminology of Mazur-Orlicz [66]). Obviously, every F-space is a B-space, with all the pseudonorms coinciding with the norm. Any convex set with a non-empty interior in an l.t.s. X is called a *convex body*.

By a *linear operator* (*linear functional*) we mean a continuous linear vector-valued (scalar-valued) function defined on a linear topological space. If any mapping is denoted by a capital letter, it will be assumed to be a linear operator or functional. If X is an l.t.s., X^* will denote its *conjugate space*, i.e. the space of all linear functionals defined on X, equipped with its *strong topology* (Bourbaki [1], IV, 3.1). The conjugate of a normed space X is a Banach space under the (admissible) norm $\|F\| = \sup\{Fx\colon \|x\| \leqslant 1\}$. X is called *reflexive* iff every linear functional Φ defined on X^* is of the form $\Phi F = Fx_\Phi$ $(x_\Phi \in X)$. If X is an l.t.s., $F \in X^*$, then the sets $\{x \in X\colon Fx = \text{const}\}$ are called *hyperplanes* in X.

A series $\sum_n x_n$ with x_n in an l.t.s. X is said to be *convergent* to x iff $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = x$; it is said to be *unconditionally convergent* iff for every permutation $\{p_n\}$ of indices the series $\sum_n x_{p_n}$ is convergent. A sequence $\{x_n\}$ is called a *basis* of X iff every x in X is uniquely representable in a form $x = \sum_{n=1}^{\infty} t_n x_n$; a system $\{x_\lambda\}_{\lambda\in\Lambda}$ is said to be an *unconditional basis* iff every $x \in X$ has a unique (up to permutation of terms) representation $x = \sum_{\lambda\in\Lambda_x} t_\lambda x_\lambda$, the series being unconditionally convergent and $\Lambda_x \subset \Lambda$, with card $\Lambda_x \leqslant \aleph_0$. A basis $\{x_n\}$ will be called *primitive* iff $\sum t_n x_n$ converges for every real sequence $\{t_n\}$.

§ 2. List of special spaces. *The spaces* $l_p(\aleph)$ $(p > 0)$. Elements of the space $l_p(\aleph)$ are real functions $x = \{\xi_\lambda\}$ defined on an abstract set Λ, with card $\Lambda = \aleph$, such that card$\{\lambda\colon \xi_\lambda \neq 0\} \leqslant \aleph_0$ and $\sum_\lambda |\xi_\lambda|^p < \infty$, the norm being defined by the formula $\|x\| = \big(\sum_\lambda |\xi_\lambda|^p\big)^\gamma$, where $\gamma = \min(1, 1/p)$.

The *space* $c_0(\aleph)$ consists of real functions $x = \{\xi_\lambda\}$ defined on a set Λ, with card $\Lambda = \aleph_0$, such that card$\{\lambda\colon |\xi_\lambda| > \delta\} < \aleph_0$ for every $\delta > 0$. Here $\|x\| = \sup_\lambda |\xi_\lambda|$.

The spaces $l_p(\aleph_0)$ and $c_0(\aleph_0)$ are shortly denoted by l_p and c_0. Instead of $l_1(\aleph)$ and l_1 we also write $l(\aleph)$ and l.

The spaces L_p $(p > 0)$. The elements of L_p are measurable functions $x = x(t)$ defined on $\mathfrak{J}$, such that

$$\int |x(t)|^p < +\infty .$$

The norm,

$$\|x\| = \Big(\int |x(t)|^p\Big)^\gamma, \qquad \text{where} \qquad \gamma = \min(1, 1/p) .$$

The *space* s, which is composed of all real sequences $x = \{\xi_n\}$, under the topology given by the pseudonorms $|\{\xi_n\}|_a = \sup_{\varkappa\leqslant a} |\xi_\varkappa|$.

The Köthe spaces [2]. Let $\{a_{an}\}_{a,n=1,2}$ be an infinite matrix of positive numbers such that

$$(\alpha) \quad a_{1n} \leqslant a_{2n} \leqslant \ldots \ (n = 1, 2, \ldots); \qquad \sum_{n=1}^{\infty} a_{an}/a_{a+1,n} < \infty \ (a = 1, 2, \ldots) .$$

By $M(a_{an})$ we shall denote the space of all real sequences $x = \{\xi_n\}$, such that $|x|_a = \sup_n a_{an}|\xi_n| < \infty$.

[2] The class of Köthe spaces coincides with that of nuclear "Stufenräume" — Köthe [57] and [58] § 30.8 (see Grothendieck [32], Chapter II, p. 59, Proposition 8).

The space $m(\Lambda)$ of all real bounded functions $x = \{\xi_\lambda\}$ defined on an abstract set Λ; $\|x\| = \sup\limits_\lambda |\xi_\lambda|$.

The spaces $l_p(\aleph)$ and L_p for $p \leqslant 1$ are complete linear metric spaces without being F-spaces. $M(a_{an})$ and s are non-normable F-spaces. All the other spaces listed above are Banach spaces.

§ 3. Auxiliary theorems on linear functional analysis.

3.1. *Suppose that X is a complete* l.m.s. *and U_n: $X \to X$ $(n = 0, 1, \ldots)$ are linear operators, with $\lim\limits_{n\to\infty} U_n x = U_0 x$ for every x in X; then this convergence is uniform on any compact subset of X.*

3.2. *Let X and Y be complete linear metric spaces. If T is a linear operator from X onto Y, then Y is isomorphic to the quotient space $X/\ker T$, where* $\ker T = T^{-1}0$.

This is an immediate consequence of the well-known closed graph theorem.

3.3 (Schauder). *Every Banach space X, with $wX \leqslant \aleph$, is a linear image of the space $l(\aleph)$.*

Let $\{x_\lambda\}_{\lambda\in\Lambda}$ with card $\Lambda = \aleph$ be an arbitrary system of points of X dense in the unit sphere of X. The required linear operator T from $l(\aleph)$ onto X can be defined by the formula $T\{\xi_\lambda\} = \sum\limits_\lambda \xi_\lambda x_\lambda$ (see Banach-Mazur [3], p. 111 and Klee [51], Prop. 2.1).

3.4. *Every F-space X is isomorphic to a subspace of the F-space $Y = \prod\limits_{\alpha=1}^{\infty} X_\alpha$, where X_α is the completion of the quotient space $Z_\alpha = X/\{z \in X$: $|z|_\alpha = 0\}$ considered as a normed space under the norm $|\cdot|$.*

Proof (Mazur and Orlicz [66], II, 232). The required embedding is given by the formula $Tx = \{[x]_1, [x]_2, \ldots\} \in Y$, where $[x]_\alpha$ denotes the element of the quotient space Z_α that is determined by x.

3.5. *If $\{|\ |_\alpha\}$ and $\{\|\ \|_\alpha\}$ are two admissible sequences of pseudonorms for an F-space X, then for every α there exist an α' and a positive number C_α such that $\|x\|_\alpha \leqslant C|x|_{\alpha'}$, $|x|_\alpha \leqslant C_\alpha\|x\|_{\alpha'}$ for every x in X* (Mazur and Orlicz [66], II, 2.2).

3.6. *Let $E = M(a_{an})$ be a Köthe space. Then the formula $U\{\xi_k\} = \{\sum\limits_{k=1}^{\infty} a_{ak}\xi_k\}$ defines a linear operator from E onto s.*

Proof. Put $F_a\{\xi_k\} = \sum\limits_{k=1}^{\infty} a_{ak}\xi_k$. F_α are in E^*, and, by § 2, formula (α), we get n: $\{\sup\limits_{|x|_\alpha\leqslant 1} |F_n x| < \infty\} = \langle 1, 2, \ldots, \alpha\rangle$. Now from Eidelheit's result [1], Satz 2 (see also Makarov [64]), follows the assertion of our proposition.

3.7. *If X is a Banach space such that $X \times s$ is reflexive, then X is reflexive.*

This follows directly from the fact that every $F \in (X\times s)^*$ has the unique representation $F(x, y) = F_1 x + F_2 y$, where $F_1 \in X^*$, $F_2 \in s^*$.

3.8. *If X is a B-space whose conjugate X^* contains a reflexive subspace Y, then there exists a linear operator U from X onto Y^*.*

Proof. Define $(Ux)F = Fx$ for $F \in Y$. Since the adjoint operator U^* with $(U^*F)x = Fx$ is the embedding operator and has, evidently, the continuous inverse, by Banach [2], p. 146, Theorem 1, U is onto X.

3.9. *If X is a non-reflexive B-space with an unconditional basis, then X contains either a subspace isomorphic with c_0 or a subspace isomorphic with l* (James [34].)

3.10. *For every infinite-dimensional F-space X the condition*

(α) $\dim X_{\alpha+1}/\{x \in X_{\alpha+1}$: $|x|_\alpha = 0\}$ *is either* 0 *or* ∞, *for sufficiently large* α,

implies the following one:

(β) *X is isomorphic to no Cartesian product $Y\times s$, Y being an B-space.*

If X is non-normable, then (β) is equivalent to the condition

(γ) *X contains a subspace isomorphic to a Köthe space.*

Proof. Non (β) implies non (α). Let $x = (y, \{\xi_n\}) \in Y\times s$. The special system of admissible pseudonorms $|x|_\alpha = \|y\| + \max\limits_{i\leqslant\alpha} |\xi_i|$ does not satisfy (α). But from 3.5 it follows that no admissible system of pseudonorms satisfies (α).

For the proof of the second assertion, see Bessaga-Pełczyński-Rolewicz [13].

3.11. *If $\{x_n\}$ is a basis in an F-space X, then every subsequence $\{x_{p_n}\}$ forms a basis in its linear hull. If $\{x_n\}$ is an unconditional basis, then $\{x_{p_n}\}$ is also unconditional.*

3.12. *Let X be an F-space with a non-primitive basis $\{x_n\}$. Then there is a subsequence $\{x_{p_n}\}$ whose closed linear hull satisfies condition (α) of* 3.10.

Proof (cf. Bessaga-Pełczyński-Rolewicz [14], proof of Theorem 7). Among the pseudonorms $|\cdot|_\alpha$ of the space X there is one $|\cdot|_{\alpha_0}$ such that $|x_p|_{\alpha_0} \neq 0$ for infinitely many p, say $p_1, p_2, \ldots$, because otherwise the series $\sum\limits_p t_p x_p$ would be convergent for any real sequence $\{t_p\}$. Put

$$Y = \text{closed linear hull of } \{x_{n_p}\}.$$

By 3.11, every y in Y can be represented in a form $y = \sum\limits_{n=1}^{\infty} t_n x_{p_n}$. The formulas

$$\Big|\sum_{n=1}^{\infty} t_n x_{p_n}\Big|'_\alpha = \sup_q \Big|\sum_{n=1}^{q} t_n x_{p_n}\Big|_\alpha$$

define an admissible system of pseudonorms on Y. Since all these pseudonorms are norms, condition (α) is fulfilled.

3.13. *Every F-space with a primitive basis $\{x_n\}$ is isomorphic to the space s.*

Proof. The required isomorphism is given by the formula $T\sum_{n=1}^{\infty} t_n x_n = \{t_n\}$.

Let us notice that, conversely, every basis in s is primitive (Banach [2], Chapter III, Theorem 13 and Bessaga-Pełczyński-Rolewicz [14], Theorem 5).

II. Infinite products of spaces and their mappings

§ 4. Coordinate spaces and coordinate products. Through this section Λ is assumed to be a fixed abstract set. $\delta^\mu = \{\delta^\mu_\lambda\} \in m(\Lambda)$, $\mu \in \Lambda$, with $\delta^\mu_\lambda = 0$ for $\lambda \neq \mu$ and $\delta^\mu_\lambda = 1$ for $\mu = \lambda$.

Definition 1. An l.m.s. E whose elements are real-valued functions $x = \{\xi_\lambda\}$ defined on Λ, satisfying the following conditions:

(α) *If $x = \{\xi_\lambda\} \in E$, $\gamma = \{\gamma_\lambda\} \in m(\Lambda)$, then $\gamma x = \{\gamma_\lambda \xi_\lambda\} \in E$, $|\gamma x| \leqslant \|\gamma\| \cdot |x|$,*

(β) card supp $x \leqslant \aleph_0$, *for every $x \in E$, where* supp$\{\xi_\lambda\} = \{\lambda\colon \xi_\lambda \neq 0\}$,

(γ) *every x in E is a sum of the uncontitionally convergent series $\sum \delta^\mu x$, where μ runs over* supp$x$,

(δ) *for every $\mu \in \Lambda$, we have $\delta^\mu \in E$,*

will be called a *coordinate space*. If $\Lambda = \langle 1, 2, \ldots\rangle$, the space E will be called a *countable-coordinate space*. We shall use also terms: *coordinate F-space, coordinate normed space*, etc. For any coordinate space E, by E^+ we shall denote the *positive cone* in E, i.e. the set of those elements in E all coordinates of which are non-negative.

The class of coordinate F-spaces coincides, up to isomorphisms, with the class of F-spaces having an unconditional basis. Let us also observe that

4.1. *Every coordinate space E can be embedded in a complete coordinate* card supp $\tilde{E}$ *consisting of all such functions $y = \{\xi_\lambda\}$ defined on Λ for which* card supp $y \leqslant \aleph_0$ *and the series $\sum_\mu \delta^\mu y$ satisfies the Cauchy conditions as a series of elements of E.*

The spaces $l_p(\aleph)$, $c_0(\aleph)$, $M(a_{\sigma n})$, s defined in section I.2 are complete coordinate spaces. As an example of an incomplete coordinate space we can consider the subset of c_0 consisting of all finite-supported sequences, or that consisting of all absolute summable sequences, etc.

Definition 2. Suppose that E is a coordinate space and $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of normed spaces with norms $\|\cdot\|_\lambda$, respectively. Denote by $\sum_\lambda{}_E X_\lambda$ the l.m.s. consisting of all systems $y = \{x_\lambda\}$, with $x_\lambda \in X_\lambda$, such $\{\|x_\lambda\|_\lambda\} \in E^+$, with the F-norm $|\{x_\lambda\}| = |\{\|x_\lambda\|_\lambda\}|$. In the case where all X_λ coincide with a fixed space X, we shall write $\Sigma_E X$ instead of $\sum_\lambda{}_E X_\lambda$.

The space $\sum_\lambda{}_E X_\lambda$ will be called a *coordinate product* of the spaces X_λ in the sense of E [3]; X_λ and E will be called the *factors* of the product and its *generating space*, respectively. Products corresponding to countable-coordinate generating spaces are said to be countable.

If E is an F-space and X_λ are B-spaces, then $Y = \sum_\lambda{}_E X_\lambda$ is an F-space, the pseudonorms $|\{x_\lambda\}|_a = |\{\|x_\lambda\|_\lambda\}|_a$ are admissible for Y.

Let us mention that for X which is a B-space the spaces $\Sigma_{c_0} X$, $\Sigma_l X$, $\Sigma_s X$, $\Sigma_{M(a_{\sigma n})} X$ are isomorphic with the *tensor products*: $X \hat{\hat{\otimes}} c_0$, $X \hat{\otimes} l$, $X \hat{\hat{\otimes}} s$, $X \hat{\hat{\otimes}} M(a_{\sigma n})$, respectively (for the definitions of tensor products and related notions, see Grothendieck [32].)

Let $Y = \sum_\lambda{}_E X_\lambda$. We shall employ the following notation: if $y = \{x_\lambda\}$ is in Y, then mod$y = \{\|x_\lambda\|_\lambda\} \in E^+$, sgn$y = \{z_\lambda\}$, where $z_\lambda = x_\lambda/\|x_\lambda\|$ if $x_\lambda \neq 0$ and $z_\lambda = 0$ if $x_\lambda = 0$. For an arbitrary system $y = \{x_\lambda\}$ with $x_\lambda \in X_\lambda$ (not necessarily belonging to Y) we write $F_\lambda y = x_\lambda$. In the case where $\Lambda = \langle 1, 2, \ldots\rangle$, we shall use also the following notation: $S^k y = \sum_{i=1}^{k} \delta^i x_i$, $S^k_j y = S^k y - S^j y$, $S_k y = y - S^k y$.

It is easy to show that

4.2. *The functions S^k, S_k, F_λ restricted to Y are linear operators, the function* mod *is continuous, the superposition F_λ*sgn *is continuous at every point y_0 such that $F_\lambda y_0 \neq 0$. For every y in Y we have $y =$ sgn$y\cdot$mod$y$, the multiplication being understood coordinatewise, i.e. $F_\lambda y = (F_\lambda$mod$y)\cdot(F_\lambda$sgn$y)$. For every $y \in \sum_{k=1}^{\infty}{}_{\tilde{E}} X_k$ ($\tilde{E}$ being the completion of E defined in 4.1) we have $\lim_{k\to\infty} S^k y = y$.*

Since every coordinate space is a coordinate product with $X_\lambda = R$, all the notations and the last proposition are valid for coordinate spaces.

§ 5. Direct sums.

Definition 3. A complete l.m.s. X will be called a *direct sum* [4] (d.s.) of its subspaces X_n $(n = 1, 2, \ldots)$ if there exist continuous linear projections T_n from X onto X_n such that

$$(5)\qquad T_n T_m = 0 \text{ for } n \neq m; \qquad x = \sum_{n=1}^{\infty} T_n x, \text{ for any } x \text{ in } X.$$

[3] Coordinate products with special generating spaces were considered in Banach [2], p. 243, and denoted by $(X_1 X_2 \ldots)_E$; see also Day [20], Chapter 2, § 2.

[4] This notion of direct sum has been introduced by Grinblyum [31]. It was studied by S. Mazur in his seminar in 1955 and independently by W. McArthur and his students.

X will be called an *unconditional direct sum* (u.d.s.) of a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of its subspaces if there are continuous linear projections $\{T_\lambda\}_\lambda$ from X onto X_λ, respectively, such that

(6) $T_\lambda T_\mu = 0$ for $\lambda \neq \mu$; the cardinality of the set $\operatorname{supp} x = \{\lambda\colon T_\lambda x \neq 0\}$ is $\leqslant \aleph_0$ for every $x \in X$; $x = \sum_{\operatorname{supp} x} T_\lambda x$, the series being unconditionally convergent.

Let us remark that the class of direct sums (unconditional direct sums) of one-dimensional subspaces coincides with the class of spaces having bases (unconditional bases).

5.1. THEOREM. *If X is an u.d.s. of its subspaces X_λ ($\lambda \in \Lambda$), Y is an l.t.s., $h_\lambda\colon X_\lambda \to Y$ are continuous and such that $h_\lambda 0 = 0$ for every λ, and for every x in X the series $\sum_\lambda h_\lambda T_\lambda x = hx$ is unconditionally convergent, then the mapping h is continuous.*

The proof makes use of the following proposition,

5.2. *Let X be a d.s. of its subspaces X_n and let $h_n\colon X_n \to Y$ be continuous and such that $h_n 0 = 0$ for all n and $hx = \sum_{n=1}^{\infty} h_n T_n x$ is convergent for every x in X. Then h is continuous.*

Proof of 5.2. Put
$$g_n x = \sum_{i=1}^{n} h_i T_i x .$$

The functions g_n are continuous as finite sums of continuous functions. We are going to show that they are equicontinuous at 0. Suppose otherwise; then one can choose a neighbourhood U of zero in Y, vectors $x_n \in X$ ($n = 1, 2, \ldots$) and increasing sequence $\{k_n\}$ of indices, such that

(a) $$\Big|\sum_{i=1}^{p} T_i x_n\Big| < 2^{-2} \quad \text{for} \quad p \leqslant k_n ,$$

and

(b) $$g_{k_{n-1}} x_n \in V, \quad g_{k_n} x_n \notin V+V \quad (n = 1, 2, \ldots) .$$

Put
$$e_n = \sum_{i=k_{n-1}+1}^{k_n} T_i x_i .$$

Since X is complete, condition (a) implies that the series $\sum_n e_n = e$ is convergent. Since according to the hypothesis the series $\sum_n h_n T_n e$ converges, we get
$$y_n = \sum_{i=k_{n-1}+1}^{k_n} h_i T_i e \to 0 .$$

But
$$\sum_{i=k_{n-1}+1}^{k_n} h_i T_i e = \sum_{i=k_{n-1}+1}^{k_n} h_i T_i e_n = \sum_{i=k_{n-1}+1}^{k_n} h_i T_i x_n = g_{k_{n-1}} x_n - g_{k_{n-1}} x_n ,$$
and by (b), $y_n \notin V$, which is contradictory to $y_n \to 0$.

We have just proved that for every neighbourhood V of zero in Y there is a number $\delta(V) > 0$ such that

(c) $$\text{if } x \in X,\ |x| < \delta(V), \text{ then } g_n x \in V .$$

Suppose $x_k \to x_0$ in X. Let V be a neighbourhood of 0 in Y. Put
$$S^k x = \sum_{i=1}^{k} T_i x, \qquad S_k x = x - S^k x .$$

By Definition 3, $S^k x \to x$, by 3.1, this convergence is uniform on the compact set $\langle x_0, x_1, \ldots\rangle$; therefore there exists an index n_0 such that

(d) $$|S_{n_0} x_k| < \delta(V) \quad \text{for} \quad k = 0, 1, \ldots$$

By (c) and (d), we have
$$hx_k - hx_0 = hS_{n_0}x_k - hS_{n_0}x_0 + (g_{n_0}x_k - g_{n_0}x_0) \in V + V + (g_{n_0}x_k - g_{n_0}x_0) .$$

Since g_{n_0} is continuous, for sufficiently large k we have $g_{n_0}x_k - g_{n_0}x_0 \in V$; thus, for such k, $hx_k - hx_0 \in V+V+V$, which means that h is continuous.

Proof of the Theorem. Let Y be a separable subspace of X. Let $\Lambda_0 = \bigcup_{x\in Y} \operatorname{supp} x$. From the separability of Y it follows that the set Λ_0 is at most countable, say $\Lambda_0 = \langle \beta_1, \beta_2, \ldots\rangle$. Thus Y is contained in a d.s. of spaces $X_{\beta_1}, X_{\beta_2}, \ldots$ and by 5.2, h is continuous on Y, but a function which is continuous on every separable subspace must be continuous on the whole space.

Remark 1. An u.d.s. X of its subspaces which is an F-space can be defined as a module over the space $m(\Lambda)$ such that: the operation of multiplication by elements of $m(\aleph)$ is bilinear over $m(\Lambda) \times X$, if Λ_n are disjoint subspaces of Λ, with $\bigcup_n \Lambda_n = \Lambda$ and χ_{Λ_n} are its characteristic functions, then $x = \sum_{n=1}^{\infty} \chi_{\Lambda_n} x$ (in norm convergence) for any x in X. Here we have $T_\lambda x = \chi_{\langle\lambda\rangle} x$, $X_\lambda = T_\lambda X$.

Such an approach suggests an idea of generalization of the concepts considered to the case in which the role of $m(\Lambda)$ would be played by a space of all bounded measurable functions defined on a space with a measure.

§ 6. Coordinate and cylindrical mappings. In this section we shall discuss the question what sorts of continuous mappings (resp. homeomorphisms) of generating spaces and of factors determine continuous mappings (resp. homeomorphisms) of coordinate products.

By 5.1, we have:

6.1. *Suppose that* E_1 *and* E_2 *are complete coordinate spaces with common* Λ, X_λ, Y_λ $(\lambda \in \Lambda)$ *are Banach spaces and* $X = \sum_\lambda{}_{E_1} X_\lambda$, $Y = \sum_\lambda{}_{E_2} Y_\lambda$. *If* $h_\lambda\colon X_\lambda \to Y_\lambda$ *are continuous and such that* $h_\lambda 0 = 0$ *and* $\{h_\lambda F_\lambda y\} \in Y$ *for any* $y \in X$, *then the mapping* $h\colon X \to Y$ *with* $hy = \{h_\lambda F_\lambda y\}$ *is continuous. If, moreover,* h_λ *are homeomorphisms and* h *is onto* Y, *then* h *is a homeomorphism from* X *onto* Y.

DEFINITION 4. Suppose that $Y_i = \sum_\lambda{}_{E_i} X_\lambda$ $(i = 1, 2)$ are coordinate products. Any mapping $g\colon Y_1 \to Y_2$ (in particular $g\colon E_1 = \Sigma_{E_1} R \to E_2 = \Sigma_{E_2} R$), or $g\colon E_1^+ \to E_2^+$, fulfilling the conditions:

(W_0) *superpositions* $F_\lambda g$ *are continuous and such that* $F_\lambda y = 0$ *implies* $F_\lambda g y = 0$ *for any* $\lambda \in \Lambda$;

(W_1) $\delta^\mu g y = g \delta^\mu y$ *for every* $y \in Y$, $\mu \in \Lambda$,

is called a *coordinate mapping*.

If Y_i are countable-coordinate products, then any mapping $g\colon Y_1 \to Y_2$ or $g\colon E_1^+ \to E_2$ satisfying condition (W_0) and the following one:

W_2) *there exists a homeomorphism* φ *from* R^+ *onto itself such that* $|S_n g y| \leqslant \varphi |S_n y|$, $y \in Y_1$, $n = 1, 2, \ldots$

will be called a *cylindrical mapping*.

A homeomorphism h from Y_1 onto Y_2 such that both h and h^{-1} are coordinate (resp. cylindrical) mappings will be called a *coordinate* (resp. *cylindrical*) *homeomorphism*.

6.2. *Suppose that* E_1 *and* E_2 *are coordinate spaces with a common* Λ, *and* X_λ $(\lambda \in \Lambda)$ *are normed spaces. If* $h\colon E_1^+ \to E_2^+$ *is a coordinate (cylindrical) mapping, then* $\hat{h}$, *with* $\hat{h} y = \operatorname{sgn} y \cdot h(\operatorname{mod} y)$, *is a coordinate (resp. cylindrical) mapping from* Y_1 *onto* Y_2. *If* h *is one-to-one and onto* E_2^+, *then* h *is one-to-one and onto* Y_2.

Proof. (i) We have

$$F_\lambda \hat{h} y = (F_\lambda \operatorname{sgn} y)(F_\lambda h \operatorname{mod} y)\,. \tag{a}$$

If h satisfies (W_0), from (a) and 4.2, we see that $F_\lambda \hat{h}$ is continuous at all points y_0 for which $F_\lambda y_0 \neq 0$. But if $F_\lambda y_0 = 0$ and $y_n \to y_0$, we have again $F_\lambda \hat{h} y_n = (F_\lambda \operatorname{sgn} y_n)(F_\lambda h \operatorname{mod} y_n) \to 0 = F_\lambda y_0$, because the sequence of the first factors is bounded and the second factors tend to zero.

(ii) Suppose that h satisfies (W_1). Then $\delta^\mu \hat{h} y = \delta^\mu(\operatorname{sgn} y \cdot h \operatorname{mod} y) = (\delta^\mu \operatorname{sgn} y)(h \delta^\mu \operatorname{mod} y) = (\operatorname{sgn} \delta^\mu y)(h \delta^\mu \operatorname{mod} y) = \hat{h} \delta^\mu y$.

(iii) Suppose that h satisfies (W_2). By 4.2, we have $|S_n \hat{h} y| = |\operatorname{mod} S_n \hat{h} y| = |S_n h \operatorname{mod} y| \leqslant \varphi |S_n y|$.

The second statement of the proposition is an obvious consequence of the first.

6.3. *Suppose that* E_i $(i = 1, 2)$ *are complete coordinate spaces,* X_i $(\lambda \in \Lambda)$ *are Banach spaces,* $Y_i = \sum_{i=1}^{\infty}{}_{E_i} X_i$. *Then every coordinate mapping* $h\colon Y_1 \to Y_2$ *or* $h\colon E_1^+ \to E_2^+$ *is continuous.*

Proof. The continuity of h in the case $h\colon Y_1 \to Y_2$ follows directly from 6.1. Assume $h\colon E_1^+ \to E_2^+$; then the mapping $\hat{h}\colon \Sigma_{E_1} R \to \Sigma_{E_2} R$ is by 6.2 a coordinate mapping again, and therefore it is continuous. But it is easy to see that in this case h is the restriction of $\hat{h}$ to E_1^+, whence it is also continuous.

6.4. *Let* $Y_i = \sum_{n=1}^{\infty}{}_{E_i} X_n$ $(i = 1, 2)$ *be two countable-coordinate products. Any cylindrical mapping* $h\colon Y_1 \to Y_2$ *is continuous.*

LEMMA 1. *Suppose* $Y = \sum_{n=1}^{\infty}{}_{E} X_n$ *and* $y_k \in Y$ $(k = 0, 1, \ldots)$. *Then* $y_k \to y_0$, *iff the following conditions are satisfied:*

$$F_n y_k \to F_n y_0 \quad (n = 1, 2, \ldots)\,, \tag{α}$$

$$\lim_{n\to\infty} (\sup_k |S_n y_k|) = 0\,. \tag{β}$$

Proof of Lemma 1. The necessity of (α) is obvious. Since $\lim_{N\to\infty} S_N y = 0$, for every y belonging to the complete l.m.s. $\sum_{n=1}^{\infty}{}_{\tilde{E}} X_n$ the tilda denotes the completion defined in 4.1), by 3.1, this convergence is uniform on $\langle y_0, y_1, \ldots\rangle$. But this means that condition (β) is necessary.

The sufficiency. Given $\varepsilon > 0$. By (β) there exists an N_ε such that

$$|y_k - y_0| \leqslant |S_{N_\varepsilon} y_k| + |S_{N_\varepsilon} y_0| + |S^{N_\varepsilon}(y_k - y_0)| \leqslant \varepsilon + \varepsilon + \sum_{n=1}^{N_\varepsilon} |F_n y_k - F_n y_0| \leqslant 3\varepsilon\,,$$

for sufficiently large k.

Proof of 6.4. If $y_k \to y_0$, then by Lemma 1, $F_n y_k \to F_n y_0$ and $\sup_k |S_n y_k| \to 0$; but by (W_0) and (W_2), we have $F_n h y_k \to F_n h y_0$ and $\sup_k |S_n h y_k| \to 0$, whence, by Lemma 1, $h y_n \to h y_0$.

6.5. *Suppose* $Y_1 = \sum_\lambda{}_E X_\lambda$, $Y_2 = \sum_\lambda{}_E Z_\lambda$, E *being normed spaces. If* $h_\lambda\colon X_\lambda \to Z_\lambda$ *are continuous and such that* $\|h_\lambda x\| \leqslant A \cdot \|x\|$, *for some constant* A, *for all* $x \in X_\lambda$ *and all* $\lambda \in \Lambda$, *then the mapping* $h\colon Y_1 \to Y_2$, *with* $h\{x_\lambda\} = \{h_\lambda x_\lambda\}$ *is continuous. If* h_λ *are homeomorphisms from* X_λ *onto* Z_λ *such that* $\|x\| \leqslant A \|h_\lambda x\| \leqslant A^2 \|x\|$, *then* h *is an homeomorphism.*

Proof. Begin with the case where Y_1 and Y_2 are countable coordinate products. It is obvious that h fulfils condition (W_0). On the other hand, by the definition of h, by condition (d) of Definition 1 and by the homogenity of norm in E, we have

$$|S_n h y_k| = |S_n \operatorname{mod} h y_k| \leqslant A|S_n \operatorname{mod} y_k| = A|S_n y_k|\,;$$

thus h fulfils also (W_2), and by 6.4, h must be continuous.

The general case of products of arbitrary power can be reduced to the case of countable products as follows: Every sequence $\{y_n\}$ in Y may be embedded in a countable coordinate product corresponding to those indices λ for which $\sum_\lambda \|F_\lambda y_k\| \neq 0$; therefore, using the countable version of the proposition, we conclude that the convergence of $\{y_k\}$ implies that of $\{hy_k\}$.

The second statement of this proposition follows directly from the first one.

In the case of coordinate products with complete factors the assumption that E is a normed space can be omitted in the formulation of 3.5. Indeed, the mappings h_λ, with $\|h_\lambda x\| \leqslant A\|x\|$ determine a mapping $\tilde{h}$: $\Sigma_{\tilde{E}} X \to \Sigma_{\tilde{E}} Y$ between complete spaces which, according to 3.1, is continuous. Hence the mapping h which is a restriction of $\tilde{h}$ to $\Sigma_E X$ is also continuous.

In particular, we have:

6.6. *Suppose that X and Z are Banach spaces, and E is a coordinate space. If there exists a homeomorphism h from X onto Z, with the property $\|hx\| = \|x\|$, then the formula $\hat{h}\{x_\lambda\} = \{hx_\lambda\}$ defines a homeomorphism from $\Sigma_E X$ onto $\Sigma_E Z$.*

From 3.6 follows

6.7. *Let X, Z be Banach spaces, and let E be a coordinate space. If $X \approx Z$, then $E \times \Sigma_E X \approx E \times \Sigma_E Z$.*

Proof. According to Klee [48], the unit sphere of an arbitrary infinite-dimensional Banach space is homeomorphic with the hyperplane (subspace of deficiency one) of this space. Hence $X \approx Z$ implies that the unit spheres in the spaces $R \times X$ and $R \times Z$ are homeomorphic. Let φ be a homeomorphism from the unit sphere of the space $R \times X$ onto that of $R \times Z$; then the formula $hx = \|x\| \cdot \varphi(x/\|x\|)$ if $x \neq 0$ and $h0 = 0$ defines a homeomorphism from $X \times R$ onto $Z \times R$. Thus by 6.6,

$$\Sigma_E(X \times R) \approx \Sigma_E(Y \times R), \quad \text{i.e.} \quad E \times \Sigma_E X \approx E \times \Sigma_E Y\,.$$

§ 7. Applications.

7.1. *The mapping $h\{\xi_\lambda\} = \operatorname{sgn}\xi_\lambda \cdot |\xi_\lambda|^{q/p}$ is a coordinate homeomorphism from $l_q(\aleph)$ onto $l_p(\aleph) \cdot (p, q) > 0)$.* (Mazur [65].)

This is an immediate consequence of 6.3.

7.2. *For every Köthe space $M(a_{an})$ there is a coordinate homeomorphism from c_0 onto $M(a_{an})$.* (Bessaga-Pełczyński [9], Lemma 1; cf. Rolewicz [76].)

Proof. Without loss of generality we may assume that the matrix $\{a_{an}\}$ satisfies the additional condition

$$(\alpha') \qquad \lim_{a\to\infty} a_{an} = \infty \qquad (n = 1, 2, \ldots)\,.$$

(for every matrix satisfying condition (α) of section I.2, we can make (α') by multiplying the rows of the matrix by some positive constants; obviously such an operation does not change the topology in $M(a_{an})$). From (α') it follows that there are homeomorphism h_n from R onto R such that $h_n(1/a) = 1/a_{an}$ $(a, n = 1, 2, \ldots)$, $h_n t = (h_n|t|)\cdot \operatorname{sgn} t$. Condition ($\alpha$) § 2 implies that the space $M(a_{an})$ can be characterized as a class of sequences $\{\xi_n\}$ for which $|\xi_n| < 1/a_{an}$, for $a = 1, 2, \ldots$ and $n > N_a$. Hence $\{\xi_n\}$ is in $M(a_{an})$ iff $|h_n^{-1}\xi_n| < 1/a$, for $a = 1, 2, \ldots$ and $n > N_a$, i.e. $\{\xi_n\} \in M(a_{an})$ iff $h_n^{-1}\xi_n \to 0$. The last condition means that the mapping $h\{\xi_n\} = \{h_n\xi_n\}$ is the required coordinate homeomorphism from c_0 onto $M(a_{an})$.

Let us remark that there is no coordinate homeomorphism from c_0 onto l. In fact: take $\{\xi_n\} \in c_0$, $\xi_n > 0$. Obviously, for every N, $\{0, \ldots, \xi_N, \xi_N, \ldots\} \in c_0$. Thus if there existed a coordinate homeomorphism $h\{\xi_n\} = \{h_n\xi_n\}$ from c_0 onto l, we would have $\sum_{i=N}^{\infty} |h_i\xi_N| = \infty$ and there would exist an increasing sequence $\{p_n\}$, with $p_1 = 1$, of positive integers, such that $\sum_{i=p_n}^{p_{n+1}-1} |h_i\xi_{p_n}| > 1$. Take $\{\eta_n\}$ in c_0, with $\eta_k = \xi_{p_n}$, for $p_n \leqslant k < p_{n+1}$; we now get $\sum_{k=1}^{\infty} |h_k\eta_k| = \infty$, which leads to a contradiction.

Now we shall describe a method of construction of cylindrical homeomorphisms due to Kadeč. Let E be a countable-coordinate B-space. Consider its positive cone E^+. Put

$$d_k x = \|S_k x\|, \qquad \text{for} \quad x \in E^+ \qquad (k = 0, 1, \ldots)\,,$$

i.e. $d_k x$ denotes the distance between x and the subspace spanned upon the first k unit vectors of E. Obviously $d_k x$ is non-increasing and tends to zero (written $d_k x \downarrow 0$) for every x in E^+. One can prove (see Bernstein [5] and Kadeč [36]) that:

For every real sequence $\{\delta_k\}$ with $\delta_k \downarrow 0$, there exists an x in E^+ such that

$$(\mathrm{a}) \qquad d_k x = \delta_k \qquad (k = 0, 1, \ldots)\,.$$

The space E is said to have the *Bernstein property* if for every $\{\delta_k\}$, with $\delta_k \downarrow 0$, there is in E^+ at most one point x (i.e. exactly one point) satisfying (a).

7.3. *If spaces E_1 and E_2 have the Bernstein property, then the mapping h, which sends any $x \in E_1^+$ to the y in E_2^+ such that $d_k x = d_k y$ $(k = 0, 1, \ldots)$, is a cylindrical homeomorphism.*

Proof (cf. Kadeč [36], Klee-Long [56]). The Bernstein property ensures that h is one-to-one and onto E_2. It is obvious that $\operatorname{sgn} he = \operatorname{sgn} e$ for $e \in E^+$, and that

(b) $\quad \|S_n he\| = \|S_n e\|$ for every $e \in E^+$ $\quad (n = 0, 1, \ldots)$.

This means that h satisfies condition (W_2).

Take $x_k \to x_0$ in E^+. By (b), $\sup_k \|hx_k\| < \infty$, and by Lemma 1,

$$\sup_k \varrho(hx_k, S^n E_2) \leqslant \sup_k \|S_n hx_k\| = \sup_k \|S_n x_k\| \to 0 .$$

Hence the set $A = \langle hx_0, hx_1, \ldots\rangle$ is bounded and can be approximated by finite-dimensional linear sets $S^n E_2$; therefore it is totally bounded (precompact). Let y be a cluster point of this set. Since $d_n hx_k = d_n x_k$ and d_n are continuous, $d_n y = d_n x_0$ $(n = 0, 1, \ldots)$, and, by the Bernstein property, $y = hx_0$ is the only cluster point of the set A. Thus h is continuous and using 4.2, we infer that h satisfies condition (W_0). The assumptions concerning h and h^{-1} being symmetrical, we infer that h^{-1} is also continuous.

7.4 (Kadeč [36]). *There is a cylindrical homeomorphism between l^+ and c^+.*

Proof. The space l has the Bernstein property: *if $x = \{\xi_n\} \in l^+$, then $\xi_k = d_{k-1}x - d_k x$.* Thus, to complete the proof, it is enough to find in c_0 an admissible norm $|||\cdot|||$ under which this space has the Bernstein property.

Let $\varphi(t) = \tan t\pi/4$ for $0 \leqslant t < 2$, and $\{\theta_n\}$ and $\{a_n\}$ be sequences of positive numbers such that

(a) $$\sum_{n=1}^{\infty} \theta_n < \infty, \qquad \lim_{n\to\infty} 2^n \varphi(2-2\theta_n) \cdot \sum_{k=n}^{\infty} a_k = 0, \qquad \sum_{k=1}^{\infty} a_k = 1 .$$

Now for any $x = \{\xi_n\} \in c_0$ we put $|||x||| = \inf\{t > 0\colon \sum_{k=1}^{\infty} a_k \varphi(\xi_k/t) \leqslant 1\}$. Since $\varphi(1) = 1$ and $\lim_{t\to 2} \varphi(t) = \infty$, we get $|||x||| \leqslant \|x\| \leqslant 2\, |||x|||$ for every x in c_0, i.e. the new norm $|||\cdot|||$ is admissible. The Bernstein property for the space c_0 equipped with this norm is, clearly, equivalent to the following statement:

(*) *For an arbitrary sequence of positive numbers $\delta_n \downarrow 0$, the system of equations*

$$\sum_{k=m}^{\infty} a_k \varphi(\xi_k/\delta_{m-1}) = 1 \qquad m = (0, 1, \ldots)$$

has at most one solution $x = \{\xi_n\}$ in c_0^+.

Proof of (*). Take a fixed n and write the first n equations in the reverse order:

(b) $$\left\{\begin{aligned} &\sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_{n-1}) = 1 ,\\ a_{n-1}\varphi(\xi_{n-1}/\delta_{n-2}) + &\sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_{n-2}) = 1 ,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ a_m\varphi(\xi_m/\delta_{m-1}) + \ldots + a_{n-1}\varphi(\xi_{n-1}/\delta_{m-1}) + &\sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_{m-1}) = 1 ,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ a_1\varphi(\xi_1/\delta_0) + \ldots + a_{n-1}\varphi(\xi_{n-1}/\delta_0) + &\sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_0) = 1 , \end{aligned}\right.$$

Let Z_n denote the set of those $x = \{\xi_k\} \in c_0^+$ which satisfy the equations (b). For any real-valued function f defined on c_0^+ we shall write

$$\Delta_n f x = \sup\{|fz - fz'|\colon z, z' \in Z_n\} .$$

Since $a_k > 0$, the first of equations (b) gives us

(c) $\quad 0 \leqslant \xi_k < 2\delta_{n-1}$ for $k \geqslant n$.

Thus, since the function φ is increasing and the sequence $\{\delta_n\}$ does not increase, we have

$$\Delta_n \sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_m) \leqslant \sup\Big\{\sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_m)\colon \{\xi_k\} \in Z_n\Big\} \leqslant \sum_{k=n}^{\infty} a_k \varphi(2\delta_{n-1}/\delta_m) ,$$

and therefore

(d) $$\Delta_n \sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_m) \leqslant \varphi(2\delta_{n-1}/\delta_{n-2}) \cdot \sum_{k=n}^{\infty} a_k \quad \text{for} \quad m \leqslant n-2 .$$

Since $\delta_j \leqslant \delta_i$ for $j \geqslant i$, we have

$$\Delta_n(\xi_m/\delta_i) \leqslant \Delta_n(\xi_m/\delta_j) .$$

This combined with the fact that the function φ is convex gives

(e) $\quad \Delta_n \varphi(\xi_m/\delta_i) \leqslant \Delta_n(\xi_m/\delta_j)$ for $j \geqslant i$, $m = 1, 2, \ldots$

We are going to estimate $\Delta_n a_m \varphi(\xi_m/\delta_{m-1})$, for $m \leqslant n-1$. By the m-th of the equations (b), we have

$$\Delta_n a_m \varphi(\xi_m/\delta_{m-1}) \leqslant \Delta_n a_{m+1}\varphi(\xi_{m+1}/\delta_{n-1}) + \ldots + \Delta_n \sum_{k=n}^{\infty} a_k \varphi(\xi_k/\delta_{m-1}) .$$

Thus, by (d) and (e),

$$\text{(f)}\quad \Delta_n a_m \varphi(\xi_m/\delta_{m-1}) \leqslant \sum_{k=1}^{n-m-1} a_{n-k}\varphi(\xi_{n-k}/\delta_{n-k-1}) + \varphi(2\delta_{n-1}/\delta_{n-2}) \cdot \sum_{k=n}^{\infty} a_k .$$

This inequality, for $m = n-1$, gives

$$\Delta_n a_{n-1}\varphi(\xi_{n-1}/\delta_{n-2}) \leqslant \sum_{k=1}^{n-m-1} (2\delta_{n-1}/\delta_{n-2}) \cdot \sum_{k=n}^{\infty} a_k .$$

Now, by induction, we obtain

$$\text{(g)}\quad \Delta_n a_m \varphi(\xi_m/\delta_{m-1}) \leqslant 2^{n-m}\varphi(2\delta_{n-1}/\delta_n) \cdot \sum_{k=n}^{\infty} a_k \quad (m \leqslant n-1) .$$

Since $\delta_n \to 0$, the product $\prod_{n=1}^{\infty} \delta_n/\delta_{n-1}$ is divergent, whence

$$\sum_{n=1}^{\infty} (1-\delta_n/\delta_{n-1}) = \infty .$$

Now, from the conditions (a), it follows that

$$1-\delta_{n_j}/\delta_{n_j-1} \geqslant \theta_{n_j}$$

for some increasing sequence $\{n_j\}$ of indices, and that

$$\liminf_{n\to\infty} 2^n \varphi(2\delta_n/\delta_{n-1}) \cdot \sum_{k=n}^{\infty} a_k = 0 \quad (m = 1, 2, \ldots) .$$

The estimation (g) now gives

$$\liminf_{n\to\infty} \Delta_n a_m \varphi(\xi_m/\delta_{m-1}) = 0 \quad (m = 1, 2, \ldots) .$$

Since $\varphi(t) = \tan t\pi/4 \geqslant t/2$, we get

$$\liminf_{n\to\infty} \Delta_n \xi_k = 0 \quad \text{for} \quad k = 1, 2, \ldots$$

But this just means that the system of equations $(*)$ has no more than one solution.

The consequences of results of § 6 and § 7 can be sumarized as follows:

7.5. THEOREM. *Assume that X_λ, Y_λ, $\lambda \in \Lambda$, are B-spaces which are pairwise homeomorphic under homeomorphisms h_λ: $X_\lambda \to Y_\lambda$, such that $A^{-1} \cdot \|x\| \leqslant \|h_\lambda x\| \leqslant A \cdot \|x\|$, A being a constant independent on λ, and let $E = l_p(\operatorname{card} \Lambda)$, $E' = l_{p'}(\operatorname{card} \Lambda)$, $p, p' > 0$. Then $\Sigma_E X_\lambda \approx \Sigma_E Y_\lambda$. Moreover, if $\operatorname{card} \Lambda = \aleph_0$, then for E and E' being any two of the spaces c_0, l_p $(p > 0)$, $M(a_{a_n})$, the assertion $\Sigma_E X_\lambda \approx \Sigma_E Y_\lambda$ holds.*

III. Spaces homeomorphic with a Hilbert space

§ 8. General theorems. We recall that the symbol $Y|X$ denotes that X is homeomorphic with a Cartesian product $Y \times Z$ for some topological space Z.

8.1. THEOREM. *Let X and Y be F-spaces. If either Y is a subspace of X or Y is an image of X under a linear operator, then $Y|X$.*

8.2. THEOREM. *Let X be an F-space, with $wX = \aleph$. If $l(\aleph)|X$, then $X \approx l(\aleph)$* (cf. Bessaga-Pełczyński [9], Theorem 1).

The first theorem is a corollary to the Bartle-Graves [4] result, which states that if Y is a subspace of X (X and Y being F-spaces) then there exists a continuous mapping φ: $X \times Y \to X$ sending any coset $[x]$ of the quotient space X/Y to a vector belonging to this coset. This mapping induces the homeomorphism $fx = ([x], x-\varphi x)$ between X and $Y \times X/Y$. To prove the sufficiency of the second assumption of 8.1, we use 3.2.

The proof of the second theorem is based on three auxiliary propositions:

8.3. *Let X and Y be topological spaces and let $Y \approx Y^{\aleph_0}$. Then the condition $X|Y$ implies $Y \approx X \times Y$; the condition $Y|X$ implies $X \approx X \times Y$. Hence the both conditions imply $X \approx Y$.*

Proof. For the cases $X|Y$ and $Y|X$: there exists a W such that $Y \approx X \times W \approx (X \times W)^{\aleph_0} \approx X^{\aleph_0} \times W^{\aleph_0} \approx X \times X^{\aleph_0} \times W^{\aleph_0} \approx X \times Y$, and there exists a Z such that $X \approx Z \times Y \approx Z \times Y^{\aleph_0} \approx Z \times Y^{\aleph_0} \times Y \approx X \times Y$, respectively.

8.4. *If X is an F-space, with $wX = \aleph$, then $X\big|\big(l(\aleph)\big)^{\aleph_0}$.*

Proof. By 3.5 and 8.1, $X\big|\prod_{q=1}^{\infty} X_q$, where X_q are B-spaces, with $wX_q \leqslant \aleph$. By 3.4 and 8.1, $X_q|l(\aleph)$ $(q = 1, 2, \ldots)$. Hence

$$\Big(\prod_{q=1}^{\infty} X_q\Big)\Big|\big(l(\aleph)\big)^{\aleph_0} \quad \text{and} \quad l(\aleph) \approx \big(l(\aleph)\big)^{\aleph_0} .$$

8.5. *Let X be a B-space. If $X = \Sigma_l X$, then $X \approx X$. In particular $l(\aleph) \approx \big(l(\aleph)\big)^{\aleph_0}$.*

To prove this proposition we shall need the following:

LEMMA 2. *Suppose that X is a B-space and $E = M(a_{an})$ is a Köthe space. Then there exists a linear operator T from $Y = \Sigma_E X$ onto $\Sigma_s X$, namely*

$$T\{x_n\} = \Big\{\sum_{k=1}^{\infty} a_{nk} x_k\Big\} .$$

Proof of Lemma 2 [5]. It is obvious that T is additive; moreover,

$$|T\{x_n\}|_a = \sup_{n\leqslant a}\Big\|\sum_{k=1}^{\infty} a_{nk}x_k\Big\| \leqslant \sup_{n\leqslant a}\Big(\Big(\sum_{k=1}^{\infty} a_{nk}/a_{n+1,k}\Big)\sup_k a_{n+1,k}\|x_n\|\Big)$$

$$\leqslant \Big(\sup_{n\leqslant a}\sum_{k=1}^{\infty} a_{nk}/a_{n+1,k}\Big)|\{x_n\}|_{a+1}.$$

Thus T is continuous. It remains to prove that T is onto $\Sigma_s X$. For any $x \in X$ and $e = \{\eta_n\} \in E$, we shall write

$$e \otimes x = \{\eta_n x\} \in \Sigma_E X.$$

Obviously

(a) $$|e\otimes x|_a = |e|_a\|x\|.$$

Let $z = \{z_n\}$ be an arbitrary element in $\Sigma_s X$. We are going to find any in $\Sigma_E X$ such that $Ty = z$. Put $y_k = \{0, \ldots, \|z_k\|, 0, \ldots\} \in s$, and let U be the linear operator from E onto s defined in 3.6. Since, obviously, $|y_k|_a = 0$ for $k > a$, from the propositions 3.2 and 3.5 it follows that for every index a there is an a' such that $\inf\{|e|_a\colon Ue = y_n\} = 0$ for $n > a'$. Thus we can choose vectors e_k $(k = 1, 2, \ldots)$. in E such that

(b) $$Ue_k = y_k = |e_k| = 2^{-a} \quad \text{for} \quad k > a'.$$

Let $u_k = z_k/\|z_k\|$ if $z_k \neq 0$, and $u_k = 0$ if $z_k = 0$. By (a) and (b), the series $y = \sum\limits_{k=1}^{\infty} e_k \otimes u_k$ is convergent and

$$Ty = \sum_{k=1}^{\infty}(Te_k)\otimes u_k = \sum_{k=1}^{\infty} y_k\otimes u_k = z. \qquad \text{Q.E.D.}$$

Proof of Proposition 8.5. By Theorem 7.5, we have $X = \Sigma_l X = \Sigma_{M(a_{an})}X$. But according to Lemma 2, and 8.1, $\Sigma_s X|\Sigma_{M(a_{an})}X$, whence $\Sigma_s X|X$. But, obviously $X^{\aleph_0} = \Sigma_s X$; therefore $X^{\aleph_0}|X$. Now taking $Y = X^{\aleph_0}$ and applying 8.3, we get $X \approx X^{\aleph_0}$.

The second assertion follows from the obvious fact that $l(\aleph) \approx \Sigma_l l(0)$.

Proof of Theorem 8:2. According to 8.4 and 8.5, from the hypothesis of the theorem it follows that the spaces X and $Y = l(\aleph)$ fulfil the assumption of 8.3, whence $X \approx Y$.

§ 9. F-spaces.

9.1. THEOREM. *Let X be an infinite-dimensional separable F-space. Each of conditions* (i)-(xviii), *listed below, is sufficient in order that X be homeomorphic with l.*

[5] This proposition can be derived from 3.6 and some of Grothendieck's results on tensor products (cf. Bessaga-Pełczyński [9], proof of Lemma 1). The present proof, however, does not make any use of the apparatus of tensor products.

(i) *there exists a B-space Y such that $X = Y^*$,*

(ii) *X is isomorphic neither to an B-space nor to any product $s \times Y$, Y being a B-space,*

(iii) *$X \approx l \times s$,*

(iv) *X is non-normable, but there exists a continuous norm defined on X.*

(v) *X is reflexive (in particular, nuclear) and non-isomorphic with s,*

(vi) *X is a B-space and X^* contains a reflexive infinite-dimensional subspace Y,*

(vii) *X has a non-primitive unconditional basis,*

(viii) *X is non-normable and has a basis no subsequence of which is primitive,*

(ix) *there exists a semicompact metric space Q having at least one cluster point and an F-space Y such that $X = Y^Q$,*

(x) *X is a linear image of a space R^Q, Q being a compact space,*

(xi) *X is a space of complex-valued functions defined on a compact space Q, under the sup-norm, and has the following property*

(G1) *For every $\varepsilon > 0$ and every $\Delta \in (R^+)^Q$ there is an $x \in X$ such that $\sup\limits_{t\in Q}\big||x(t)| - \Delta(t)\big| < \varepsilon$,*

(xii) *X is a Dirichlet algebra (in the sense of Gleason* [1]*),*

(xiii) *$X = \Sigma_E X_n$, where E is a countable coordinate space and X_n $(n = 1, 2, \ldots)$ are infinite-dimensional B-spaces,*

(xiv) *$X \approx Y^{\aleph_0}$, Y being a infinite-dimensional F-space non-isomorphic with s.*

(xv) *$X \neq s$, $X \approx T^{\aleph_0}$, where T is an arbitrary topological space,*

(xvi) *X is a subspace of a B-space with an unconditional basis,*

(xvii) *X is a subspace of L.*

(xviii) *X is a subspace of a space R^Q, Q being a countable compact space.*

Proof. (i) The sufficiency of this condition has been proved by Kadeč [38] and Klee [51].

(ii) If this condition is satisfied, then, according to 3.10, X contains a subspace Y isomorphic to a Köthe space $M(a_{an})$, whence, by 8.1, $M(a_{an})|X$, and by 7.2 and 7.4, $l|X$. Using 8.2, we get $X \approx l$.

(iii) Since $w(l\times s) = \aleph_0$ and $l|(l\times s)$, Theorem 8.2 gives $l\times s \approx l$.

(iv) Since the space s admits no continuous norm (no convex body in s is linearly bounded), condition (iv) implies (ii).

(v) If X is a B-space, condition (v) implies (i) and therefore is sufficient. So it is enough to restrict our attention to the case where

X is non-normable and (ii) is not fulfilled, i.e. $X \approx s \times Y$, where Y is a B-space. But, by 3.7, Y must be reflexive, whence, by (i), $Y \approx l$, and therefore $X \approx l \times s$. Now, by (iii), $X \approx l$.

(vi) Under this condition, by 3.8 and 8.1, $Y^*|X$, but by (i), $Y^* \approx l$, i.e. $l|X$. Now, by 8.2, $X \approx l$.

(vii) If X is a B-space, to get the statement, we apply in turn: (v), 3.9, 7.4, 8.1 and 8.2.

If X is non-normable, according to 3.11, 3.12 and 3.10, there is a subspace Y of X, with an unconditional basis, non-isomorphic to any product s time a B-space. We have $Y \approx l$, by (ii), or by what we have just proved. Hence, by 8.1 and 8.2, $X \approx l$.

(viii) Under this assumption, according to Bessaga-Pełczyński [12], Lemma 4, no subspace of X is isomorphic to s, whence, by (ii), $X \approx l$.

(ix) Begin with the case $Y = R$. Let $\{q_n\}$ be a sequence in Q, with $q_n \to q_0$. From Tietze's extension theorem it follows that the formula $Tx = \{x(q_n) - x(q_0)\}$ defines a linear operator from R^Q onto c_0. Hence, by 8.1 and 7.4, $l|X$, and, by 8.2, $X \approx l$. The statement in the general case follows from the fact that $R^Q|Y^Q$ and from 8.2.

(x) If X contains a subspace isomorphic to c_0, the assertion follows from 7.4, 8.1, 8.2. Otherwise, by Pełczyński [10], Theorem 5, and Eberlein [23], X is reflexive. Therefore, by (v), $X \approx l$.

(xi) Since X is infinite-dimensional, Q is infinite and therefore there is a sequence of non-empty disjoint open subsets $G_n \subset Q$. According to Urysohn's Lemma and condition (G1), one can find a sequence $\{x_n\}$ in X such that

$$1 \geqslant \sup_{t \in G_n} |x_n(t)| \geqslant 1 - 2^{-n-2}, \qquad \sup_{t \in Q \setminus G_n} |x_n(t)| \leqslant 2^{-n-2} \qquad (n = 1, 2, \ldots).$$

Let Y be the closed linear hull of x_n. It is easy to see that $\sum_{n=1}^{\infty} \xi_n x_n$ is norm-convergent iff $\lim \xi_n = 0$; moreover

$$\frac{1}{2} \sup_n |\xi_n| \leqslant \Big\| \sum_n \xi_n x_n \Big\| \leqslant 2 \sup_n |\xi_n|,$$

i.e. Y is isomorphic to c_0. Now, by 7.4, 8.1, 8.2, $X \approx l$.

(xii) It is easily seen that every Dirichlet algebra has the property (G1) (cf. Glicksberg [29]).

(xiii) If the generating space E does not consist of all real sequences, the unit vectors constitute a non-primitive basis for E, whence, by (vii), $E \approx l$. But since E is isomorphic to a subspace of X, from 8.1 and 8.2 follows $X \approx l$. In the other case the formulas $\|\{x_n\}\| = \max\{\|x_i\|_a\colon i \leqslant a\}$ define an admissible system of pseudonorms for X such that

$\dim X_{a+1}/\{x \in X_{a+1}\colon \|x\|_a = 0\} = \infty$ $(a = 1, 2, \ldots)$, X_a being defined in 3.10. Hence, by 3.10 and (ii), $X \approx l$.

(xiv) If Y is a Banach space, the assertion follows from (xiii). If Y satisfies (ii), then $Y \times l$, and, by 8.1, 8.2, $X \approx l$. If $Y \approx Z \times s$, where Z is an infinite-dimensional B-space, then $Z^{\aleph_0}|X$, but $Z^{\aleph_0} \approx l$, and by 8.2, we get again $X \approx l$.

(xv) $X \approx T^{\aleph_0} \approx (T^{\aleph_0})^{\aleph_0} \approx X^{\aleph_0}$, i.e. condition (xiv) is satisfied.

(xvi) Under this assumption, according to Bessaga-Pełczyński [11], Theorem 5, X contains an infinite-dimensional subspace Y with an unconditional basis. Hence, by (vii), 8.1 and 8.2, $X \approx l$.

(xvii) By Kadeč-Pełczyński [42], any subspace of L is either reflexive or contains a subspace isomorphic to l. In both cases, by (v), 8.1, 8.2, $X \approx l$.

(xviii) Under this condition, according to Pełczyński-Semadeni [74], p. 214, X contains a subspace isomorphic to c_0. Now, by 7.4, 8.1 and 8.2, $X \approx l$.

Observe that, in view of Theorems 8.1 and 8.2, to show that a separable F-space X is homeomorphic with the Hilbert space (or with l), it is sufficient to find a subspace of X satisfying one of the conditions of 9.1. As we have mentioned in the Introduction, for any concrete known Banach space it is enough to use condition (vi). The classification of F-spaces is closely related to that of Banach spaces; this is due to the following proposition (Bessaga-Pełczyński [9], Remark 1).

9.2. *Under the conjecture that all separable infinite-dimensional B-spaces are homeomorphic with l, every separable infinite-dimensional F-space X, with $X \neq s$, is homeomorphic with l.*

From the conjecture that there is an F-space Y with $Y \neq s$ and $Y \approx s$, it follows that all non-normable separable F-spaces are homeomorphic with l.

Proof. The first statement follows directly from 9.1 (ii), (iii). The assumption that Y is non isomorphic with s and $Y \approx s$ gives $s \approx s^{\aleph_0} \approx Y^{\aleph_0}$, whence, by (xiv), $s \approx l$. This implies that every non-normable F-space which does not fulfil (ii) is homeomorphic with l.

We see that if it is true that all separable infinite-dimensional Banach spaces are homeomorphic, then all the topological types among infinite-dimensional separable F-spaces are represented either by two spaces, l and s, or only by l. If the first situation occurs, the class of F-spaces homeomorphic with s consists of one members. Hence it would be worthwhile to solve:

PROBLEM 1. Is the space s homeomorphic with l? In Banach's monograph [2], p. 233, it is mentioned that $s \not\approx l$, but no proof of this fact is known.

It is interesting to know whether or not an analogical method can be applied to separable complete linear metric spaces. This is connected with the following three questions:

PROBLEM 2. Suppose that X is a complete linear metric space and Y is a subspace of X. Is the formula $X \approx Y \times X/Y$ valid?

PROBLEM 3. It is true that $X|l$ for every separable complete linear metric space X?

PROBLEM 4. Is it true that every separable infinite-dimensional complete linear metric space with an unconditional basis is homeomorphic with l?

We know that the answer to Problem 2 is positive if Y is an F-space (Michael [67]). Perhaps a good approach to Problem 4 is to modify Kadeč's [40] method in a suitable way.

Also the following question is still open.

PROBLEM 5. Is the space S of all measurable real functions on J, with the topology of measure convergence, homeomorphic to l?

Let us come back to F-spaces. In the non-separable case all that we know can be summarized in the following theorem.

9.3. THEOREM. *Let X be an infinite-dimensional F-space, with density character $\aleph$. Each of the conditions* (xix)-(xxiv) *listed below is sufficient in order that X be homeomorphic with $l(\aleph)$.*

(xix) *X^* contains a reflexive subspace homeomorphic with $l(\aleph)$.*

(xx) *X is an abstract L-space* (Kakutani [43]).

(xxi) *$X = R^Q$, where Q is a compact space which admits a sequence $\{\mu_n\}^{n=1}_{\infty}$ of Baire measures such that measure algebras $\mathfrak{B}(\mu_n, Q)$ are homogeneous in the sense of Maharam* [63] *and* $\sup_n w\,\mathfrak{B}(\mu_n, Q) = \aleph$ (cf. Bessaga-Pełczyński [9], Corollary 6).

(xxii) *$X = R^G$, where G is a compact topological group.*

(xxiii) *$X = R^Q$, Q being a Stone-Čech compactification of a discrete set.*

(xxiv) *$X = R^Q$, where Q is a semicompact space containing a closed subspace Q_1, with $R^{Q_1} \approx l(\aleph)$.*

The proof of the sufficiency of condition (xix) is the same as that of (vi). To prove (xx), we represent the space X as a space $L(\mu)$ (according to Kakutani's [43] representation theorem) and then we use Mazur's [65] homeomorphism which sends any function x belonging to $L(\mu)$ to the function $|x|^{1/2}\operatorname{sgn} x$ in $L_2(\mu)$.

The proof of (xxi), (xxii) and (xxiii) is given in Pełczyński [71]. The sufficiency of (xxiv) follows from the fact that, according to Urysohn-Tietze extension theorem, the linear operator of restriction of functions defined on Q to the subset Q_1 is onto R^{Q_1}, and from Theorems 8.1, 8.2.

In particular, the following problems remain open.

PROBLEM 6. Is the space $c_0(\aleph)$ homeomorphic with $l(\aleph)$, for $\aleph > \aleph_0$?

PROBLEM 7. Is it true that every reflexive Banach space of density character $\aleph$ $(\aleph > \aleph_0)$ is homeomorphic with $l(\aleph)$?

PROBLEM 8. Is it true that every Banach space with an unconditional basis of density character $\aleph$ $(\aleph > \aleph_0)$ is homeomorphic with $l(\aleph)$?

§ 10. Spaces $l \times W$ and W^Q. From 8.3 and 8.5 it follows that

10.1. *For an arbitrary topological space W the conditions $W \times l \approx l$ and $W|l$ are equivalent.*

Hence the problem for which W, $W \times l \approx l$ is reduced to the study of divisors of l. We have

10.2. *If $W|l$, then $W^Q|l$ for any metric semi-compact space Q. If $W_\lambda|l$ $(\lambda \in \Lambda$, card $\Lambda \leqslant \aleph_0)$, then $\prod_\lambda W_\lambda|l$.*

Proof. The condition $W|l$ implies $W^Q|l^Q$, but l^Q is a separable F-space, with $l|l^Q$ and by 8.2, $l^{Q} \approx l$.

The condition $W_\lambda|l$ $(\lambda \in \Lambda)$ gives $\prod_\lambda W_\lambda|l^{\aleph_0}$. Now Proposition 8.5 gives our second assertion.

10.3. *Any closed convex body W in an arbitrary separable F-space X is a divisor of l.*

Proof. If X is one-dimensional, there are only three, up to homeomorphisms, convex bodies in X, namely: J, R^+, R. We have

(a) $$R|l, \quad R^+|l, \quad J|l\,.$$

The first fact is trivial. The second and the third have been proved by Klee [48].

In the general case, by Bessaga-Klee [7] (cf. Corson-Klee [19]), W has a representation $W \approx Y \times J^q \times (R^+)^r$, where Y is a subspace of X of a finite deficiency, $r = 0$ or 1, and q is a non-negative integer. Since Y is a separable F-space, by 8.4 and 8.5, $Y|l$. Now according to (a) and 10.2, $W|l$.

10.4. *Any locally compact closed convex subset of W an arbitrary F-space X is a divisor of l.*

Proof. By Klee [49], W is representable as $J^p \times (R^+)^r \times R^q$, with $p \leqslant \aleph_0$, $q < \aleph_0$, $r < \aleph_0$. Now, by 10.2, $W|l$.

Let us consider another example of a divisor of l:

10.5. *Let $\Gamma = \{(u, v) \in R \times R\colon 0 \leqslant v \leqslant g(u),\ |u| \leqslant 1\}$, with $g(t) = 0$ for $t < 1$, $g(t) = 1$ for $t \geqslant 1$. Then $\Gamma \times J^{\aleph_0} \approx J^{\aleph_0}$, whence in particular $\Gamma|l$.*

Proof. Put

$$A = \big\{\{\xi_n\} \in R^{\aleph_0}\colon\ -1 \leqslant \xi_1 \leqslant 1,\ \xi_k \in J \quad \text{for} \quad k > 1\big\},$$
$$B = \big\{\{\xi_n\} \in R^{\aleph_0}\colon\ (\xi_1, \xi_2) \in \Gamma,\ \xi \in J \quad \text{for} \quad k > 2\big\}.$$

Our proposition is obviously equivalent to the statement $A \approx B$.

By Klee [53], there exists a homeomorphism h from $\mathfrak{I}^{\aleph_0}$ into itself such that $h\{\{\xi_n\} \in I^{\aleph_0}\colon \xi_1 = 0\} = \{\{\xi_n\} \in \mathfrak{I}^{\aleph_0}\colon \xi_1 = \xi_2 = 0\}$. Now it is easy to check that the formula

$$f\{\xi_1, \xi_2, \dots\} = \begin{cases} h\{\xi_1, \xi_2, \dots\} & \text{if} \quad \xi_1 \geqslant 0, \\ \{\xi_1, 0, 0, \dots\} + h\{0, \xi_2, \xi_3, \dots\} & \text{if} \quad \xi_1 < 0 \end{cases}$$

(addition being understood coordinatewise) defines a required homeomorphism from A onto B.

We have just proved that $\Gamma|\mathfrak{I}^{\aleph_0}$, but since, by 10.4, $\mathfrak{I}^{\aleph_0}|l$, we get $\Gamma|l$.

Now we shall consider some sufficient conditions for the space Q, in order that function spaces W^Q be homeomorphic with l.

10.6. *If Q is a semicompact metric space and $W, \mathfrak{J}$ are metric spaces such that $l|\mathfrak{J}^Q$, $\mathfrak{J}|W$, $W|l$, then $W^Q \approx l$.*

Proof. Since $W|l$, by 10.2, $W^Q|l^Q$. On the other hand, the conditions $l|\mathfrak{J}^Q$, $\mathfrak{J}|W$ imply $l|W^Q$. Now, by 8.3 and 8.5, $W^Q \approx l$.

10.7. *Let $\mathfrak{J}$ be an arbitrary interval. If W has the properties $W|l$, $\mathfrak{J}|W$, then for arbitrary semicompact metric space Q having at least one cluster point, we have $W^Q \approx l$.*

Proof. The space R^Q itself, $(R^+)^Q$ and $\mathfrak{I}^Q$ are closed convex bodies in the space R^Q. Hence, by Corson-Klee [19], in any of the three cases: $\mathfrak{J} = R, R^+, \mathfrak{I}$, we have $\mathfrak{I}^Q = l$; applying 10.6 we get $W^Q \approx l$.

From 10.3, 10.4, 10.5, 10.7 and from previously used representations of closed convex bodies and convex locally compact sets, follows:

10.8. *Let X be a separable F-space. If W is either a closed convex body in X or a closed convex locally compact subset of X or $W = \Gamma \times \mathfrak{I}$, Γ being defined in 10.5, then $W^Q \approx l$ for any semicompact metric space Q having at least one cluster point.*

Denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the classes of all metric spaces which are divisors of the Hilbert cube $\mathfrak{I}^{\aleph_0}$ and of the Hilbert space, respectively. Let $\mathfrak{R}_3$ be the class of all metric spaces W such that there exists an infinite compact metric space Q, with $W^Q \approx l$; and let $\mathfrak{R}_4$ be the class of all metric spaces such that $W^Q \approx l$ for every infinite compact metric space Q.

Problem 9. Do the classes $\mathfrak{R}_2$ and $\mathfrak{R}_3$ or $\mathfrak{R}_3$ and $\mathfrak{R}_4$ coincide?

Problem 10. Does the class $\mathfrak{R}_1$ contain all the compact spaces belonging to $\mathfrak{R}_2$?

Problem 11. Does every closed convex set in an arbitrary B-space belong to $\mathfrak{R}_4$?

Problem 12. Is it true that every closed convex set in an arbitrary B-space is divisible by an interval?

Problem $12+i$. Does every compact metric absolute retract belong to the class $\mathfrak{R}_i$ $(i = 1, 2, 3, 4)$? (Cf. Klee [54] and [55].)

IV. Supplement

§ 11. Special homeomorphisms. Begin with the following:

Definition 5. Let Y and X be linear topological spaces. A mapping $\varphi\colon X \to Y$ will be called:

(a) *Lipschitzian*,

(b) *Lipschitzian for small distances*,

(c) *Lipschitzian for large distances*,

(d) *uniformly continuous*

iff for every neighbourhood V of zero in Y there is a neighbourhood U in X such that $x_1 - x_2 \in tU$ implies $\varphi x_1 - \varphi x_2 \in tV$ (a) for all real t, (b) for all $t \leqslant 1$, (c) for all $t \geqslant 1$, (d) for $t = 1$;

(e) φ is called *locally uniformly continuous* iff every x in X has a neighbourhood in which φ is uniformly continuous.

A homeomorphism h from X onto Y will be called *uniform*, *Lipschitzian*, *Lipschitzian for small distances*, etc., iff both h and h^{-1} are of the corresponding type.

From Definition 5 it follows that

11.1. (1) *A mapping φ is Lipschitzian iff φ is Lipschitzian for both small and large distances.*

(2) *If φ is Lipschitzian for either small or large distances, then φ is uniformly continuous.*

(3) *If φ is uniformly continuous and homogeneous* (i.e. $\varphi(tx) = t\varphi(x)$) *then φ is Lipschitzian* (cf. Corson-Klee [19], Corollary 5.5).

(4) *If Y is a locally convex linear topological space, then every uniformly continuous mapping φ from a linear topological space X into Y is Lipschitzian for large distances. In particular, the notion of a Lipschitzian mapping and of a Lipschitzian mapping for small distances, for locally convex spaces, coincide* (cf. Corson-Klee [19], Proposition 5.3).

The statements (1)-(3) are obvious. To prove (4), we shall need:

Lemma 3. *Let U be a neighbourhood of zero in an* l.t.s. *X. Write $U^n = \{x \in X\colon$ there are $x_0, \dots, x_n \in X$, with $x_0 = 0$, $x_n = x$, $x_i - x_{i-1} \in U\}$. Then we have $U^n \supset nU$; moreover, if U is convex, then $U^n = nU$.*

Proof of Lemma 3. We obviously have

$$U^n = \underbrace{U + \dots + U}_{n \text{ times}}.$$

This immediately gives the first statement: $U^n \supset nU$. Now suppose that U is convex. If $y = x_1 + \dots + x_n$ with $x_i \in U$ $(i = 1, \dots, n)$ is an arbitrary vector in nU, then

$$y = n\left(\frac{1}{n}x_1 + \dots + \frac{1}{n}x_n\right),$$

and $\left(\frac{1}{n}x_1+\ldots+\frac{1}{n}x_n\right)$ is in U as a convex linear combination of vectors $x_1, \ldots, x_n$. This means that $y \in nU$, which completes the proof of the lemma.

Now assume that $\varphi\colon X \to Y$ is uniformly continuous, Y is a locally convex l.t.s., and X is an arbitrary l.t.s. Let V be a neighbourhood of zero in Y. Without loss of generality we may assume that V is convex. Take a neighbourhood U_1 in X such that $x_1 - x_2 \in U_1$ implies $\varphi x_1 - \varphi x_2 \in V$, and another neighbourhood U, with $\tau U \subset U_1$ for any $0 \leqslant \tau \leqslant 2$. Let $n \leqslant t < n+1$, where n is a positive integer. We have $tU \subset t\tau U_1$ for any $0 \leqslant \tau \leqslant 2$, whence $tU \subset nU_1 \subset U_1^n$. Now from the definition of U_1 it follows that the condition $x_1 - x_2 \in tU_1$ implies $x_1 - x_2 \in U_1^n$, which implies $\varphi x_1 - \varphi x_2 \in V^n$, whence, by the lemma, $\varphi x_1 - \varphi x_2 \in nV \subset tV$.

The notion of a Lipschitzian mapping defined above generalizes that of a mapping fulfilling the Lipschitz condition in the sense of norm, for normed spaces. In general, for linear metric spaces these two notions do not coincide. To observe this, consider in the one-dimensional space R two F-norms: the absolute value $|x|$ and the following one $\|x\| = |x|/(1+|x|)$. The identity operator $T\colon \langle X, |x|\rangle \to \langle X, \|x\|\rangle$ obviously fulfils condition (a) of Definition 5 but does not satisfy the metrical Lipschitz condition. The reason we use Definition 5 is that we want any linear operator to be Lipschitzian.

The above consideration suggest an abstract approach to the definition of a "Lipschitz structure" as a uniform space with an operation of multiplying neighbourhoods of the diagonal by real numbers, subjected to some axioms. Another axiomatic definition of "Lipschitz structure" can be found in Sandberg [77]. The readers interested in abstract problems of Lipschitzian mappings are also referred to Katetov [44] and Efremovič [24].

Let us observe that if spaces X and Y are equivalent under a homeomorphism f such that f and f^{-1} are differentiable, then obviously they are linearly isomorphic. The differential $[df]_{x=x_0}$ is a required linear operator (isomorphism) from X onto Y. Thus there is no reason to study in this paper the differentiable homeomorphisms, C^p-isomorphisms, analytical isomorphisms, etc.

With each of the types of homeomorphisms considered above a classification of linear topological spaces is connected. We shall discuss in more detail only the case of uniform homeomorphisms and the related classification. To begin with, we shall recall the terminology on uniform spaces (cf. Bourbaki [17] and Kelley [46]).

By a *uniform space* $\langle X, \mathfrak{U}\rangle$ (shortly X) we mean a pair consisting of a topological space X and a filter $\mathfrak{U}$ of subsets of $X \times X$ satisfying the following axioms:

(i) *every* $\mathcal{U} \in \mathfrak{U}$ *contains the diagonal* $\{(x, x)\colon x \in X\}$,

(ii) *if* $\mathcal{U} \in \mathfrak{U}$, *then* $\{(y, x)\colon (x, y) \in \mathcal{U}\} \in \mathfrak{U}$,

(iii) *for every* $\mathcal{U} \in \mathfrak{U}$ *there exists a* $\mathcal{W} \in \mathfrak{U}$ *with* $\mathcal{W} \circ \mathcal{W} \in \mathfrak{U}$, *the operation* "$\circ$" *being defined as follows:* $\mathcal{U} \circ \mathcal{V} = \{(x, y)\colon$ *there is a* z *with* $(x, z) \in \mathcal{U}$, $(z, y) \in \mathcal{V}\}$.

(iv) *for every* $x \in X$ *the family of all the sets* $\mathcal{U}(x) = \{y \in X\colon (x, y) \in \mathcal{U}\}$, *with* $\mathcal{U} \in \mathfrak{U}$, *is a basis of neighbourhoods of the point* x *in the topology of* X.

The filter $\mathfrak{U}$ is called the *uniformity* of X and its members are called the *neighbourhoods of the diagonal* or *entourages*.

A mapping φ between two uniform spaces $\langle X, \mathfrak{U}\rangle$ and $\langle Y, \mathfrak{V}\rangle$ is said to be uniformly continuous iff for every $\mathcal{V} \in \mathfrak{V}$ we have $\{(x, x') \in X \times X\colon (\varphi x, \varphi x') \in \mathcal{V}\} \in \mathfrak{U}$. A one-to-one mapping h from X onto Y is called a *uniform homeomorphism* iff both h and h^{-1} are uniformly continuous.

In the sequel we shall write $\mathcal{U}^n = \underbrace{\mathcal{U} \circ \mathcal{U} \circ \ldots \circ \mathcal{U}}_{n \text{ times}}$.

A uniform space $\langle X, \mathfrak{U}\rangle$ is called *uniformly bounded* iff for every $\mathcal{U} \in \mathfrak{U}$ there is a positive integer n, with $\mathcal{U}^n = X \times X$ ([6]). X is said to be *locally uniformly bounded* iff there exists an open set V such that for every $\mathcal{U} \in \mathfrak{U}$ we have $V \times V \subset \mathcal{U}^n$ for some n.

There exist uniformly bounded linear metric spaces; for instance the space S of all measurable functions defined on $\mathfrak{I}$. No locally convex l.t.s. is uniformly bounded. As we shall see later, among the locally convex spaces only the normed spaces are locally uniformly bounded.

Of course, uniform boundedness (local uniform boundedness) is an invariant under uniform homeomorphisms; moreover, it is easy to show that

11.2. *If* X *is uniformly bounded (locally uniformly bounded), then the image of* X *under an arbitrary uniformly continuous (uniformly continuous and open) mapping is uniformly bounded (locally uniformly bounded).*

DEFINITION 6. Let $\langle X, \mathfrak{U}\rangle$ be a uniform space and let $\mathcal{U}, \mathcal{V} \subset \mathfrak{U}$. Write

$$\tilde{M}(\mathcal{U}, \mathcal{V}, \varepsilon) = \sup\{n\colon \text{for every } x \in X \text{ there are } x_1, \ldots, x_n \in \mathcal{U}^{[1/\varepsilon]}(x) \text{ such that } (x_i, x_j) \notin \mathcal{V} \text{ for } i \neq j\}.$$

Let $\tilde{\mathcal{M}}(\mathcal{U}, \mathcal{V})$ be the class of all non-negative functions $\varphi(\cdot)$, defined for $\varepsilon > 0$, such that there is an $\varepsilon_0 > 0$, with $\varphi(\varepsilon) \geqslant M(U, V, \varepsilon)$ for $\varepsilon < \varepsilon_0$. Write

$$\tilde{\Phi}(X) = \bigcap_{\mathcal{V} \in \mathfrak{U}} \bigcup_{\mathcal{U} \in \mathfrak{U}} \tilde{\mathcal{M}}(\mathcal{U}, \mathcal{V}).$$

$\tilde{\Phi}(X)$ will be called the *approximative dimension* of the uniform space X.

(6) Spaces with this property were considered in Atsuji [1] and were called "finitely chainable". The term "uniformly bounded" has been proposed by Isbell [33].

Observe that the conditions $\mathfrak{V}_1 \subset \mathfrak{V}_2$ and $\mathfrak{U}_2 \subset \mathfrak{U}_1$ imply $M(\mathfrak{U}_1, \mathfrak{V}_1) \subset M(\mathfrak{U}_2, \mathfrak{V}_2)$. Hence

11.3. *If $\mathfrak{B}$ is any basis for the uniformity $\mathfrak{B}$ of a uniform space X, then* $\Phi(X) = \bigcap_{\mathfrak{V} \in \mathfrak{B}} \bigcup_{\mathfrak{U} \in \mathfrak{B}} M(\mathfrak{U}, \mathfrak{V})$.

We can also easily check that

11.4. *If Y is uniformly homeomorphic with X, then* $\tilde{\Phi}(Y) = \tilde{\Phi}(X)$.

Now assume that X is a locally convex linear topological space. The uniformity $\mathfrak{U}$ of X is then determined by the class $\mathfrak{C}$ of all convex symmetric neighbourhoods of zero. Namely, the family $\{\hat{U}: U \in \mathfrak{C}\}$, with $\hat{U} = \{(x, y) \in X \times X: x - y \in U\}$, then is a basis of the filter $\mathfrak{U}$. From Lemma 3 it follows that

$$(*) \qquad \widehat{n \cdot U} = \hat{U}^n \quad (n = 1, 2, \ldots).$$

This means that *X is locally uniformly bounded* iff X is locally bounded as a linear topological space, i.e., according to Kolmogorov [60], *iff X is isomorphic to a normed space.* Hence, by 11.2,

11.5. *If X is a normed space and Y is a non-normable F-space, then Y is not uniformly homeomorphic with X.*

Assume that U and V are in $\mathfrak{C}$. Put

$$M(U, V, \varepsilon) = \{\sup n: \text{there are } x_1, \ldots, x_n \in \frac{1}{\varepsilon} U, \text{ with } x_i - x_j \notin V \ (i \neq j)\}.$$

Now define $M(U, V)$ and $\Phi(X)$ in the same way as in Definition 6, with $\tilde{M}(\ldots)$ replaced by $M(\ldots)$ and $\mathfrak{U}$ replaced by $\mathfrak{C}$.

Since

$$\frac{1}{2\varepsilon} \leqslant \left[\frac{1}{\varepsilon}\right] \leqslant \frac{1}{\varepsilon} \quad \text{for} \quad 0 < \varepsilon < 1,$$

from $(*)$ it follows that

$$M(U, 4V, \varepsilon) \leqslant \tilde{M}(U, V, \varepsilon) \leqslant M(U, V, \varepsilon).$$

Hence, by 11.3, we get

11.6. *If X is a locally convex linear topological space, then* $\tilde{\Phi}(X) = \Phi(X)$.

The functor $\Phi(X)$ is a known invariant under isomorphisms of linear topological spaces, called the *Kolmogorov approximative dimension.* It has been studied and computed for concrete spaces, in Pełczyński [73], Kolmogorov [53], Mitiagin [63], Rolewicz [75]. According to 11.4 and 11.6 the Kolmogorov approximative dimension is also an invariant under uniform homeomorphisms. We know many examples of separable infinite-dimensional F-spaces distinguishable by means of $\Phi(\cdot)$. For instance, if H_n denotes the space of all entire functions of n complex variables, then $\Phi(H_n) \neq \Phi(H_m)$ for $n \neq m$ (cf. Rolewicz [75]). It has been shown also that such properties of F-spaces as being a Schwartz space or being nuclear can be characterized by means of approximative dimension.

The above method cannot be applied to infinite-dimensional B-spaces, because $\Phi(\cdot)$ does not distinguish between such spaces. The uniform structures of Banach spaces have been studied in a recent paper [62] by Lindenstrauss. Examples of non-uniformly homeomorphic separable infinite-dimensional B-spaces are given in it. All this implies that classification with respect to uniform homeomorphisms does not coincide with the topological classification as well in the case of Banach spaces as in the case of F-spaces. The following problem is still open.

PROBLEM 17. Do there exist two non-isomorphic F-spaces, which are uniformly homeomorphic?

§ 12. Abstract decomposition scheme; invariant infinite powers; radial homeomorphism. It is easily seen that proposition 8.3 can be generalized as follows:

12.1. *Let $\mathfrak{A}$ be an algebra consisting of objects $X, Y, \ldots$ with two operations $X \times Y$ and X^∞ and one relation $\sim$. Assume that the following axioms are satisfied:*

(α) *$X \sim X$; $X \sim Y$ and $Y \sim Z$ imply $Z \sim X$.*

(β) *$X \sim X_1$ and $Y \sim Y_1$ implies $X \times Y \sim X_1 \times Y_1$.*

(γ) *$X \sim Y$ implies $X^\infty \sim Y^\infty$.*

(δ) *$(X \times Y) \times Z \sim X \times (Y \times Z)$; $(X \times Y)^\infty \sim X^\infty \times Y^\infty$.*

If X and Y are elements of $\mathfrak{A}$ such that $X \sim Z \times Y$, $Y \sim X \times W$ for some Z, $W \in \mathfrak{A}$, and $Y \sim Y^\infty \sim Y \times Y$, then $X \sim Y$.

Pełczyński in his paper [70] used a concrete model of scheme 12.1, in which $\mathfrak{A}$ is the class of all Banach spaces, with $\sim$ and $X \times Y$, X^∞ being interpreted as the relation of isomorphism and Cartesian product and $\Sigma_{l_p} X$, respectively. For the purposes of topological classification, the symbol $\sim$ should denote the relation of being homeomorphic; the natural interpretation of $X \times Y$ is as a Cartesian product. The infinite power X^∞ can be interpreted in different ways. For instance, as in 8.3, we may assume $X^\infty = X^{\aleph_0}$; obviously $X^{\aleph_0}$ is invariant (i.e. fulfils axiom (γ)) in the class of all topological spaces. In Bessaga-Pełczyński [7] we assumed $X^\infty = \Sigma_{c_0} X$; this is, of course, invariant in the class of all Banach spaces. Similarly, from Theorem 7.5 it follows that $X^\infty = \Sigma_{l_p} X$ is also invariant in the class of all B-spaces.

It is natural to ask if every coordinate product is invariant; in other words:

PROBLEM 18. Suppose that E is a coordinate space, and X and Y are B-spaces. Does the condition $X \approx Y$ imply $\Sigma_E X \approx \Sigma_E Y$?

Let us mention another similar question:

Problem 19. Let X, Y, X_1, Y_1 be B-spaces. Do the conditions $X\approx X_1$, $Y\approx Y_1$ imply that the tensor products $X\hat{\otimes} Y$ and $X_1\hat{\otimes} Y_1$ or $X\hat{\hat{\otimes}} Y$ and $X_1\hat{\hat{\otimes}} Y_1$ are homeomorphic?

According to Proposition 6.6, if X and Y are homeomorphic by means of a norm-preserving homeomorphism, then $\Sigma_E X\approx\Sigma_E Y$ holds.

We have

12.2. *Let X and Y be normed spaces. The following conditions are equivalent:*

(a) *There is a non-preserving homeomorphism between X and Y.*

(b) *The unit spheres in X and Y are homeomorphic.*

(c) *If X_1 and Y_1 are hyperplanes in X and in Y, then $X_1\approx Y_1$.*

(d) *There is a homeomorphism h from X onto Y such that $h\lambda x=\lambda hx$, $\|hx\|=\|x\|$ for every real λ and every $x\in X$.*

Proof. The implication (a)→(b) is obvious.

(b)→(c). This follows from Klee's [48] result stating that hyperplanes of infinite dimensional normed spaces are homeomorphic with unit spheres.

(b)→(d). If φ is a homeomorphism from the unit sphere of X onto that of Y, then $hx=\|x\|\varphi(x/\|x\|)$ for $x\neq 0$ and $h0=0$ is a homeomorphism from X onto Y with the required properties.

(d)→(a) is obvious.

Definition 7. Two normed spaces X and Y satisfying any of the equivalent conditions (a)-(d) of 12.2 are called *radially homeomorphic*.

To solve Problem 18 it would be sufficient to show that any two homeomorphic B-spaces are radially homeomorphic.

§ 13. Conjectures weaker than that of topological equivalence of all infinite-dimensional Banach spaces. Consider the following sentences:

(*1*) Every separable infinite-dimensional B-space is homeomorphic with l.

(*1'*) Every separable infinite-dimensional B-space is radially homeomorphic with l.

(*2*) Every separable infinite-dimensional B-space is divisible by s.

(*2'*) Every separable infinite-dimensional B-space is homeomorphic with a certain non-normable F-space.

(*3*) In every infinite-dimensional B-space all the closed convex bodies are homeomorphic.

(*3'*) In every infinite-dimensional F-space all the closed convex bodies are homeomorphic.

(*4*) For every infinite-dimensional B-space X, we have $X\times R\approx X$.

(*5*) Every two homeomorphic B-spaces are radially homeomorphic.

(*6*) There are only finitely many topologically different separable infinite-dimensional B-spaces.

(*7*) For every B-space X, the condition $X\times R\approx l$ implies $X\approx l$.

(*7'*) Every B-space homeomorphic with l is radially homeomorphic with l.

(*8*) In every B-space homeomorphic with l all the closed convex bodies are homeomorphic with l.

None of the above sentences has been proved. All the known implications between them can be expressed by the following diagram:

$$\begin{tikzcd}
(1') \arrow[d, leftrightarrow] & (2') \arrow[d, leftrightarrow] & (3) \arrow[r, leftrightarrow] & (3') \arrow[drr] & & \\
(1) \arrow[r] \arrow[drrr] & (2) \arrow[r] \arrow[ur] & (4) \arrow[r] & (5) \arrow[r] & (7) \arrow[r] \arrow[d, leftrightarrow] & (8) \\
 & & & (6) \arrow[ur] & (7') &
\end{tikzcd}$$

Proof. (*1*)↔(*1'*). This follows from 12.2.

(*1*)→(*2*). To prove this, it is sufficient to note that by 9.1 (iii), $s|l$.

(*2*↔(*2'*). This follows from 9.1 (ii) and (iii).

(*2*)→(*3*). Let W be a convex body in X. By Corson-Klee [19], $W\approx Z\times\mathfrak{J}^p\times(R^+)^r$, where p, r are non-negative integers and Z is a subspace of X of the deficiency $p+r$. Let Y be a separable infinite-dimensional subspace of Z. By 8.1, $W\approx(Z/Y)\times Y\times\mathfrak{J}^p\times(R^+)^r$. Now assuming (*2*), we get $W\approx(Z/Y)\times Y\times\mathfrak{J}^p\times(R^+)^r\times s$. But, by Bessaga-Klee [1], $s\approx(R^+)^r\times\mathfrak{J}^p\times s\approx s\times R^{p+r}$, whence $W\approx Z\times R^{p+r}\approx X$.

(*3*)→(*3'*). According to Bessaga-Klee [7], if X is a non-normable F-space, then all the closed convex bodies in X are homeomorphic with X.

(*4*)→(*5*)→(*7*)→(*7'*). This follows from 12.2.

(*7*)→(*8*). Let $X\approx l$ and let W be a closed convex body in X. We have $W\approx Y\times\mathfrak{J}^p\times(R^+)^r$. Since Y is of a finite deficiency in X, from the hypothesis (*7*) follows $Y\approx l$. Now, by Klee [2], $Y\times\mathfrak{J}^p\times(R^+)^r\approx l\times\mathfrak{J}^p\times(R^+)^r\approx l$.

(*6*)→(*7*). Assume $X\times R\approx l$. Put $X_0=X\times R$, $X_1=X$, $X_{n+1}=$ *a subspace of X_n of deficiency one* $(n=1,2,\ldots)$. Under hypothesis (*6*) there are p, q $(p<q)$, with $X_p\approx X_q$. But this implies that $X_{q-p}\approx X_p\times R^q\approx X_q\times R^q\approx X_0=l$. Since X is separable and $X_{q-p}|X$, by 8.2, we get $X\approx l$.

All the other arrowed implications are trivial.

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On Egoroff's theorem

by

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I. Although Egoroff's theorem [6] is usually stated for sequences one finds it used in certain instances when the collection of functions involved is non-denumerable ([5], [7]). However, several counter-examples exist in the literature which show that the conclusion of the theorem does not in general follow in this case ([2], [8], [9], [10]). Hahn and Rosenthal [3] must have realized this, although no reference to a counter-example is mentioned, since they state and prove a non-denumerable analogue to Egoroff's theorem, but by placing certain restrictions on the functions not found in the original form of the theorem. Essentially, they prove:

Let m be a measure function on an additive class of sets $\boldsymbol{A}$ of a space X, A an element of $\boldsymbol{A}$ such that $m(A) < +\infty$ and F a real function defined on $A \times (0,1)$ such that for each $x \in A$, $F(x,\cdot)$ is continuous on $(0,1)$ and for each $t \in (0,1)$, $F(\cdot,t)$ is measurable on A. If

$$\lim_{t\to 0} F(x,t) = G(x)$$

a.e. on A, where G is finite a.e. on A, then, for each $\eta > 0$, there exists a set $B \subset A$ such that $m(A-B) < \eta$ and the convergence of $F(\cdot,t)$ to G is uniform on B.

It is the purpose of this note to weaken the hypotheses of the above theorem. In what follows F, m, A, G and $\boldsymbol{A}$ are to have the same significance as above as well as the notation $F(x,\cdot)$ and $F(\cdot,t)$. We obtain our results by replacing the set $(0,1)$ with an infinite set M and varying its nature.

II. We first suppose that M is an infinite subset of a topological space Y which is Hausdorff and second countable while its closure, $\mathrm{cl}\,M$, is countably compact (see Hall and Spencer [4]). This allows us to assume without any loss that if we let M' denote the derived set of M and H a countable subset of M dense in M, then, if $p \in \mathrm{cl}\,M$ but $p \notin H$, then $p \in M'-M$. Let l.s.c. (u.s.c.) denote lower [upper] semi-continuous. If f is a real function defined on a set E and $H \subset E$ then the