Connected chains in quasi-ordered spaces

by

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We establish here a theorem on the existence of connected chains in quasi-ordered spaces. This is a partial generalization to quasi-ordered spaces of a previous result on partially ordered spaces, and furnishes an order theoretic extension of a theorem of Whyburn on the lifting of arcs through light open mappings. Throughout the paper, arc is used in the sense of "continuum irreducibly connected between two points". We do not assume metrizability of the spaces, but all spaces are assumed to be Hausdorff.

Recall that $(X, \leqslant)$ is a quasi-ordered space if X is a space and $\leqslant$ is a reflexive transitive binary relation on X. If $\leqslant$ is also antisymmetric, then $(X, \leqslant)$ is a partially ordered space. A chain in X is an ordered subset of $(X, \leqslant)$. We denote by Graph$(\leqslant)$ the set of pairs (x, y) in $X \times X$ with $x \leqslant y$. Let $L(x) = \{y | y \leqslant x\}$ and $L_x = \{y | L(y) = L(x)\}$. By the notation $y < x$ we mean that $y \in L(x) \setminus L_x$. We say that $A \subset (X, \leqslant)$ has no local minima if for each $x \in A$ and any open set V about x, there exists $y \in V \cap A$ with $y < x$. We say that A has no proper local minima if the set of elements of A which are not minimal in A has no local minima. We denote by $A \setminus B$ the complement of B in A; closure is denoted by $*$, $F(A)$ denotes the boundary of A, and $\square$ denotes the empty set.

LEMMA 1. *Let $(X, \leqslant)$ be a compact quasi-ordered space, and let V be an open set in X. If*

(1) *For each $x \in X$, $\{y\colon y \leqslant x\}$ is closed, and*

(2) *V has no local minima,*

then if C is a component of V, $C^* \cap F(V) \neq \square$.

Proof. If $C^* \cap F(V) = \square$, then there is an open and closed set N with $C_1^* \subset N \subset V$. Let T be a maximal chain in N; it follows from (1) that T has an inf in N^*, which by maximality and (2) must lie in $F(N)$, a contradiction. We note that this argument is essentially the same as that given in [1].

THEOREM 1. *Let* $(X, \leqslant)$ *be a compact quasi-ordered space, and let* W *be an open set in* X. *If*

(1) W *is a chain,*

(2) Graph $(\leqslant)$ *is closed,*

(3) W *contains no local minima,*

(4) L_x *is totally disconnected for each* $x \in X$,

then any element a *of* W *belongs to an ordered arc* K *with* $K \cap F(W) \neq \square$ *and* $a = \sup K$.

Proof. Let W be as above, and fix $a \in W$. Since $[W \cap L(a)]^*$ is a compact quasi-ordered space and $W \cap L(a)$ satisfies the above hypotheses, we may assume that $X = L(a)$ and that W is an open set in X with $a \in W \subset W^* = X$. Let δ be an open cover of X. Let V_1 be an open subset of some member of δ, with $a \in V_1 \subset W$ and $F(V_1) \cap L_a = \square$. Let C_1 be the component of V_1 containing a. By Lemma 1, $C_1^* \cap F(V_1) \neq \square$. Let $z_1 \in \inf C_1$ with $z_1 < a$. Let V_2 be an open subset of some member of δ with $a \in V_2 \subset W$ and $F(V_2) \cap L_{z_1} = \square$. Let $W_2 = V_2 \cap L(z_1)$ and let C_2 be the component of W_2 containing z_1. Then as above, $C_2^* \cap F(W_2) \neq \square$, and we choose $z_2 \in \inf C_2$ with $z_2 < z_1$. If α is a limit ordinal, set $C_\alpha = \limsup\limits_{\beta<\alpha} C_\beta^*$; we show next that C_α is a single point.

Let $x \in C_\alpha$, and note that for $\beta < \alpha$, $C_\beta^* \subset L(z_\beta)$. There is a net $\{x_\beta\} \to x$ where $x_\beta \in C_\beta^*$. Let $z_\beta \in \sup C_\beta^*$; then $\{z_\beta\}$ clusters at z, and we may assume that $\{z_\beta\}$ is strictly monotone decreasing. We show that $C_\alpha \subset L_z$. Note that $C_\alpha = \limsup\limits_{\beta<\alpha} C_\beta^* \subset \bigcap\limits_{\beta>\alpha} L(z_\beta)$, so $x \leqslant z_\beta$ for each β; hence by (2), $x \leqslant z$. If $x < z$, let V be an open set about x with $V \cap L(z) = \square$. There exists $x_\beta \in V \cap C_\beta^*$ for some $\beta < \alpha$. Then for $\beta_1 > \beta$, $C_{\beta_1}^* \leqslant C_\beta^*$, so $z < z_{\beta_1} < x_\beta < z$, a contradiction. Hence $C_\alpha \subset L_z$; but C_α is a continuum and L_z is totally disconnected, so C_α is a point. This argument also shows that $\bigcap\limits_{\beta\leqslant\alpha} C_\beta^*$ is a continuum.

Hence by transfinite induction there is a continuum $K_\delta \subset X$ with $a \in K_\delta$, $K_\delta \cap F(W) \neq \square$, and K_δ is the union of subcontinua each of which is contained in a member of δ.

Let $\mathfrak{D} = \{\delta$: δ is an open cover of $X\}$, $\mathfrak{K} = \{C | C$ is a continuum $\subset X$, $a \in C$, $C \cap F(W) \neq \square\}$. We give $\mathfrak{K}$ the finite topology, i.e., for the open sets U and V of X; let $N(U, V) = \{A |\ A$ closed $\subset X$, $A \subset U$, $A \cap V \neq \square\}$ and take $\{N(U, V)$: U, V open$\}$ as a sub-basis for the open sets in $\mathcal{S}(X)$ (the space of non-empty closed subsets of X). Then $\mathcal{S}(X)$ is compact Hausdorff, and $\mathfrak{K}$ is closed in $\mathcal{S}(X)$.

Let K be a cluster point of $\{K_\delta\}_{\delta\in\mathfrak{D}}$. We claim that K is an arc from a to $F(W)$. Note that $K \in \mathfrak{K}$. We show first that $L_x \cap K = x$ for each $x \in K$. Suppose $y \in L_x \cap K$ with $y \neq x \in K$.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be the neighborhood systems of x and y respectively, and give $\mathfrak{R} \equiv \mathfrak{D} \times \mathfrak{U} \times \mathfrak{V}$ the product direction. For $(\delta, U, V) \in \mathfrak{R}$, there exists $(\delta_1, U_1, V_1) \succcurlyeq (\delta, U, V)$ where δ_1 satisfies the condition: $O \in \delta_1$, $O \cap U_1 \neq \square \to O \subset U$, and $O' \in \delta_1$, $O' \cap V_1 \neq \square \to O' \subset V$. Then $N \equiv N(X, U_1) \cap N(X, U_2) \cap \mathfrak{K}$ is an open set containing K, so there is a refinement δ_2 of δ_1 with $K_{\delta_2} \in N$. Hence there exist C_α^*, C_β^* contained in K_{δ_2} with $C_\alpha^* \cap U_1 \neq \square$ and $C_\beta^* \cap V_1 \neq \square$, and hence $C_\alpha^* \subset U$, $C_\beta^* \subset V$. Let $K(\delta, U, V) = \bigcup \{C_\gamma^*$: γ between $\alpha, \beta\}$; then $K(\delta, U, V)$ is a continuum which meets U and V, and lies between an element of U and an element of V. Let $K(x, y)$ be a cluster point of $K(\delta, U, V)$. Then $K(x, y)$ lies between x and y (using (2)), and is a continuum containing x and y. Therefore $x = K(x, y) = y$, a contradiction.

Thus $L_x \cap K = x$, for each $x \in K$; since K is an ordered continuum, it follows that K is an arc, and the proof is complete.

Let $(X, \leqslant)$ be a quasi-ordered space. Define $\mathfrak{L} \subset X \times X$ by $(x, y) \in \mathfrak{L}$ iff $L(x) = L(y)$. If X is compact and Graph $(\leqslant)$ is closed, then $\mathfrak{L}$ is closed, and $X/\mathfrak{L}$ is a partially ordered space with closed graph. Denote the natural map by φ: $X \to X/\mathfrak{L}$. Note that φ is order preserving.

We say that a subset C of X is *biconnected* if (i) C is connected, and (ii) $L_x \cap C$ is connected, for each $x \in C$.

COROLLARY 1. *Let* $(X, \leqslant)$ *be a compact quasi-ordered space, and let* θ *be the set of minimal elements of* X. *If*

(1) *Graph* $(\leqslant)$ *is closed,*

(2) $X \backslash \theta$ *has no local minima, and*

(3) $X/\mathfrak{L}$ *is an arc*

then each element a *of* X *can be joined to* θ *with a biconnected chain.*

Proof. Let $a \in X$; consider the monotone-light factorization of φ

$$X \xrightarrow{m} M \xrightarrow{l} X/\mathfrak{L}\,.$$
$$\xrightarrow{\qquad\varphi\qquad}$$

It can be seen that M inherits a quasi-ordering from $X/\mathfrak{L}$, and has closed graph. Now $M \backslash m(\theta)$ satisfies the conditions on W in Theorem 1, so $m(a)$ belongs to an ordered arc B with $m(\theta) \cap B \neq \square$ and $m(a) \in \sup B$. Thus $m^{-1}B$ is a biconnected chain in X joining a to θ.

In attempting to weaken (3) of Corollary 1, we are faced with the problem of finding conditions on X which insure that if B is an arc in $X/\mathfrak{L}$, then $\varphi^{-1}B$ has no proper local minima. In this connection we have

LEMMA 2. *Let* $(X, \leqslant)$ *and* $(Y, \leqslant)$ *be compact quasi-ordered spaces with closed graphs, and let* f: $X \to Y$ *be continuous and satisfy* $x < y \leftrightarrow f(x) < f(y)$. *The following are equivalent*

(1) *If B is a closed chain in Y with no proper local minima, then $f^{-1}B$ has no proper local minima.*

(2) *If B is a closed subset of Y with no proper local minima, then $f^{-1}B$ has no proper local minima.*

(3) *If B is a closed subset of Y with no proper local minima, then for $y \in B$, (y not minimal), and $t \in f^{-1}y$, there exists $\{y_\alpha\} \subset B$ with each $y_\alpha < y$, $\{y_\alpha\} \to y$, and $t \in \left(\bigcup f^{-1}(y_\alpha)\right)^*$.*

Proof. (1)$\to$(2). Let B be as in (2), and let $x \in f^{-1}B$ with x not minimal in $f^{-1}B$. Let α: $Y \to Y/\mathfrak{L}$ be the natural map; note that $\alpha(B)$ has no proper local minima, so by [1] there is an arc $C \subset \alpha(B)$ with $\alpha f(x) = \sup C$. Then $f(x) \in \alpha^{-1}C$, a closed chain, and by (1), $x \in f^{-1}\alpha^{-1}C$ which has no proper local minima. Note that x is not a minimal element of $f^{-1}\alpha^{-1}C$; otherwise let $t \in C$ with $t < \alpha f(x)$. Then it follows that for any $y \in f^{-1}\alpha^{-1}t$ we have $y < x$, a contradiction.

(2)$\to$(3). Let B be as in (3). Choose a non-minimal element $y \in B$ and $t \in f^{-1}(y)$. Since B satisfies (2), for each neighborhood W of t there is an element $t_W \in W \cap f^{-1}B$ such that $t_W < t$. Hence $\{t_W\} \to t$ so $f(t_W) \to f(t) = y$. Note that $f(t_W) < f(t)$ and $t \in \left(\bigcup f^{-1}f(t_W)\right)^*$.

(3)$\to$(1). Let B be a closed chain in Y with no proper local minima; let $t \in f^{-1}y$ for some non-minimal $y \in B$, and let V be an open set about t. By (3), there is a net $\{y_\alpha\} \subset B$ with each $y_\alpha < y$, $\{y_\alpha\} \to y$, and $t \in \left(\bigcup f^{-1}(y_\alpha)\right)^*$. Then $V \cap f^{-1}(y_\alpha) \neq \square$ for some α. Let $x \in V \cap f^{-1}(y_\alpha)$; then $f(x) = y_\alpha < y = f(t)$, so $x \in t$. Hence $f^{-1}B$ has no proper local minima, and the proof is complete.

DEFINITIONS. 1) Let X, Y, and f be as above. If f satisfies one of the conditions of Lemma 2 we say that f is *dense from below*.

2) We say that f is *open from below* if: $(y_\alpha) \subset Y$, each $y_\alpha < y$, $\{y_\alpha\} \to y$ imply $f^{-1}(y_\alpha) \to f^{-1}(y)$ (i.e., $\limsup f^{-1}(y_\alpha) = f^{-1}(y) = \liminf f^{-1}(y_\alpha)$).

COROLLARY 2. *Let X, Y, and f be as above. If f is open from below, then f is dense from below.*

Proof. Let B be a closed subset of Y with no proper local minima. Let y be a non-minimal element of B, and let $t \in f^{-1}(y)$. Since y is not a local minimum, there is a net $\{y_\alpha\} \subset B$ with each $y_\alpha = y$ and $\{y_\alpha\} \to y$. Since f is open from below, $f^{-1}(y_\alpha) \to f^{-1}(y)$. In particular we have $f^{-1}(y) = \limsup f^{-1}(y_\alpha) \subset \left(\bigcup f^{-1}(y_\alpha)\right)^*$, so f is dense from below.

COROLLARY 3. *Let $(X, \leqslant)$ and $(Y, \leqslant)$ be compact quasi-ordered spaces with closed graphs. Let f: $X \to Y$ be continuous and onto, with $x < y \leftrightarrow f(x) < f(y)$. If f is dense from below, then for any ordered arc $B \subset Y$ there is a biconnected chain $T \subset X$ with $f(T) = B$.*

Proof. Since f is dense from below, $f^{-1}B$ has no proper local minima. The conclusion now follows from Corollary 1, where X is replaced by $f^{-1}B$. Note that a similar conclusion holds if B has no proper local minima and is minimal with respect to being a closed chain.

We note that Corollary 3 contains the arc-lifting theorem of Whyburn ([2], p. 186). For, let X and Y be compact spaces with f: $X \to Y$ continuous, open, and onto, and let A be an arc in Y. Then the natural ordering on A induces a quasi-ordering on $f^{-1}A$, and the graph is closed since f is continuous. Let $f_1 = (f|f^{-1}A)$: $f^{-1}A \to A$; then f_1 is open and hence dense from below. By Corollary 3, there is a biconnected chain $T \subset f^{-1}A$ with $f(T) = A$. If further f is light, then T is an arc.

COROLLARY 4. *Let $(X, \leqslant)$ be a compact quasi-ordered space with unique minimal element 0. If*

(1) *Graph $(\leqslant)$ is closed,*

(2) *$L(x)$ is connected, for each $x \in X$, and*

(3) *φ: $X \to X/\mathfrak{L}$ is dense from below,*

then each element a of X lies in a biconnected chain T with $0 \in T$ and $a \in \sup T$.

Proof. Since $L(x)$ is connected, $\varphi L(\mathfrak{S})$ is connected. Thus by [1] there is an ordered arc $A \subset \varphi(X)$ from $\varphi(a)$ to $\varphi(0)$. The conclusion follows from Corollary 3.

Remarks. 1. It is conjectural that (1) of Theorem 1 may be deleted; perhaps an argument of the sort given in Theorem 2 of [1] may reveal this.

2. It would be of interest to have further information about the biconnected chain T in Corollary 1. For example, does there exist T with no proper local minima?

References

[1] R. J. Koch, *Arcs in partially ordered spaces*, Pac. J. Math. 9 (1959), pp. 723-728.

[2] G. T. Whyburn, *Analytic topology*, American Mathematical Society, New York, 1952.

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Reçu par la Rédaction le 21. 12. 1963