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Confluent mappings and unicoherence of continua

by

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§ 1. Introduction. A new kind of continuous mappings will be introduced and studied in this paper, namely the notion of *confluent mapping* (¹), which comprises, among other mappings, interior, monotone and, for locally connected continua, also quasi-monotone mappings.

In particular, some theorems concerning the invariance of unicoherence and of hereditary unicoherence, known only for some of the three kinds of continuous mappings quoted above, will be generalized to confluent mappings (theorems X and XIV). This task has arisen from investigations of dendroids (see [1] and [2]) and is intended to be applied to them (see § 6).

The essentiality of the hypotheses assumed will be shown by examples (see the final parts of § 3 and § 5).

I am very much indebted to Professor B. Knaster, who contributed to my investigations his kind advice and valuable improvements.

§ 2. Definitions and preliminary properties. A continuous mapping f of a topological space X onto a topological space Y is *confluent* if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q.

Confluent mappings have the following properties I-VII:

I. *If f is a confluent mapping of X onto Y, B is a subset of Y, and A is the union of some components of $f^{-1}(B)$, then the partial mapping $g = f|A$ is a confluent mapping of A onto $f(A)$.*

Proof (²). Let Q be a subcontinuum of $f(A)$ and C a component of $g^{-1}(Q)$. Since

$$g^{-1}(Q) = A \cap f^{-1}(Q) , \tag{1}$$

C lies in a component C' of $f^{-1}(Q)$. It follows from $C \subset A$ that

$$0 \neq C = A \cap C \subset A \cap C' , \tag{2}$$

(¹) denomination by Professor B. Knaster.
(²) simplified by A. Lelek.

and from $Q \subset f(A) \subset B$ that

(3) $$C' \subset f^{-1}(B).$$

According to the hypothesis regarding A, conditions (2) and (3) give $C' \subset A$, whence $C' \subset g^{-1}(Q)$ by (1). Thus $C' = C$ and $g(C) = f(C) = f(C') = Q$, f being a confluent mapping.

The following corollary is a particular case of I:

II. *If f is a confluent mapping of X onto Y and*

$$A = f^{-1}f(A) \ (^3),$$

then the partial mapping $g = f|A$ is a confluent mapping of A onto $f(A)$

Obviously,

III. *If f_1 and f_2 are confluent mappings of X onto Y and Y onto Z respectively, then $f = f_2 f_1$ is a confluent mapping of X onto Z.*

IV. *If $f = f_2 f_1$ is confluent, so is also f_2.*

Remark that f_1 need not then be confluent, as is shown by simple examples.

Now, recall that a continuous mapping f is:

interior if f maps every set open in X onto a set open in Y (see [11], p. 348);

quasi-monotone if X is a continuum and if for any continuum Q in Y with a non-vacuous interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q (see [12], p. 136);

monotone if $f^{-1}(y)$ is a continuum for each point $y \in Y$ (see [15], p. 127); or, which is equivalent, if for any continuum Q in Y the set $f^{-1}(Q)$ has only one component (see [9], p. 123).

Hence

V. *Any monotone mapping of a continuum is confluent,*

and it follows by Whyburn's theorem (7.5) in [15], p. 148 that also

VI. *Any interior mapping of a compact space is confluent,*

while by Wallace's theorem (2.3) in [12], p. 138

Any quasi-monotone mapping of a locally connected continuum is confluent,

and inversely (see here theorem IX, p. 215).

The class of confluent mappings is essentially larger than the classes of monotone and interior mappings: it is easy to find confluent mappings (even among those of dendroids) which belong to none of the above-mentioned particular classes of confluent ones (even in a certain local sense).

(³) According to Whyburn ([15], p. 137) such a subset A of X is said to be an *inverse* set under the mapping f.

A direct consequence of Whyburn's factorization theorem (see [14], (2.3), p. 297) and of IV is the following factor theorem for confluent mappings:

VII. *If X is compact and if f is a confluent mapping of X onto Y, then there exists a unique factorization of f into two confluent mappings:*

$$f(x) = f_2 f_1(x), \quad x \in X,$$

where f_1 is monotone and f_2 is 0-dimensional (⁴).

§ 3. Invariants in locally connected continua. Henceforth the topological spaces under consideration will be assumed to be metric continua.

VIII. *If a continuum X is locally connected and a mapping f of X onto Y is confluent, then for every subcontinuum $Q \subset Y$ such that* $\mathrm{Int}(Q) \neq 0$ *the number of components of $f^{-1}(Q)$ is finite.*

Proof. Assume that $f^{-1}(Q)$, where Q is a subcontinuum of Y, has infinitely many components. Thus there exists a sequence of those components $C_1, C_2, \ldots$ convergent to the limit C_0 which is disjoint with them. In fact, C_0 is a continuum as a limit of continua; consequently, there is at most one n such that $C_0 \cap C_n \neq 0$; hence it suffices to omit C_n in this sequence. $f^{-1}(Q)$ being a compactum, $C_0 \subset f^{-1}(Q)$. Further, C_0 being a continuum, there exists a component C of $f^{-1}(Q)$ such that $C_0 \subset C$. Since C_0 is a limit of continua C_n which are disjoint with C_0, thus, C_n being components of $f^{-1}(Q)$, also with C, we have $C_0 \subset \overline{X-C}$. Consequently, $C_0 \subset C \cap \overline{X-C}$, i.e. $C_0 \subset \mathrm{Fr}(C)$. But $\mathrm{Fr}(C) \subset \mathrm{Fr}\big(f^{-1}(Q)\big)$ by Kuratowski's theorem on locally connected spaces (see [9], § 44, III, 3, p. 169). Therefore $C_0 \subset \mathrm{Fr}\big(f^{-1}(Q)\big) = f^{-1}(Q) \cap \overline{X - f^{-1}(Q)}$, whence $C_0 \subset \overline{X-f^{-1}(Q)}$, which implies that $f(C_0) \subset f\big(\overline{X-f^{-1}(Q)}\big)$. Since $f(C_0) = Q$ by the confluence of the mapping f as well as by the continuity of f, and since $f\big(\overline{X-f^{-1}(Q)}\big) \subset \overline{f\big(X-f^{-1}(Q)\big)}$ by the continuity of f, we conclude that $Q \subset \overline{f\big(X-f^{-1}(Q)\big)}$, i.e. that $Q \subset \overline{Y-Q}$, whence $\mathrm{Int}(Q) = 0$.

It follows from VIII by the definitions of confluence and of quasi-monotoneity that

IX. *If f is a confluent mapping of a locally connected continuum, then f is quasi-monotone.*

Consequently, *all the invariants of locally connected continua under quasi-monotone mappings are also invariants under confluent mappings.*

(⁴) i.e. such that $\dim f_2^{-1}(y) = 0$ for each $y \in Y$ (see [15], p. 130, "light transformations").

In particular, unicoherence being an invariant under quasi-monotone mappings (see [12], p. 144), we conclude that

X. *The unicoherence of locally connected continua is an invariant under confluent mappings.*

It should be observed that the hypothesis of local connectedness is essential in corollaries IX and X. The following example proves this essentiality.

Let S be the curve composed of circumferences $\varrho = 1$ and $\varrho = 2$, and of the spiral line $\varrho = (2+e^{\varphi})/(1+e^{\varphi})$, where $-\infty < \varphi < +\infty$, which approximates both these circumferences. It is easy to state that the curve S is unicoherent and not locally connected, while its projection from the origin onto the circumference, e.g. $\varrho = 1$ (thus onto a non-unicoherent continuum), is an interior mapping, hence by VI a confluent one, but it is not a quasi-monotone mapping, because each arc of the circumference $\varrho = 1$ is under this projection an image of infinitely many disjoint arcs of the spiral part of the curve S.

§ 4. Mapping into the circumference. First, recall some known notions. Let $\mathcal{S}$ be the circumference $|z| = 1$ and $\mathcal{E}$ the straight line $\mathrm{Im}(z) = 0$.

A continuous mapping f of a separable metric space X into $\mathcal{S}$ is said to be:

inessential or *essential* according as it does or does not belong to the same component of the functional space $\mathcal{S}^X$ as the mapping $f_0(x) = 1$, where $x \in X$ (see [5], p. 161);

equivalent to 1 on a set $A \subset X$, written $f \sim 1$ on A, provided that there exists a continuous mapping $\varphi(x)$ of A into $\mathcal{E}$ such that $f(x) = e^{i\varphi(x)}$ for $x \in A$ (see e.g. [9], § 51, II, p. 310);

irreducibly non-equivalent to 1 on $A \subset X$, written f irr non ~ 1 on A, provided that f non ~ 1 on A and that $f \sim 1$ on every closed and proper subset $F \subset A$ (see [9], § 51, VII, p. 322).

A space X is said to be *discoherent* provided that it is connected and that for every decomposition $X = A \cup B$ on closed connected sets A and B such that $A \neq X \neq B$ the intersection $A \cap B$ is not connected (see [9], § 41, X, p. 104).

It follows from known theorems that if f is a continuous and essential mapping of a continuum X into $\mathcal{S}$, then there exists in X a subcontinuum C with property $f|C$ irr non ~ 1 (see [5], p. 162 and [9], § 51, VIII, 3, p. 325), and every such C is discoherent (see [9], § 51, VII, 1, p. 322), i.e. either indecomposable or decomposable into only two subcontinua such that neither of them is contained in the other and that their intersection is non-connected. Hence in particular

XI. *Every continuous mapping of a hereditarily decomposable and hereditarily unicoherent continuum into a circumference is inessential.*

The following theorem will now be proved.

XII. *There is no confluent mapping of a hereditarily decomposable and hereditarily unicoherent continuum onto a circumference.*

Proof (5). Let X be such a continuum and suppose that f is a confluent mapping of X onto the circumference. Since f is inessential by XI, there exists by Eilenberg's theorem 1 in [5], p. 162, a continuous mapping $\varphi\colon X \to \mathcal{E}$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in X$. Thus f can be factored as follows:

$$f(x) = \psi\varphi(x) \quad \text{for} \quad x \in X, \tag{4}$$

where $\varphi(x)$ is the continuous real-valued mapping defined above, and $\psi(t)$ is the expotential function $\psi(t) = e^{it}$ for $t = \varphi(x) \in \mathcal{E}$.

As a subcontinuum of $\mathcal{E}$, the image $\varphi(X)$ is a straight segment. Denote it by I:

$$\varphi(X) = I.$$

It follows from (4) that the partial mapping $\psi|I$ is onto, because f is onto by hypothesis, and that $\psi|I$ is confluent by IV, because f is confluent by hypothesis. Thus the circumference $\psi(I)$ were unicoherent by X, because the segment I is a unicoherent and locally connected continuum. But this contradicts the discoherence of circumference.

§ 5. Invariance of hereditary decomposability and hereditary unicoherence. First we have the following theorem:

XIII. *The hereditary decomposability of continua is an invariant under confluent mappings.*

In fact, let Q be an indecomposable subcontinuum of the continuous image $f(X)$ of a continuum X, and C an arbitrary component of $f^{-1}(Q)$. If the mapping f were confluent, we should have $f(C) = Q$, and therefore C would contain an indecomposable subcontinuum (see [9], § 43, V, 4, p. 146).

Now we prove the following main theorem:

XIV. *The conjunction of the hereditary decomposability and the hereditary unicoherence of continua is an invariant under confluent mappings.*

Proof. If X is such a continuum, then by XIII its confluent image $f(X)$ is also hereditarily decomposable. It remains to show that $f(X)$ is hereditarily unicoherent.

Suppose that it is not so. Thus there exists a subcontinuum M of $f(X)$ which is hereditarily decomposable but not unicoherent, and

(5) I am indebted to A. Lelek for the first proof of this theorem.

therefore contains (see [10], theorem 2.6, p. 187, and apply a homeomorphism of a simply closed curve onto the circumference) a subcontinuum N which has an upper semi-continuous decomposition

$$N = \bigcup_{t \in S} N_t \tag{5}$$

on mutually disjoint continua N_t such that the hyperspace of this decomposition is the circumference S. That means (see [15], (3.1), p. 125) the existence of a continuous mapping ϑ of N onto S such that $N_t = \vartheta^{-1}(t)$ for each $t \in S$; thus ϑ is monotone, which implies by V that

(6) the mapping ϑ is confluent.

Let C be a component of $f^{-1}(N)$. Thus $C \subset X$, whence by hypothesis

(7) C is hereditarily decomposable and hereditarily unicoherent.

Hence according to theorem I the mapping $f|C$ is confluent by the confluence of f. Thus the superposition $\vartheta(f|C)$ is confluent by (6) and III, and it maps C onto S, which contradicts XII by (7).

In theorems XI, XII and XIV the hypothesis of hereditary decomposability is essential. It can be proved by the following example, described here in van Dantzig's notation (see [3], p. 106). Let Σ_n be van Dantzig's solenoid (see also [4], pp. 73-76) lying in a torus T_0 with centre of symmetry p. Let E be an arbitrary meridial half-plane of T_0, ${}_EK$ the circumference $T_0 \cap E$, ${}_EM$ its centre, and O the equatorial circumference composed of all centres ${}_EM$ for mobile E. As is well known, Σ_n is indecomposable and thus unicoherent; it is even hereditarily unicoherent, because every proper subcontinuum of Σ_n is an arc or a point. The projection of T_0 onto the circumference O, i.e. the mapping of each ${}_EK$ onto its centre ${}_EM \in O$, is evidently an essential mapping transforming open subsets of the solenoid $\Sigma_n \subset T_0$ onto open subsets of O. Hence it is an essential and confluent mapping of the hereditarily unicoherent continuum Σ_n onto the non-unicoherent continuum O.

Also the hypothesis of hereditary unicoherence is essential in theorems XI, XII and XIV. In fact, consider the same example as that described in the final part of § 3. As regards theorem XI, it is easy to show that the projection, considered in it of S onto the circumference $\varrho = 1$ is an essential mapping. Further, as regards theorems XII and XIV, the same projection is an interior, hence by VI a confluent mapping, of a unicoherent but not hereditarily unicoherent continuum S onto the circumference, and thus onto a non-unicoherent continuum.

§ 6. Applications to dendroids. Problems and remarks.

Recall that a dendroid is an arcwise connected and hereditarily unicoherent continuum (see, e.g. [1], p. 239). Since every dendroid is hereditarily decomposable (see [1], (47), p. 239), the following corollaries are special cases of theorem XIV:

Corollary 1. *Every confluent image of a dendroid is a dendroid*, whence in particular by V and VI

Corollary 2. *Every interior, or monotone image of a dendroid is a dendroid.*

Eilenberg proved (see [5], theorems 5, p. 165 and 14, p. 176) that if the space X is compact, then in two cases, namely if f is a monotone and if f is an interior mapping of X, the connectedness of the space S^X implies the connectedness of the space $S^{f(X)}$. The problem arises whether, more generally, the theorem remains true for confluent mappings f. In equivalent words: does the inessentiality of all continuous mappings of a continuum X into S imply that of all continuous mappings of confluent images $f(X)$ into S?

Ward (see [13], p. 13) calls a continuous mapping $f\colon X \to Y$ *pseudo-monotone* if for each pair of closed connected sets $A \subset X$ and $B \subset f(A)$, some component of $A \cap f^{-1}(B)$ is mapped by f onto B. Simple examples show that the pseudo-monotoneity of f neither implies nor is implied by its confluence, by its interiority, by its monotoneity, and so by its quasi-monotoneity, even if X is a hereditarily decomposable or a hereditarily unicoherent continuum. The problem arises whether the confluence of f implies its pseudo-monotoneity if the continuum X is simultaneously hereditarily decomposable and hereditarily unicoherent?

Remark that the positive answer to this problem would imply at once by Ward's fixed point theorem (see [13], Corollary 1.1, p. 14) that each continuum X, all non-trivial subcontinua of which have cut-points (the *hereditary divisibility* by points), has a fixed point under confluent mapping of X onto X.

Remark, however, that the problem of such fixed point theorem for continua X which are simultaneously hereditarily decomposable and hereditarily unicoherent would remain nevertheless unsolved, because the class of such continua is really larger than that of continua which are hereditarily divisible by points. Indeed, on the one hand, there exist continua hereditarily decomposable and hereditarily unicoherent without cut-points [6]. On the other hand, every X which contains an indecomposable subcontinuum L is not hereditarily divisible by points, because L has no cut-point (see [9], § 43, V, 1, p. 145). Similarly, every continuum X which is hereditarily decomposable but contains a non-uni-

[6] Such is, for instance, Knaster's continuum $\mathfrak{N}_Y$ irreducible between two points (see [7], p. 296) which arises from his continuum $\mathfrak{N}_V$ (see [6], p. 570 and 571) by replacing its strata in form V by strata in form Y, vertical segments of which have lengths tending to zero.

coherent continuum M, contains also by already quoted Miller's theorem (see [10], p. 187, theorem 2.6) a subcontinuum $N \subset M$ which has an upper semi-continuous decomposition (15) on mutually disjoint continua, such that the hyperspace of this decomposition is the circumference. Thus N contains no cut-point.

B. Knaster even asks (New Scottish Book, problem 526) whether the simultaneously hereditarily decomposable and hereditarily unicoherent continua (which he calls "λ-dendroids") have fixed points under arbitrary continuous mappings.

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A correction to my paper: "Continua meeting an orbit at a point"

(Fundamenta Mathematicae 52 (1963), pp. 319-321)

by

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As pointed out to me by L. W. Anderson, there is an error in the proof of lemma 2, and in fact the statement is false as it stands. While this can be partially rectified, the author is at present unable to retrieve the theorem in its stated generality. The extra condition we must impose is as follows: (We do not assume that X is connected.)

($*$) *Let* $\varphi\colon X \to X/G$ *be the natural projection. There is an invariant naighborhood* V *of* $G(p)$ *in* X *such that if* U *is any neighborhood of* $\varphi(p)$ *in* X/G, *there is a neighborhood* $W \subset U$ *such that the component* A *of* $\varphi(p)$ *in* W^- *meets* $X/G-W$ *and if* $a \in A \cap X/G-W$, *the component of* a *in* $\varphi(V^-)-W$ *meets* $X/G-\varphi(V)$.

Condition ($*$) is implied by condition (ii) of Anderson and Hunter, and its form is more suitable for the induction argument. Our proof now is in a similar vein to that of Anderson and Hunter, but uses a slice instead of a local cross section to eliminate condition (i) and a revised form of lemma 2 to overcome the finite induction in their argument. The proof of the *theorem* remains unchanged except in one detail.

Since ($*$) is satisfied for any compact connected semigroup, the corollary remains valid.

We now describe how lemma 2 may be altered in order to give the desired result.

First, in place of (iii) of lemma 2 we must have

(iii) $M \cap (X-V) \neq \emptyset$ *for a fixed invariant neighborhood* V *in* X,

and in place of (iv), we have

(iv) M/K *is connected and satisfies* ($*$) *with respect to* $M \cap V$ *and* K.

Now in the proof we proceed exactly as before, but taking $U \subset \pi(V)$ and such that W satisfies ($*$) with respect to U, up to the point where we define M_0. Let A be the component of W^- containing $\pi(p)$. Since M/K is connected, A meets $U^- - W$. Let B_a be the component of $M/K-W$ containing a, where a is any point of $A \cap (M/K-W)$. Then B_a meets