Thus for $x \in E$ and $y \in E$, we have

$$|T_k U(x) - T_k(x)| < \tfrac{1}{2}\delta\big(T_k(E), E\big), \qquad |V(y) - y| < \tfrac{1}{4}\delta\big(T_k(E), E\big).$$

On the other hand,

$$|T_k(x) - y| \geqslant \delta\big(T_k(E), E\big).$$

Hence

$$|T_k U(x) - V(y)| > \tfrac{1}{4}\delta\big(T_k(E), E\big),$$

and therefore the sets $T_k U(E)$ and $V(E)$ are disjoint, as was to be shown.

Remark. If $\mathfrak{F}$ is a closed group of rigid motions, and $E$ is a bounded closed set, then (as we have just proved) we can place $\mathfrak{c}$ disjoint transforms of $E$ in any neighborhood of $E$ if $\mathfrak{F}$ contains arbitrarily small motions taking $E$ into sets disjoint from $E$, and we see that otherwise only a finite number of disjoint transforms of $E$ can be placed in any bounded region.

For other closed semigroups of affine transformations, this result can fail in various ways. On the one hand, if $\mathfrak{F}$ is the family of all translations of the form $T(x) = x + ht$ with $t$ fixed and $h = 0$ or $h \geqslant 1$, then we may be able to place $\mathfrak{c}$ transforms of $E$ in a bounded region even though $\mathfrak{F}$ contains no small motions. On the other hand, if $\mathfrak{F}$ is the group of all similarity transformations, then we can always place at least $\aleph_0$ disjoint transforms of $E$ in any open set; the maximum number is $\mathfrak{c}$ or $\aleph_0$ according as there do or do not exist similarities arbitrarily near the identity which take $E$ into sets disjoint from $E$.

Finally, we mention one additional case where the result *is* correct. If $\mathfrak{F}$ is the family of all similarity transformations with magnification at least 1, then the maximum number of copies of $E$ which can be placed in a bounded neighborhood of $E$ is $\mathfrak{c}$ or finite according as $\mathfrak{F}$ does or does not contain similarities arbitrarily near the identity which take $E$ into sets disjoint from $E$. Notice that allowing enlargement may help in packing; we can place $\mathfrak{c}$ disjoint enlarged copies of a spherical surface in a bounded region, whereas there is room for only a finite number of congruent copies.

### References

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[2] R. M. Robinson, *Conjugate algebraic integers in real point sets*, Math Zeitschrift 83 (1964), pp. 415-417.

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# Correction to "On classes of abelian groups"

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by

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The purpose of the present paper is to correct the relations between invariants of a $p$-primary group of bounded order, its subgroup and its factor group that were stated in Lemmas 1 and 2 of [1]. These relations do not determine all factor groups involved. Elements $g_\lambda$, $a_\lambda$ considered in the proof of Lemma 1 (p. 157) were erroneously assumed to form bases of groups $G$ and $A$. This fact was pointed out by Dr. E. James Peake, Jr., I wish to thank Dr. Peake for his comments.

Lemmas 1, 2, 4, Definition 3 and Remark on p. 165 of [1] should be replaced by the following Lemmas 1′, 2′, 4′, Definition 3′ and Remark′.

Let $G$ be any $p$-primary group of bounded order, i.e., $p^M G = 0$ for some natural number $M$. Then there exists a direct decomposition

$$G = G_1 + G_2 + \ldots + G_M$$

such that the groups $G_n$ are direct sums of groups $Z(p^n)$, $n = 1, 2, \ldots, M$. We denote by $\mathfrak{g}$ the invariant of $G$, which is a function defined by

$$\mathfrak{g}(n) = \begin{cases} \dim G_n & \text{for} \quad n = 1, 2, \ldots, M, \\ 0 & \text{for} \quad n > M. \end{cases}$$

LEMMA 1′. *If $A$, $G$ are $p$-primary groups $p^M G = p^M A = 0$ and $\mathfrak{a}$, $\mathfrak{g}$ are invariants of $A$ and $G$ respectively, then $A \underset{\approx}{\subset} G$ iff there exist cardinal numbers $\mathfrak{h}_n(k)$, $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$, such that*

(1) $$\mathfrak{g}(n) = \sum_{k=0}^{n} \mathfrak{h}_n(k),$$

(2) $$\mathfrak{a}(n) = \sum_{k=n}^{M} \mathfrak{h}_k(n)$$

*for $n = 1, 2, \ldots, M$.*

*The invariants of factor groups* $A^1 = A/A[p]$, $G^1 = G/A[p]$ *are determined by*

(3) $$\mathfrak{g}^1(n) = \mathfrak{h}_n(0) + \sum_{k=1}^{n+1} \mathfrak{h}_{n+1}(k),$$

(4) $$\mathfrak{a}^1(n) = \sum_{k=n+1}^{M} \mathfrak{h}_k(n+1)$$

*for* $n = 1, 2, \ldots, M$.

Proof. If $A \subset G$, then there exists a direct decomposition $G = G_0 + G_1$ *such that* $G_0[p] = A[p]$ (see [2] p. 94). Let us write

$$D_{kn} = A[p] \cap p^{k-1}A \cap p^{n-1}G_0,$$
$$\mathfrak{h}_n(k) = \dim D_{kn}/(D_{k+1,n} + D_{k,n+1}),$$
$$\mathfrak{h}_n(0) = \mathfrak{g}_1(n)$$

for $k, n = 1, 2, \ldots$ ($\mathfrak{g}_1$ is the invariant of $G_1$). Then formulae (1), (2) follow immediately.

If (1) and (2) are satisfied, then we put $H_{nk} = Z(p^n)^{\mathfrak{h}_n(k)}$ and thus

$$G \approx \sum_{n=1}^{M} \sum_{k=0}^{n} H_{nk}, \quad A \approx \sum_{n=1}^{M} \sum_{k=0}^{n} p^{n-k} H_{nk},$$

whence $A \subsetapprox G$.

It is obvious that $\mathfrak{a}^1(n) = \mathfrak{a}(n+1)$ and

$$G^1 = G/A[p] \approx G_0/G_0[p] + G_1;$$

then $\mathfrak{g}^1(n) = \mathfrak{g}_0(n+1) + \mathfrak{g}_1(n)$ and formulae (3), (4) follow by appropriate substitutions.

Lemma 2'. *If the formulae* (1)-(4) *hold and the invariants of groups* $A^1$, $G^1$, $A^1 \subset G^1$, *are* $\mathfrak{a}^1$, $\mathfrak{g}^1$, *then there exist groups* $A$, $G$, $A \subset G$, *having invariants* $\mathfrak{a}$, $\mathfrak{g}$ *and an isomorphism* $\varphi$: $G/A[p] \to G^1$ *such that* $\varphi(A/A[p]) = A^1$.

Proof. There exists a direct decomposition $G^1 = G_0^1 + G_1^1$ such that $G_0^1[p] = A^1[p]$. If $\Lambda_0$, $\Lambda_1$, $\Lambda_2$ are disjoint sets of indices, $|\Lambda_2| = \mathfrak{a}(1)$ and

$$G_i^1 = \sum_{\lambda \in \Lambda_i} \{g_\lambda^1\} \quad \text{for} \quad i = 0, 1,$$

then we define $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ and

$$G = \sum_{\lambda \in \Lambda} \{g_\lambda\}$$

and the orders of elements $g$ are

$$O(g_\lambda) = \begin{cases} pO(g_\lambda^1) & \text{if} \quad \lambda \in \Lambda_0, \\ O(g_\lambda^1) & \text{if} \quad \lambda \in \Lambda_1, \\ p & \text{if} \quad \lambda \in \Lambda_2. \end{cases}$$



The formula

$$\psi(g_\lambda) = \begin{cases} g_\lambda^1 & \text{if} \quad \lambda \in \Lambda_0 \cup \Lambda_1, \\ 0 & \text{if} \quad \lambda \in \Lambda_2 \end{cases}$$

defines an epimorphism $\psi$: $G \to G^1$. Let us write $A = \psi^{-1}(A^1)$, $G_i = \sum_{\lambda \in \Lambda_i} \{g_\lambda\}$.

Then $\operatorname{Ker}\psi = A[p]$; in fact, if $g = g_0 + g_1 + g_2$ $(g_i \in G_i)$ and $\psi(g) = 0$, then $g \in A$ and $g_0 = 0$. Moreover, $pg_1 = 0$ and thus $pg = 0$. Conversely, if $g \in A[p]$, then $pg = 0$ and $\psi(g_0) + \psi(g_1) = \psi(g) \in A^1[p] = G_0^1[p]$. Consequently $\psi(g_0) = 0$; thus $g_0 = 0$ and the equality $pg_1 = 0$ implies $\psi(g_1) = 0$ and $g \in \operatorname{Ker}\psi$.

The homomorphism $\psi$ induces an isomorphism $\varphi$: $G/A[p] \to G^1$ such that $\varphi(A/A[p]) = A^1$.

Definition. If $\mathfrak{a}$, $\mathfrak{g}$, $\mathfrak{a}^1$, $\mathfrak{g}^1$ are invariants of groups $A$, $G$, $A^1$, $G^1$ such that $p^M A = p^M G = p^M A^1 = p^M G^1 = 0$, then the relation $T(\mathfrak{a}, \mathfrak{g}, \mathfrak{a}^1, \mathfrak{g}^1)$ holds iff there exist cardinal numbers $\mathfrak{h}_n(k)$, $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$ such that equalities (1)-(4) are satisfied.

Let $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ be functions defined on $P \times N$ and taking cardinal numbers as values. We put $\mathfrak{g}_p(n) = \mathfrak{g}(p, n)$, $\mathfrak{a}_p(n) = \mathfrak{a}(p, n)$, $\mathfrak{b}_p(n) = \mathfrak{b}(p, n)$.

Definition 3'. The relation $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ holds iff the following conditions are satisfied.

(4.6') *There exist integers* $M(p)$ *such that* $\mathfrak{g}(p, n) = \mathfrak{a}(p, n) = \mathfrak{b}(p, n) = 0$ *for* $n > M(p)$.

(4.7') *For each prime* $p$ *there exists a sequence of invariants of* $p$-*primary groups of bounded order* $\mathfrak{a}_p^0 = \mathfrak{a}_p$, $\mathfrak{g}_p^0 = \mathfrak{g}_p$, $\mathfrak{a}_p^1$, $\mathfrak{g}_p^1$, $\ldots$, $\mathfrak{a}_p^{M(p)} = 0$, $\mathfrak{g}_p^{M(p)} = \mathfrak{b}_p$ *such that the relations* $T(\mathfrak{a}_p^i, \mathfrak{g}_p^i, \mathfrak{a}_p^{i+1}, \mathfrak{g}_p^{i+1})$, $i = 0, 1, \ldots, M(p)-1$, *hold.*

Lemma 4'. *If* $G, A, B$ *are torsion groups having primary components of bounded orders and* $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ *are their invariants, then the relation* $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ *holds iff there exists a subgroup* $A' \subset G$ *such that* $A' \approx A$ *and* $G/A' \approx B$.

Proof. It is clear that we can restrict our considerations to the case of primary groups.

Let us suppose that invariants $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ of $p$-primary groups satisfy the relation $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$. Then there exists a sequence of invariants of $p$-primary groups of bounded orders $\mathfrak{a}^0, \mathfrak{g}^0, \mathfrak{a}^1, \mathfrak{g}^1, \ldots, \mathfrak{a}^M, \mathfrak{g}^M$ such that $\mathfrak{a}^0 = \mathfrak{a}$, $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{a}^M = 0$, $\mathfrak{g}^M = \mathfrak{b}$ and the relations $T(\mathfrak{a}, \mathfrak{g}, \mathfrak{a}^{i+1}, \mathfrak{g}^{i+1})$, $i = 0, 1, \ldots, M-1$, hold. We put $A^M = 0$ and let $G^M$ be any group having invariant $\mathfrak{g}^M$. Proceeding by induction, if groups $A^m \subset G^m$ with invariants $\mathfrak{a}^m, \mathfrak{g}^m$ and isomorphisms $\varphi^m$: $G^m/A^m[p] \to G^{m+1}$ such that $\varphi^m(A^m/A^m[p]) = A^{m+1}$ are defined for $m > n \geqslant 0$ $(m \leqslant M)$, then by Lemma 2' there exist groups $A^n, G^n$, $A^n \subset G^n$ with invariants $\mathfrak{a}^n, \mathfrak{g}^n$ and an isomorphism $\varphi_n$: $G^n/A^n[p] \to G^{n+1}$ such that $\varphi_n(A^n/A^n[p]) = A^{n+1}$. Consequently $G^n/A^n \approx G^n/A^n[p]/A^n/A^n[p] \approx G^{n+1}/A^{n+1}$ and by induction on $n$ $G/A = G^0/A^0 \approx G^n/A^n \approx G^M/A^M \approx B$. The first part of the proof is finished.

If $G/A = B$, $p^M G = 0$ and invariants of groups $G, A, B$ are $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$, then we put $G^0 = G$, $A^0 = A$, $G^{n+1} = G^n/A^n[p]$, $A^{n+1} = A^n/A^n[p]$ and let $\mathfrak{a}^n, \mathfrak{g}^n$ be invariants of $A^n, G^n$, $n = 0, 1, \ldots, M$. It is clear that the relations $T(\mathfrak{a}^i, \mathfrak{g}^i, \mathfrak{a}^{i+1}, \mathfrak{g}^{i+1})$, $i = 0, 1, \ldots, M-1$, hold. Moreover, $G^n/A^n \approx G^{n+1}/A^{n+1}$ and $A^M = 0$; thus $B = G/A \approx G^0/A^0 \approx G^M/A^M \approx G^M$ and finally $\mathfrak{g}^M = \mathfrak{b}$. Hence the relation $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ hold.

Remark′. If $A \subset G$ and $p$ primary components of $G$ are of bounded order, then by relations (1) and (2) it follows that there exists a homomorphic mapping of $G$ onto $A$. Since a group $G/G[p]$ may be embedded in $G$, it is easy to see that the group $B = G/A$ may be embedded in $G$. Consequently the conditions (4.9) and (4.10) of [1] are equivalent.

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# Universalmengen bezüglich der Lage im $E^n$

von

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**1. Einleitung, die Mengen $M_n^m$.** Mit $E^n$ bezeichnen wir stets den $n$-dimensionalen euklidischen Raum, den wir auf ein festes kartesisches Koordinatensystem bezogen denken. Sind $A$ und $B$ zwei homöomorphe Teilmengen von $E^n$, so sagen wir, daß $A$ und $B$ die gleiche Lage im $E^n$ haben, falls es einen Homöomorphismus von $E^n$ auf sich gibt, der $A$ auf $B$ abbildet. Gibt es einen Homöomorphismus von $E^n$ auf sich, der $A$ auf eine Teilmenge von $B$ abbildet, so sagen wir, daß sich $A$ im $E^n$ lagetreu in $B$ einbetten läßt.

Ist $\mathfrak{S}$ ein System von topologischen Räumen, so nennt man einen Raum $U$ aus $\mathfrak{S}$ Universalmenge von $\mathfrak{S}$, falls sich jeder Raum aus $\mathfrak{S}$ topologisch (d.h. homöomorph) in $U$ einbetten läßt. Ist $\mathfrak{S}^*$ ein System von Teilmengen des $E^n$ ($n$ fest), so liegt es nahe, nach Universalmengen von $\mathfrak{S}^*$ bzgl. der Lage zu fragen. Dabei soll eine Menge $U^*$ aus $\mathfrak{S}^*$ eine Universalmenge von $\mathfrak{S}^*$ bzgl. der Lage im $E^n$ heißen, falls sich jede Menge aus $\mathfrak{S}^*$ lagetreu in $U^*$ einbetten läßt. Ähnlich wie bei den gewöhnlichen Universalmengen liegt die Frage nach $m$-dimensionalen Universalmengen bzgl. der Lage im $E^n$ nahe. Eine kompakte $m$-dimensionale Teilmenge $U$ von $E^n$ soll $m$-dimensionale Universalmenge bzgl. der Lage im $E^n$ heißen, falls sich jedes $m$-dimensionale Teilkompaktum $A$ von $E^n$ lagetreu in $U$ einbetten läßt, falls es also stets einen Homöomorphismus $h$ von $E^n$ auf sich gibt, der $A$ auf eine Teilmenge von $U$ abbildet. In dieser Arbeit soll die Frage nach der Existenz derartiger Universalmengen in einigen Fällen beantwortet werden. Wir betrachten nämlich naheliegende Verallgemeinerungen der bekannten Mengerschen Universalmengen und entscheiden, in welchen Fällen diese Mengen $m$-dimensionale Universalmengen bzgl. der Lage im $E^n$ sind. Ehe wir jedoch die genauen Ergebnisse angeben, sollen die zu betrachtenden Mengen (wir bezeichnen sie mit $M_n^m$) definiert werden.

Mit $W_e^n$ bezeichnen wir stets den $n$-dimensionalen Einheitswürfel im $E^n$, d.h. die Menge aller Punkte $p = (\xi_i, \ldots, \xi_n)$, deren Koordinaten die Ungleichungen $0 \leqslant \xi_1 \leqslant 1$ $(i = 1, \ldots, n)$ befriedigen. Ist $i \geqslant 1$ eine ganze Zahl, so sei $D_i$ die Vereinigung aller offenen Intervalle