Packing closed sets *

by

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1. Introduction. We consider throughout point sets in a q-dimensional Euclidean space, where q is a fixed positive integer. We shall prove the following theorem: *Let E be a bounded closed set and G a bounded open set. Then the maximum number of disjoint copies of E which can be placed in G is either finite or* $\mathfrak{c}$. (Here $\mathfrak{c}$ denotes the cardinal number of the continuum.) This remains true no matter whether we insist that the copies of E be obtained by translation alone, or whether we allow both translation and rotation. The result for translation is proved in § 2, the one for translation and rotation, and indeed some more general results, in § 5. We suppose throughout that the set E is non-empty.

If there do not exist arbitrarily small motions which take E into a set disjoint from E, then clearly only a finite number of disjoint congruent copies of E can be placed in any bounded portion of the space. What we shall prove is that if there are arbitrarily small motions which take E into sets disjoint from E, then there are $\mathfrak{c}$ arbitrarily small motions which take E into mutually disjoint sets. In this case, we can place $\mathfrak{c}$ disjoint copies of E in any open set containing one copy. In particular, the maximum number of disjoint copies of E which can lie in a bounded open set is either $\mathfrak{c}$ or finite. If we consider placing copies of E in the whole space, then we have an alternative between $\mathfrak{c}$ copies when there exist arbitrarily small motions which take E into a disjoint set, and $\aleph_0$ copies otherwise.

We shall use vector addition and subtraction of the points in the q-dimensional Euclidean space, and multiplication by real numbers. The origin will be denoted by 0, and $|x|$ will mean the distance from x to 0. We also define vector addition and subtraction of point sets, and multiplication by real numbers, by the formulas

$$A+B = \{x+y \mid x \in A,\ y \in B\},$$
$$A-B = \{x-y \mid x \in A,\ y \in B\},$$
$$hA = \{hx \mid x \in A\}.$$

Notice that $2A$ is not the same as $A+A$.

* This work was supported by the Miller Institute for Basic Research in Science.

In the study of the packing of copies of E obtained by translation alone, the difference set $D = E - E$ plays a central role. Indeed, translation by a vector t takes E into a set disjoint from E if and only if $t \notin D$. Hence there will be arbitrarily small translations which take E into sets disjoint from E if and only if 0 is not an inner point of D.

Although the same alternative between packing a finite number of copies of E and $\mathfrak{c}$ copies of E into a bounded region exists whether we allow translation alone, or translation and rotation, the answer (finite or $\mathfrak{c}$) may very well be different for the two cases. We shall give a simple example in the plane illustrating this. The set E will lie on a denumerable infinity of rays from 0. It follows that $\mathfrak{c}$ disjoint copies of E can be obtained by rotation about the origin. But we can choose E so that 0 is an inner point of $D = E - E$, and hence only a finite number of copies of E obtained by translation can be placed in a bounded region.

For example, let E consist of all points (x, y) on the infinite sequence of segments

$$2 \leqslant x \leqslant 4, \quad y = x/2^n \quad (n = 2, 3, \ldots),$$

and the one additional segment $1 \leqslant x \leqslant 5$, $y = 0$. Then E is a closed set. We shall verify that D contains the entire square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$. Let (x, y) be any point in the square. If $y = 0$, we can obtain this point as the difference of the two points $(3+x, 0)$ and $(3, 0)$. Otherwise, because of the symmetry of D, we may suppose that $y > 0$. Choose n so that $2 \leqslant 2^n y \leqslant 4$. Then (x, y) can be obtained as the difference of $(2^n y, y)$ and $(2^n y - x, 0)$, which both belong to E.

2. Packing translates of a closed set. Let E be a bounded closed set, and $D = E - E$ the difference set. If 0 is an inner point of D, then we cannot find arbitrarily small translations which take E into a set disjoint from E. Hence there can be only a finite number of disjoint copies of E in a bounded region, and only a denumerable infinity in the whole space.

Suppose now that 0 is not an inner point of D, so that there are arbitrarily small translations which take E into sets disjoint from E. We shall show that there are in fact $\mathfrak{c}$ arbitrarily small translations which take E into mutually disjoint sets. Hence $\mathfrak{c}$ disjoint copies of E can be placed in any neighborhood of E.

We shall choose vectors $t_1, t_2, \ldots$ so that the $\mathfrak{c}$ series

$$t = \sum_{n=1}^{\infty} a_n t_n \quad (a_n = 0, 1)$$

converge and represent arbitrarily small vectors, and the translates of E by these vectors are all disjoint. The first condition is satisfied if we suppose that $|t_n| \leqslant \varepsilon/2^n$, where ε is an arbitrarily small positive number. To satisfy the second condition, we need only require that $t_n \notin D$, and that $|t_n|$ is small enough, compared to the distance $\delta(t_k, D)$ of t_k from D for the various $k < n$. We shall assume that

$$|t_n| < \frac{\delta(t_k, D)}{2 \cdot 2^{n-k}} \quad \text{for} \quad k < n.$$

We are to show that the $\mathfrak{c}$ translations above lead to mutually disjoint copies of E.

Equivalently, we may show that the $\mathfrak{c}$ translations

$$t = \sum_{n=1}^{\infty} d_n t_n \quad (d_n = 0, \pm 1)$$

all take E into sets disjoint from E. Consider any such series, and let d_k be the first non-vanishing coefficient. We may suppose that $d_k = 1$, so that

$$t = t_k + \sum_{n=k+1}^{\infty} d_n t_n.$$

Then we find that

$$|t - t_k| \leqslant \sum_{n=k+1}^{\infty} |t_n| < \frac{\delta(t_k, D)}{2}.$$

Hence $t \notin D$. Thus the translation of E through the vector t yields a set disjoint from E.

Remark. The following observation is sometimes useful in computing difference sets. Let $E = \bigcap_{n=1}^{\infty} A_n$, where the A_n are bounded closed sets with $A_1 \supset A_2 \supset A_3 \supset \ldots$ Put $D = E - E$ and $B_n = A_n - A_n$. Clearly, $D \subset \bigcap_{n=1}^{\infty} B_n$. On the other hand, we see that if $\varepsilon > 0$, then $A_n \subset E(\varepsilon)$ for n sufficiently large, where $E(\varepsilon)$ is the set of all points at a distance at most ε from E. In this case, $B_n \subset E(\varepsilon) - E(\varepsilon) = D(2\varepsilon)$. It follows that $\bigcap_{n=1}^{\infty} B_n \subset D$, and hence $D = \bigcap_{n=1}^{\infty} B_n$.

3. Some examples on the line. We shall now consider some one-dimensional examples of the result proved in § 2. Suppose that $0 < \varrho < 1/2$, and construct the set E as follows: Start with the interval $[0, 1]$ and delete the middle portion $(\varrho, 1-\varrho)$. Then delete similar portions of each of the remaining intervals $[0, \varrho]$ and $[1-\varrho, 1]$. Continue this process indefinitely. The points remaining ultimately constitute E. If $\varrho = 1/3$, then E is the Cantor set.

How many copies of E can be placed on the line? According to § 2, we must check whether 0 in an inner point of $D = E-E$. We shall use the remark at the end of § 2 to compute D.

The set E was obtained as the intersection of a sequence of sets A_n, where

$$A_0 = [0, 1],$$
$$A_1 = [0, \varrho] \cup [1-\varrho, 1],$$
$$A_2 = [0, \varrho^2] \cup [\varrho-\varrho^2, \varrho] \cup [1-\varrho, 1-\varrho+\varrho^2] \cup [1-\varrho^2, 1],$$

and so forth. Each set is obtained from the preceding by replacing each interval of length h by two subintervals of length ϱh at its ends. If we put $B_n = A_n - A_n$, then we find that

$$B_0 = [-1, 1],$$
$$B_1 = [-1, -1+2\varrho] \cup [-\varrho, \varrho] \cup [1-2\varrho, 1],$$
$$B_2 = [-1, -1+2\varrho^2] \cup [-1+\varrho-\varrho^2, -1+\varrho+\varrho^2] \cup [-1+2\varrho-2\varrho^2, -1+2\varrho]$$
$$\cup [-\varrho, -\varrho+2\varrho^2] \cup [-\varrho^2, \varrho^2] \cup [\varrho-2\varrho^2, \varrho]$$
$$\cup [1-2\varrho, 1-2\varrho+2\varrho^2] \cup [1-\varrho-\varrho^2, 1-\varrho+\varrho^2] \cup [1-2\varrho^2, 1],$$

and so forth. Here we obtain B_{n+1} from B_n by replacing each interval of length h by *three* subintervals of length ϱh at the left end, center, and right end.

If $\varrho \geqslant 1/3$, then the intervals overlap, so that each $B_n = [-1, 1]$ and hence $D = [-1, 1]$. In this case, two copies of E are disjoint only when one lies completely to the right of the other.

On the other hand, if $\varrho < 1/3$, then intervals are deleted at each step, and the intersection D, like E, is nowhere dense and of measure 0. In particular, 0 is not an inner point of D. Thus the set E corresponding to any $\varrho < 1/3$ admits $\mathfrak{c}$ arbitrarily small translations which all produce disjoint copies of E.

I used the above set with $\varrho = 1/5$ in [2], § 2, as an example of a bounded closed set E on the line having positive transfinite diameter and such that $\mathfrak{c}$ disjoint copies of E could be placed on the line. It should also be remarked that the set $D = E-E$ was shown to have measure 0 by Piccard [1], p. 92, when $\varrho = 1/p$ $(p = 4, 5, 6, \ldots)$; however, she drew no conclusion about packing copies of E.

When $\varrho < 1/3$, we can find quite explicitly $\mathfrak{c}$ translations which give disjoint copies of E. It is easily seen that E is similar to the set

$$E_1 = \Big\{\sum_{n=1}^{\infty} a_n \varrho^n \;\Big|\; a_n = 0, 1\Big\}.$$

Indeed, $E = [(1-\varrho)/\varrho]E_1$. Also, introduce the set $E_2 = 2E_1$, that is,

$$E_2 = \Big\{\sum_{n=1}^{\infty} b_n \varrho^n \;\Big|\; b_n = 0, 2\Big\}.$$

We see that

$$E_1 + E_2 = \Big\{\sum_{n=1}^{\infty} c_n \varrho^n \;\Big|\; c_n = 0, 1, 2, 3\Big\},$$

and each number in E_1+E_2 is obtained just once in this form. Now if $\varrho < 1/4$, then these numbers are all distinct. Thus the copies of E_2 obtained using translations through distances in E_1, or the copies of E_1 using translations from E_2, are all disjoint. In other words, for the set E we can use as translation numbers the elements of E either halved or doubled.

The situation is not quite so simple if $1/4 \leqslant \varrho < 1/3$, but we can proceed as follows. Let

$$E_1^{(m)} = \Big\{\sum_{n=1}^{\infty} a_n \varrho^{mn} \;\Big|\; a_n = 0, 1\Big\}.$$

Then $E_1^{(m)} + E_2$ consists of all numbers of the form

$$\sum_{n=1}^{\infty} c_n \varrho^n \quad \text{with} \quad \begin{cases} c_n = 0, 1, 2, 3 & \text{if } m|n, \\ c_n = 0, 2 & \text{otherwise}, \end{cases}$$

each number being obtained just once in this form. A simple argument shows that these numbers are all distinct if

$$2\varrho/(1-\varrho) + \varrho^m/(1-\varrho^m) < 1.$$

Indeed, this inequality prevents a "carry" into the nth position when $m|n$; for other values of n, we need only prevent a "carry" of 2 units, which is a weaker condition. The inequality is satisfied when $\varrho < 1/3$ and m is sufficiently large. We find in this way $\mathfrak{c}$ translations which give disjoint copies of E_2, corresponding to the elements of $E_1^{(m)}$. Thus the elements of $[(1-\varrho)/2\varrho]E_1^{(m)}$ furnish suitable translations for the set E.

Remark. In some cases, it may be simpler to exhibit $\mathfrak{c}$ translations which give disjoint copies of a set E than to calculate the difference set D and verify that 0 is not an inner point. For example, if

$$E = \Big\{\sum_{n=1}^{\infty} \frac{a_n}{9^n} \;\Big|\; a_n = 0, 1, 4, 5\Big\},$$

then the $\mathfrak{c}$ translations chosen from

$$F = \left\{ \sum_{n=1}^{\infty} \frac{b_n}{9^n} \,\middle|\, b_n = 0, 2 \right\}$$

are seen at once to give disjoint copies of E. The computation of $D = E - E$ is more complicated.

4. Affine transformations. In this section, we derive some elementary properties of affine transformations which are needed in § 5.

Let T be an affine transformation. Then for any point x and any real h, we have

$$T(hx) - T(0) = h[T(x) - T(0)].$$

For $h = -1$, this yields $T(x) + T(-x) = 2T(0)$. Solving for $T(0)$ and substituting in the previous equation, we see that

$$T(hx) = \tfrac{1}{2}(h+1)\,T(x) - \tfrac{1}{2}(h-1)\,T(-x).$$

In particular, for $h \geq 1$ we have

$$|T(hx)| \leq h \max[|T(x)|, |T(-x)|].$$

We now introduce the norm of an affine transformation by the equation

$$N(T) = \max_{|x| \leq 1} |T(x)|.$$

In terms of this norm, we can estimate $T(x)$ in general:

$$|T(x)| \leq N(T) \max(1, |x|).$$

This is clear if $|x| \leq 1$. Now suppose that $|x| > 1$. Let $h = |x|$ and $x_1 = x/h$. Then $x = hx_1$ where $h > 1$ and $|x_1| = 1$. Applying the preceding inequality yields the desired conclusion.

If U and V are affine transformations, we define $U \pm V$ by the equations $(U \pm V)(x) = U(x) \pm V(x)$. Then $U \pm V$ are also affine transformations. Clearly $N(U \pm V) \leq N(U) + N(V)$.

We say that a sequence T_n of affine transformations converges if $T_n(x)$ converges uniformly on bounded sets. Since

$$|T_m(x) - T_n(x)| \leq N(T_m - T_n) \max(1, |x|),$$

we see that a necessary and sufficient condition for convergence is that $N(T_m - T_n) \to 0$ as $m \to \infty$ and $n \to \infty$. (Actually, convergence of $T_n(x)$ at $q+1$ points which span the space is already sufficient to insure convergence of T_n.) If the sequence T_n converges, let $T(x) = \lim_{n\to\infty} T_n(x)$. Then T is also affine, and we say that $T = \lim_{n\to\infty} T_n$.

For a difference of the form $T(x) - T(y)$, we can find a better sort of estimate than for $T(x)$ itself. Let x and y be any points. Put $h = |x-y|$. Then $x - y = hz$ with $|z| = 1$. We see that

$$2[T(x) - T(y)] = T(x-y) - T(y-x) = T(hz) - T(-hz) = h[T(z) - T(-z)],$$

and hence

$$|T(x) - T(y)| \leq N(T)\,|x-y|.$$

The product UV of two affine transformations U and V will be defined by the equation $UV(x) = U(V(x))$, so that the multiplication is from right to left. Then

$$|TU(x) - TV(x)| \leq N(T)\,|U(x) - V(x)|,$$

and so

$$N(TU - TV) \leq N(T) N(U - V).$$

It should be noticed that in general $TU - TV \neq T(U - V)$.

We also define the deviation $\Delta(T)$ of an affine transformation T from the identity transformation I by the equation

$$\Delta(T) = N(T - I) = \max_{|x| \leq 1} |T(x) - x|.$$

Clearly $N(T) \leq 1 + \Delta(T)$. Also, we see that the inequality $\Delta(T) < 1$ insures that T is one-to-one. Indeed, if T is not one-to-one, then it maps the whole space onto a space of lower dimension, and so we must have $\Delta(T) \geq 1$.

We shall now prove a basic inequality which shows that a product of several transformations near the identity is also near the identity, and gives an explicit estimate for the deviation. We see that

$$N(UV - U) \leq N(U) N(V - I) \leq [1 + \Delta(U)]\Delta(V).$$

Adding the equation $N(U - I) = \Delta(U)$, we find that

$$\Delta(UV) = N(UV - I) \leq [1 + \Delta(U)]\Delta(V) + \Delta(U),$$

and so

$$1 + \Delta(UV) \leq [1 + \Delta(U)][1 + \Delta(V)].$$

We can then obtain by induction the general inequality

$$1 + \Delta(T_1 T_2 \dots T_n) \leq \prod_{k=1}^{n} [1 + \Delta(T_k)].$$

Applying the above estimate for $N(UV - U)$ with $U = T_1 T_2 \dots T_n$ and $V = T_{n+1} \dots T_{n+p}$, and using the product inequality just proved, we see that

$$N(T_1 T_2 \dots T_{n+p} - T_1 T_2 \dots T_n) \leq \prod_{k=1}^{n+p} [1 + \Delta(T_k)] - \prod_{k=1}^{n} [1 + \Delta(T_k)].$$

Thus the sequence of products $T_1 T_2 \dots T_n$ will certainly converge whenever the infinite product $\prod_{k=1}^{\infty}[1+\Delta(T_k)]$ converges.

Whenever the sequence $T_1 T_2 \dots T_n$ converges, we say that the infinite product $T_1 T_2 T_3 \dots$ exists, and define it by the formula

$$T_1 T_2 T_3 \dots = \lim_{n\to\infty} T_1 T_2 \dots T_n .$$

However, we shall not say that the infinite product converges unless a stronger condition is satisfied, namely that for some m the product $T_{m+1} T_{m+2} \dots$ exists and is one-to-one. With this definition, a convergent infinite product of affine transformations is one-to-one whenever all the factors are one-to-one.

The product inequality proved above can be extended to infinite products:

$$1+\Delta(T_1 T_2 T_3 \dots) \leqslant \prod_{k=1}^{\infty}[1+\Delta(T_k)] .$$

If the product on the right converges, then we see that $\Delta(T_{m+1} T_{m+2} \dots) < 1$ for m sufficiently large. This insures that the transformation $T_{m+1} T_{m+2} \dots$ is one-to-one. Hence the infinite product $T_1 T_2 T_3 \dots$ converges in the sense of the above definition.

We should keep in mind that multiplication is from right to left, so that T_1 is the last transformation to be carried out in the infinite product $T_1 T_2 T_3 \dots$

5. Packing affine transforms of a closed set. Let $\mathfrak{F}$ be a family of one-to-one affine transformations which is a semigroup with identity and which is closed in the space of all such transformations. Thus we assume that (1) $I \in \mathfrak{F}$, (2) if $T \in \mathfrak{F}$ and $U \in \mathfrak{F}$ then $TU \in \mathfrak{F}$, and (3) if $T_n \in \mathfrak{F}$ for $n = 1, 2, \dots$ and $T_n \to T$ as $n \to \infty$, where T is one-to-one, then $T \in \mathfrak{F}$. In particular, if the factors of a convergent infinite product lie in $\mathfrak{F}$, then the product does also. Some examples of possible families $\mathfrak{F}$ are the following, each of which includes the preceding: all translations; all translations and rotations; all similarity transformations with magnification at least 1; all similarity transformations; all one-to-one affine transformations.

Let E be a bounded closed set. Suppose that there exist transformations $T \in \mathfrak{F}$ arbitrarily near I such that $T(E)$ is disjoint from E. We shall then prove that there are $\mathfrak{c}$ transformations $T \in \mathfrak{F}$ arbitrarily near I which produce mutually disjoint sets $T(E)$.

We may suppose that E lies in the sphere $|x| \leqslant 1$. Given any ε with $0 < \varepsilon < 1$, we shall choose a sequence of transformations $T_n \in \mathfrak{F}$ such that $\Delta(T_n) \leqslant \varepsilon/2^{n+1}$. Then, using the fact that $e^x < 1+2x$ for $0 < x < 1$, we see that

$$\prod_{n=1}^{\infty}[1+\Delta(T_n)] \leqslant \exp \sum_{n=1}^{\infty} \Delta(T_n) \leqslant \exp \tfrac{1}{2}\varepsilon < 1+\varepsilon .$$

Thus any product of the form

$$T = T_1^{a_1} T_2^{a_2} T_3^{a_3} \dots \qquad (a_n = 0, 1)$$

converges, and we have $\Delta(T) < \varepsilon$. We obtain in this way $\mathfrak{c}$ transformations $T \in \mathfrak{F}$ which are arbitrarily near to I.

We shall now show that the transformations T_n can be chosen so that all $\mathfrak{c}$ of these products produce disjoint transforms of E. Indeed, we need only insure that $T_n(E)$ is disjoint from E, and that $\Delta(T_n)$ is small enough, compared to the distance $\delta\big(T_k(E), E\big)$ of $T_k(E)$ from E for the various $k < n$. The following inequality will be sufficient:

$$\Delta(T_n) \leqslant \frac{\delta\big(T_k(E), E\big)}{8 \cdot 2^{n-k}} \qquad \text{for} \qquad k < n .$$

From this inequality, it follows that if U is any product of the form

$$U = T_{k+1}^{a_{k+1}} T_{k+2}^{a_{k+2}} \dots \qquad (a_n = 0, 1),$$

then

$$1+\Delta(U) \leqslant \prod_{n=k+1}^{\infty} [1+\Delta(T_n)] < 1+\tfrac{1}{4}\delta\big(T_k(E), E\big) .$$

It is now easy to see that all $\mathfrak{c}$ of the transformations

$$T = T_1^{a_1} T_2^{a_2} T_3^{a_3} \dots \qquad (a_n = 0, 1)$$

produce disjoint transforms $T(E)$ of E. Indeed, let any two of these transformations be given. Suppose that the first place that the exponents disagree is for T_k. Then the two transformations have the form

$$T' = S T_k U, \qquad T'' = SV ,$$

where

$$S = T_1^{a_1} T_2^{a_2} \dots T_{k-1}^{a_{k-1}} \qquad (a_n = 0, 1),$$
$$U = T_{k+1}^{a_{k+1}} T_{k+2}^{a_{k+2}} \dots \qquad (a_n = 0, 1),$$
$$V = T_{k+1}^{b_{k+1}} T_{k+2}^{b_{k+2}} \dots \qquad (b_n = 0, 1) .$$

Thus we have

$$\Delta(U) < \tfrac{1}{4}\delta\big(T_k(E), E\big), \qquad \Delta(V) < \tfrac{1}{4}\delta\big(T_k(E), E\big) .$$

We want $T'(E)$ and $T''(E)$ to be disjoint. This is equivalent to saying that $T_k U(E)$ and $V(E)$ are disjoint. Now

$$N(T_k U - T_k) \leqslant N(T_k)\Delta(U) \leqslant 2\Delta(U) < \tfrac{1}{2}\delta\big(T_k(E), E\big) .$$

Thus for $x \in E$ and $y \in E$, we have

$$|T_k U(x) - T_k(x)| < \tfrac{1}{2}\delta\big(T_k(E), E\big), \qquad |V(y) - y| < \tfrac{1}{4}\delta\big(T_k(E), E\big).$$

On the other hand,

$$|T_k(x) - y| \geqslant \delta\big(T_k(E), E\big).$$

Hence

$$|T_k U(x) - V(y)| > \tfrac{1}{4}\delta\big(T_k(E), E\big),$$

and therefore the sets $T_k U(E)$ and $V(E)$ are disjoint, as was to be shown.

Remark. If $\mathfrak{F}$ is a closed group of rigid motions, and E is a bounded closed set, then (as we have just proved) we can place $\mathfrak{c}$ disjoint transforms of E in any neighborhood of E if $\mathfrak{F}$ contains arbitrarily small motions taking E into sets disjoint from E, and we see that otherwise only a finite number of disjoint transforms of E can be placed in any bounded region.

For other closed semigroups of affine transformations, this result can fail in various ways. On the one hand, if $\mathfrak{F}$ is the family of all translations of the form $T(x) = x + ht$ with t fixed and $h = 0$ or $h \geqslant 1$, then we may be able to place $\mathfrak{c}$ transforms of E in a bounded region even though $\mathfrak{F}$ contains no small motions. On the other hand, if $\mathfrak{F}$ is the group of all similarity transformations, then we can always place at least $\aleph_0$ disjoint transforms of E in any open set; the maximum number is $\mathfrak{c}$ or $\aleph_0$ according as there do or do not exist similarities arbitrarily near the identity which take E into sets disjoint from E.

Finally, we mention one additional case where the result *is* correct. If $\mathfrak{F}$ is the family of all similarity transformations with magnification at least 1, then the maximum number of copies of E which can be placed in a bounded neighborhood of E is $\mathfrak{c}$ or finite according as $\mathfrak{F}$ does or does not contain similarities arbitrarily near the identity which take E into sets disjoint from E. Notice that allowing enlargement may help in packing; we can place $\mathfrak{c}$ disjoint enlarged copies of a spherical surface in a bounded region, whereas there is room for only a finite number of congruent copies.

References

[1] Sophie Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, Mémoires de l'Université de Neuchatel, vol. 13 (1939).

[2] R. M. Robinson, *Conjugate algebraic integers in real point sets*, Math Zeitschrift 83 (1964), pp. 415-417.

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Reçu par la Rédaction le 22. 11. 1963

Correction to "On classes of abelian groups"

(Fundamenta Mathematicae 51(1962), pp. 149-178)

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The purpose of the present paper is to correct the relations between invariants of a p-primary group of bounded order, its subgroup and its factor group that were stated in Lemmas 1 and 2 of [1]. These relations do not determine all factor groups involved. Elements g_λ, a_λ considered in the proof of Lemma 1 (p. 157) were erroneously assumed to form bases of groups G and A. This fact was pointed out by Dr. E. James Peake, Jr., I wish to thank Dr. Peake for his comments.

Lemmas 1, 2, 4, Definition 3 and Remark on p. 165 of [1] should be replaced by the following Lemmas 1′, 2′, 4′, Definition 3′ and Remark′.

Let G be any p-primary group of bounded order, i.e., $p^M G = 0$ for some natural number M. Then there exists a direct decomposition

$$G = G_1 + G_2 + \ldots + G_M$$

such that the groups G_n are direct sums of groups $Z(p^n)$, $n = 1, 2, \ldots, M$. We denote by $\mathfrak{g}$ the invariant of G, which is a function defined by

$$\mathfrak{g}(n) = \begin{cases} \dim G_n & \text{for} \quad n = 1, 2, \ldots, M, \\ 0 & \text{for} \quad n > M. \end{cases}$$

Lemma 1′. *If A, G are p-primary groups $p^M G = p^M A = 0$ and $\mathfrak{a}, \mathfrak{g}$ are invariants of A and G respectively, then $A \subsetneqq G$ iff there exist cardinal numbers $\mathfrak{h}_n(k)$, $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$, such that*

$$\mathfrak{g}(n) = \sum_{k=0}^{n} \mathfrak{h}_n(k), \tag{1}$$

$$\mathfrak{a}(n) = \sum_{k=n}^{M} \mathfrak{h}_k(n) \tag{2}$$

for $n = 1, 2, \ldots, M$.