Topological analysis of analytic functions

by

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Introduction. Our approach is that of Whyburn's *Topological analysis* [8]. In section 1, making no use of the notion of rectifiability, we develop a topological analogue of the line integral, and by use of "local" inverses of the exponential function we obtain Whyburn's topological (circulation) index. In section 2, adapting a standard proof of the Riemann Mapping Theorem [6], we obtain a new proof of the power series expansion independent of the two proofs of Porcelli and Connell ([1], [2], [5]). As corollaries we have the one point removable singularity theorem and trivially the infinite differentiability and antidifferentiability of an analytic function. Making use of antiderivatives of analytic functions, we obtain close analogues of the classical line integral and the Cauchy Integral Formula. This enables us to obtain the Laurent expansion on an annulus.

In section 3 we are concerned with the removable singularity problem. The setting of this problem consists of having a closed nowhere dense subset A of the closure $\overline{U}$ of the unit disk U, and a continuous function f defined on $\overline{U}$ and differentiable on $U-A$. The problem is to find conditions on f or A so that f is differentiable on U. Examples developed by Denjoy [3] show that conditions need be imposed. Our main results in this direction consist of two sets of conditions which enable us to conclude that f is differentiable on U. One consequence of our conditions is a new proof for the case when A is a rectifiable arc dividing U into two disjoint regions.

0. Notation. Let K denote the complex plane, and ω the positive integers. For $r>0$, let U_r denote the interior of the circle C_r with center 0 and radius r. We shall generally write U for U_1 and C for C_1. For $z \in K$, let $P_x(z)$ denote the real part of z, and $P_y(z)$ the imaginary part. Let I denote the interval $[0, 1]$, and let Q denote $I\times I$. If M and N are subsets of K, we let $\delta(M, N)$ denote the set $\sup\{|x-y| \mid x\in M, y\in N\}$. We shall call a set $R\subset K$, a *circular region*, if R is the interior of some circle T. For $H\subset K$, $m(H)$ shall denote the *planar Lebesgue measure* of H. If J is a simple closed curve, let $E(J)$ denote the exterior of J, and $I(J)$ the interior of J.

Let $A \subseteq B \subseteq K$ and let f be a function defined on B. Then $f|A$ shall denote the function g on A, such that $g(z) = f(z)$ for all $z \in A$. Let f and g be functions on subsets of K, such that range g lies in domain f. Then fg shall denote the function h, such that $h(z) = f(g(z))$ for all z in domain g. For $z \in K$, let $I_0(z) = z$. Let f be a function defined on a set S in K, into K. If f is continuous, we shall call f a *map of S into K*; f is called an *open map*, if $f(V)$ is open in K for all open sets $V \subseteq S$; f is called a *light map*, if f is non-constant on all non-degenerate continua of S.

1. Line integral analogue. We shall first derive a topological analogue to the line integral. The analogue is motivated by the notion of analytic continuation.

DEFINITION 1.1. Let S be an open set and F a collection of functions on subsets of K. Then the statement that F is a C_s *collection* shall mean that for $f \in F$, f is a map defined on an open set $S_f \subseteq S$, that $S = \bigcup_{f\in F} S_f$, and that $f, g \in F$, $S_f \cap S_g \neq \emptyset$ implies that there exists $c \in K$ such that $f(x) = g(x) + c$ for all $x \in S_f \cap S_g$. Let h be a map of the interval $[a, b]$ into S, and $a = t_0 < t_1 < \ldots < t_{n+1} = b$ be a subdivision of $[a, b]$. Then we say that $t_0 < \ldots < t_{n+1}$ is a $C_{F,h}$ *subdivision of* $[a, b]$, if $h([t_i, t_{i+1}]) \subseteq S_f$ for some $f \in F$, for $i = 0, 1, \ldots, n$.

THEOREM 1.1. *Let S be an open set, F a C_s collection, and h a map of I into S. Then there exists a $C_{F,h}$ subdivision of I; moreover, there exists a unique number J denoted by $I_0^1 F\,dh$, such that if $0 = t_0 < t_1 < \ldots < t_{n+1} = 1$ is a $C_{F,h}$ subdivision of I, and $f_0, f_1, \ldots, f_n$ is a collection of elements of F such that $h([t_i, t_{i+1}]) \subseteq S_{f_i}$ for $i = 0, 1, \ldots, n$, then*

$$J = \sum_0^n f_i h(t_{i+1}) - f_i h(t_i)\,.$$

Proof. Since $\{S_f\}_{f\in F}$ is a collection of open sets covering the compact set $H = h(I)$, there exists a finite subcollection G of F such that $H \subseteq \bigcup_{g\in G} S_g$. $\{h^{-1}(S_g \cap H)\}_{g\in G}$ is a finite collection of open sets in I covering I, and there exists a subdivision $T_1 = \{t_i\}_0^{n+1}$ of I such that for $i = 0, 1, \ldots, n$, $h([t_i, t_{i+1}]) \subseteq S_g$ for some $g \in G$. Thus T_1 is a $C_{F,h}$ subdivision of I.

Let $T_2 = \{v_i\}_0^{q+1}$ be a $C_{f,h}$ subdivision of I, and $g_0, g_1, \ldots, g_q$ elements of F such that $h([v_i, v_{i+1}]) \subseteq S_{g_i}$ for $i = 0, 1, \ldots, q$. Let $T_3 = \{w_i\}_0^{p+1}$ be the mesh of T_1 and T_2. Trivially

$$J_1 = \sum_0^p k_i h(w_{i+1}) - k_i h(w_i) = \sum_0^q g_i h(v_{i+1}) - g_i h(v_i)\,,$$

where for $i = 0, 1, \ldots, p$, $k_i = g_j$ and j is the integer such that $[w_i, w_{i+1}] \subseteq [v_j, v_{j+1}]$. Also,

$$J_2 = \sum_0^p m_i h(w_{i+1}) - m_i h(w_i) = \sum_0^m f_i h(t_{i+1}) - f_i h(t_i)\,,$$

where for $i = 0, 1, \ldots, p$, $m_i = f_j$ and j is the integer such that $[w_i, w_{i+1}] \subseteq [t_j, t_{j+1}]$. Let $i \in [0, 1, \ldots, p]$. Then from Definition 1.1, there exists $c \in K$, such that $m_i(x) = k_i(x) + c$ for all $x \in S_{m_i} \cap S_{k_i} \supseteq h([w_i, w_{i+1}])$. Thus $m_i h(w_{i+1}) - m_i h(w_i) = k_i h(w_{i+1}) - k_i h(w_i)$, and consequently J_1 must be equal to J_2. Thus $I_0^1 F\,dh$ is uniquely determined.

The following theorem shows that J is invariant under homotopic deformation.

THEOREM 1.2. *Let h be a map of Q into an open set S such that h satisfies one of the following: (1) $h(0, 0) = h(0, t)$ and $h(1, 0) = h(1, t)$ for $t \in I$; or (2) $h(0, t) = h(1, t)$ for $t \in I$. Let $h_t(x) = h(x, t)$ for all $(x, t) \in Q$, and let F be a C_s collection. Then $I_0^1 F\,dh_0 = I_0^1 F\,dh_1$.*

Proof. If $t \in I$, then, by Theorem 1.1, there exists a subdivision $0 = x_0 < \ldots < x_{n+1} = 1$ of I and $f_0, f_1, \ldots, f_n \in F$, such that $h_t([x_i, x_{i+1}]) \subseteq S_{f_i}$ for $i = 0, 1, \ldots, n$. Thus $\{h_t^{-1}(S_{f_i} \cap h(Q))\}_n^0$ is a collection of open sets in Q covering $\{t\} \times I$, and hence there exists an open set $S_t \subseteq I$ containing t, such that $S_t \times [x_i, x_{i+1}] \subseteq h^{-1}(S_{f_i} \cap h(Q))$ for $i = 0, 1, \ldots, n$.

Let $u, v \in S_t$. Then for $i = 0, 1, \ldots, n$, there exists $c_i \in K$ such that $f_i(z) = f_{i+1}(z) + c_i$ for $z \in S_{f_i} \cap S_{f_{i+1}}$. Hence we have

$$\begin{aligned} I_1^0 F\,dh_u - I_0^1 F\,dh_v &= \Big[\sum_0^n f_i h_u(x_{i+1}) - f_i h_u(x_i)\Big] - \Big[\sum_0^n f_i h_v(x_{i+1}) - f_i h_v(x_i)\Big] \\ &= \sum_0^n [f_i h_u(x_i) - f_i h_u(x_i)] + [f_i h_u(x_{i+1}) - f_i h_u(x_i)] + \\ &\quad + [f_i h_u(x_{i+1}) - f_i h_u(x_{i+1})] + [f_i h_v(x_i) - f_i h_v(x_{i+1})] \\ &= 0\,. \end{aligned}$$

Since I is compact, and $\bigcup_{t\in I} S_t = I$, there exists a subdivision $t_0 < \ldots < t_{m+1}$ of I such that for $i = 0, 1, \ldots, m$, $[t_i, t_{i+1}] \subseteq S_t$ for some $t \in I$. Thus clearly $I_0^1 F\,dh_0 = I_0^1 F\,dh_1$.

THEOREM 1.3. *Let S be an open set, F a C_S collection, T a simple closed curve, P a circle with center z_0 and radius r lying in $I(T)$, and h a map of $M = \overline{I(T)} - I(P)$ into S. Let a and b be distinct points of T and let A and B be distinct subarcs of T with endpoints a and b. Then*

$$J = I_a^b F\,dh_1 + I_a^b F\,dh_2 = \pm I_0^1 F\,dhk\,,$$

where $h_1 = h|A$, $h_2 = h|B$, and $k(t) = rE(2\pi it) + z_0$ for $t \in I$; moreover, if v is a map of $T \cup I(T)$ into S, then, upon replacing h by v, $J = 0$.

Proof. Let $0 < r < 1$, and g be a homeomorphism of $N = \overline{U} - U_r$ onto M. (Cf. [8], pp. 34-35.) For $(x, t) \in Q$, let $w(x, t) = hg[(t+r-rt) \times \times E(2\pi i x)]$. Then $w(x, 1) = hgE(2\pi i x)$ for $x \in I$, and $w(0, t) = w(1, t)$ for $t \in I$. Hence from Theorem 1.2, since w_0 and w_1 are homeomorphisms,

$$J = \pm I_0^1 F\,dw_1 = I_0^1 F\,dw_0 = \pm I_0^1 F\,dhk\,.$$

To handle the case involving v, we take $w(x, t) = hg[t \cdot E(2\pi i x)]$ for $(x, t) \in Q$. In this case $J = \pm I_0^1 F\,dw_0 = 0$, where $w_0(x) = w(x, 0) = hg(0)$ for all $x \in I$.

In theorem 1.4 we show to what extent J is independent of the choice of path. In Theorem 1.5, we apply Theorem 1.4 to obtain an analogue to the monodromy theorem.

THEOREM 1.4. *Let S be a connected and simply connected open set, F a C_S collection, and h and k maps of I into S, such that $h(0) = k(0)$ and $h(1) = k(1)$. Then*

$$I_0^1 F\,dh = I_0^1 F\,dk\,.$$

Proof. $M = h(I) \cup k(I)$ is a subcontinuum of S and $\delta(M, K-S) = \varepsilon > 0$. It follows from the Zoretti Theorem (cf. [8], p. 35) that there exists a simple closed curve T such that $M \subseteq I(T)$, and $\delta(x, K-S) > \varepsilon/2$ for $x \in T$. Thus $T \subset S$ and hence $I(T) \subset S$. There exists a homeomorphism w of $I(T)$ onto U (cf. [8], p. 38). For $(x, t) \in Q$, let $h(x, t) = w^{-1}[t \cdot wk(x) + (1-t) wh(x)]$. Clearly h is a map of Q into S. Now $h_0(x) = w^{-1}[0 \cdot wk(x) + (1-0) wh(x)] = w^{-1}[wh(x)] = h(x)$ for $x \in I$. Similarly $h_1(x) = k(x)$ for $x \in I$. Hence from Theorem 1.2,

$$I_0^1 F\,dh = I_0^1 F\,dh_0 = I_0^1 F\,dh_1 = I_0^1 F\,dk\,.$$

THEOREM 1.5. *Let S be a connected and simply connected open set and F a C_S collection. Then there exists a map g of S into K such that if $f \in F$ and R is a component of S_f, then there exists $c \in K$ such that $f(x) = g(x) + c$ for all $x \in R$.*

Proof. Let $z_0 \in S$, $g(z_0) = 0$, and $z \in S - \{z_0\}$. Since S is connected, there exists an arc $A_z \subseteq S$ with endpoints z_0 and z. Let h_z be a homeomorphism of I onto A_z such that $h_z(0) = z_0$ and $h_z(1) = z$. Set $g(z) = I_0^1 F\,dh_z$.

Let $x_0 \in S$. There exists $f \in F$ such that $x_0 \in S_f$. Let y be a point of the component R of S_f containing x_0. There exists an arc $B \subseteq R$ with endpoints x_0 and y. If k is a homeomorphism of I onto B such that $k(0) = x_0$ and $k(1) = y$, then from Theorem 1.4,

$$g(y) = I_0^1 F\,dh_y = I_0^1 F\,dh_{x_0} + I_0^1 F\,dk = I_0^1 F\,dh_{x_0} + [f(y) - f(x_0)]\,.$$

If $c = I_0^1 F\,dh_{x_0} - f(x_0)$, then for all $y \in R$, we have $g(y) = f(y) + c$. Thus g is the desired function.

So far strictly speaking we have been dealing with an analogue of the Stieltjes integral. We shall now show that an analogue to the integral along an arc falls out of our initial definition.

THEOREM 1.6. *Suppose that S is an open set, F a C_S collection, and A an arc in S with endpoints a and b. Then there exists a unique number $I_{aA}^b F\,dz$ such that if h is a map of I onto A satisfying $h(0) = a$ and $h(1) = b$, then*

$$I_{aA}^b F\,dz = I_1^0 F\,dh\,.$$

Proof. If $n \in \omega$, then from the Zoretti Theorem, there exists a simple closed curve C_n such that $A \subseteq (C_n)$ and $\delta(x, A) < 1/n$ for all $x \in C_n$. Suppose $x \in K - A$. Since $K - A$ is connected (cf. [8], p. 29), there exists an arc B with endpoints x and y, such that $B \cap A = \emptyset$ and $|y| > \sup_{t \in A} |t| + 2$. Hence $y \notin H_n = C_n \cup I(C_n)$ for all $n \in \omega$. For $n > \delta(a, b)^{-1}$, $n \in \omega$, we have $\delta(z, A) < 1/n < \delta(A, B)$ for all $z \in C_n$, and hence $C_n \cap B = \emptyset$; consequently $x \notin H_n$. Thus $\bigcap_1^\infty H_n = A$. Now

$$\bigcap_1^\infty [H_n \cap (K-S)] = (K-S) \cap [\bigcap_1^\infty H_n] = (K-S) \cap A \subseteq (K-S) \cap S = \emptyset\,;$$

also $H_{n+1} \cap (K-S) \subseteq H_n \cap (K-S)$ for $n \in \omega$. Hence there exists $n_0 \in \omega$ such that $H_{n_0} \cap (K-S) = \emptyset$, and consequently $H_{n_0} \subseteq S$. Let h and k be maps of I onto A such that $h(0) = k(0) = a$ and $h(1) = k(1) = b$. Then from Theorem 1.4, $I_0^1 F\,dh = I_0^1 F\,dk$, and hence $I_{aA}^b F\,dz$ is uniquely defined.

DEFINITION 1.2. Let S be an open set, $z_0 \in K$, and F a collection of functions on subsets of K. Then the statement that F is an L_{S,z_0} collection means that F is a $C_{S-\{z_0\}}$ collection such that $Ef(z) = z - z_0$ for $f \in F$ and $z \in S_f$, where $E(z) = e^z$ for $z \in K$.

THEOREM 1.7. *If S is an open set in K and $z_0 \in K$, then there exists an L_{S,z_0} collection. Moreover if h is a map of I into K and $z_0 \in K - h(I)$, then there exists a unique number J, such that if S is an open set containing $h(I)$ and F is an L_{S,z_0} collection, then*

$$J = I_0^1 F\,dh = 2\pi i \mu_I(h, z_0)\,,$$

where μ denotes Whyburn's topological index (cf. [8], p. 58). *Finally, if $h(0) = h(1)$ then*

$$J = 2n\pi i \quad \textit{for some integer } n.$$

Proof. For $z \in K$, let $P(z) = z - z_0$ and $y \in S - \{z_0\}$. Then $P(y) \neq 0$, and there exists $x \in K$ such that $E(x) = P(y)$. Now there exists an open set V containing x_0 such that E is a homeomorphism on V. Since E is an open map, $E(V)$ is an open set containing $P(y)$, so there exists a circular region R_y containing y and lying in $S \cap P^{-1}E(S)$. Hence there exists a map f of $P(R_y)$ into K, such that $Ef(z) = z$ for all $z \in P(R_y)$.

Let f_y denote the function fP. Then $Ef_y(z) = EfP(z) = P(z) = z-z_0$ for all $z \in R_y$.

Let $u, v \in S$ and $z \in R_u \cap R_v$. Then $Ef_u(z) = Ef_v(z) = z-z_0$, and thus $E[f_u(z)-f_v(z)] = (z-z_0)/(z-z_0) = 1$, and consequently $f_u(z) = f_v(z)+2\pi n(z)i$ where $n(z)$ is an integer. Then $n = (2\pi i)^{-1}(f_u-f_v)$ is a continuous function on the connected set $R_u \cap R_v$ and hence is constant. Thus $F = \{f_y|\ y \in S\}$ is an L_{S,z_0} collection.

Let F and G be L_{S,z_0} collections and for $t \in I$, set $v(t) = I_0^t F\,dh$ and $w(t) = I_0^1 G\,dh$. Then clearly v and w are continuous. Now for $t \in I$, there exists a collection $f_0, f_1, \ldots, f_n$, and a subdivision $0 = x_0 < \ldots < x_{n+1} = t$ of $[0, t]$ such that $I_0^t F\,dh = \sum_0^n f_i h(x_{i+1}) - f_i h(x_i)$, and hence

$$Ev(t) = E\Big[\sum_0^n \big(f_i h(x_{i+1}) - f_i h(x_i)\big)\Big] = \prod_0^n Ef_i h(x_{i+1})/Ef_i h(x_i)$$

$$= \prod_0^n (x_{i+1}-z)(x_i-z)^{-1} = [h(t)-z_0]\cdot[h(0)-z_0]^{-1}.$$

Similarly for $t \in I$,

$$Ew(t) = [h(t)-z_0]\cdot[h(0)-z_0]^{-1},$$

and hence $E[v(t)-w(t)] = 1$. Since $v-w$ is continuous, there exists an integer n such that $v(t)-w(t) = 2n\pi i$ for all $t \in I$. Now $v(0) = 0$ and $w(0) = 0$, and hence $n = 0$. Thus $v = w$ and J is uniquely defined.

Suppose $h(0) = h(1)$. Then $E(J) = Ev(1) = [h(1)-z_0]\cdot[h(0)-z_0]^{-1} = 1$ and $J = 2n\pi i$ for some integer n.

Let c be a number such that $E(c) = h(0)-z_0$. Then for $t \in I$, let $Q(t) = v(t)+c$. Then

$$EQ(t) = E[v(t)+c] = Ev(t)\cdot E(c)$$

$$= [h(t)-z_0]\cdot[h(0)-z_0]^{-1}\cdot[h(0)-z_0] = h(t)-z_0.$$

Thus $J = 2\pi i\mu_I(h, z_0)$ (cf. [8], pp. 56-58).

THEOREM 1.8. *Let $z_0 \in K$, and let S be a connected and simply connected open set excluding z_0. Then there exists a map g of S into K, such that $Eg(z) = z-z_0$ for all $z \in S$. Furthermore there exists a map k of S into K, such that $k(z)^2 = z-z_0$ for all $z \in S$.*

Proof. From Theorem 1.7, there exists an L_{S,z_0} collection. Since $z_0 \notin S$, the existence of the desired function g follows from Theorem 1.5.

Let $k(z) = E[2^{-1}g(z)]$ for all $z \in S$. Then

$$k(z)^2 = E[2^{-1}g(z)]^2 = Eg(z) = z-z_0 \quad \text{for} \quad z \in S.$$

THEOREM 1.9. *Suppose that T is a simple closed curve, a and b distinct points of T, and A and B distinct subarcs of T with endpoints a and b. Then there exists $n = \pm 1$, such that for $z_0 \in I(T)$,*

$$J = \mu_{aA}^b(z, z_0) + \mu_{bB}^a(z, z_0) = -\mu_{aB}^b(z, z_0) - \mu_{bA}^a(z, z_0) = n,$$

where $\mu_{aA}^b(z, z_0)$ denotes the unique number $I_{aA}^b L\,dz$, where L is an L_{K,z_0} collection.

Proof. Let L be an L_{K,z_0} collection. Then from Theorem 1.3, $J = I_{aA}^b L\,dz + I_{bB}^a L\,dz = nI_0^1 L\,dk = n\mu_I(k, z_0)$, where $k(t) = rE(2\pi it)+z_0$ for some $r > 0$, $n = \pm 1$, for all $t \in I$. Let $p(t) = \log r + 2\pi it$ for $t \in I$. Then $Ep(t) = E(\log r + 2\pi it) = rE(2\pi it) = k(t)-z_0$ for $t \in I$. Thus $J = n[p(1)-\\ -p(0)] = 2n\pi i$.

Let $z_1 \in I(T)$, $z_1 \neq z_0$, and let W be an arc in $I(I)$ with endpoints z_0 and z_1. Let h be a homeomorphism of I onto W such that $w(0) = z_0$ and $w(1) = z_1$, and v a continuous function on I onto T such that $v(0) = a$, $v(1/4) \in A$, $v(1/2) = b$, and v is one-to-one on $[0, 1)$. Set $w(x, t) = v(x)-h(t)+z_0$ for $(x, t) \in Q$. Now $z_0 \notin w(Q)$, since $W \cap T = \emptyset$, and $w(x, 0) = v(x)$, and $w(x, 1) = v(x)+z_0-z_1$ for $x \in I$. Then from Theorem 1.2, $J = I_0^1 L\,dw_0 = I_0^1 L\,dw_1 = \mu_I(v+z_0-z_1, z_0) = \mu_I(v, z_1)$. Thus J is uniquely determined.

THEOREM 1.10. *Let S be an open set, F a C_S collection, T a simple closed curve, and h a map of T into S. Then there exists a unique number $J = I_T F\,dh$ such that if a and b are distinct points of T, and A and B are distinct subarcs of T with endpoints a and b, then*

$$I_a^b F\,dh_1 + I_b^a F\,dh_2 = J[I_{aA}^b L\,dz + I_{bB}^a L\,dz]$$

for all L_{K,z_0} collections L, where $z_0 \in I(T)$, and $h_1 = h|A$ and $h_2 = h|B$.

Proof. The proof follows readily from Theorem 1.9.

Remark. Let L be an $L_{K,0}$ collection and h and k maps of I into $K-\{0\}$. Then for $t \in I$,

$$E(I_0^t L\,dh\cdot k) = [h(t)k(t)][h(0)k(0)]^{-1} = [h(t)/h(0)]\cdot[k(t)/k(0)]$$

$$= E(I_0^t L\,dh)\cdot E(I_0^t L\,dk) = E(I_0^1 L\,dh + I_0^1 L\,dk),$$

and thus

$$I_0^t L\,dh\cdot k = I_0^t L\,dh + I_0^t L\,dk + 2n\pi i$$

for some integer n. Now $I_0^t L\,dp = 0$, when $t = 0$ for $p = h, k, h\cdot k$ and hence n must equal zero. With this observation we have completed the derivation of the machinery of Whyburn's topological index μ.

THEOREM 1.11. *Let A be a closed and nowhere dense subset of $\overline{U}$, and f a map of $\overline{U}$, such that f is differentiable on $U-A$ and $f(A)$ is nowhere dense in K. Then if f is light, f is an open map on U. Moreover if f is* *light and* $p \in f(\overline{U})-f(C)$, *then* $f^{-1}(p)$ *has at most* $m = \mu_C(f, p)$ *elements. Finally if* f *is light on* $U-A$, *then* $\sup_{x\in\overline{U}}|f(x)| \leqslant \sup_{t\in C}|f(t)|$, *and if* f *is light* $|f(x)| < \sup_{t\in C}|f(t)|$ *for all* $x \in U$.

Proof. Suppose that f is light on $U-A$ and T is a simple closed curve in $\overline{U}$. Let $q \in f(H)-f(T)$, where $H = T \cup I(T)$, and let S be the component of $K-f(T)$ containing q. We shall show that $S \subseteq f(H)$. Set $Q = f(H) \cap S$. Then Q is open in the relative topology of $f(H)$, and hence $f^{-1}(Q)$ is open in H. Since $f^{-1}(Q) \cap T = \emptyset$, $f^{-1}(Q)$ is open in K. Now $Q_0 = f^{-1}(Q)-A$ is a non-empty open set in $U-A$, and hence $f(Q_0)$ is open in K. Let G be the set of all points $x \in U-A$ such that $f'(x) = 0$. Then (cf. [8], pp. 72-73) $m(G) = 0$ and since $f(A)$ is nowhere dense in K, we have $Q_1 = f(Q_0)-G-f(A) \neq \emptyset$. Let $y \in Q_1$. Then paralleling Whyburn (cf. [8], pp. 67-68 and 72-74, and cf. Theorem 1.9) we see that $f^{-1}(y)$ has a finite number n_0 of elements, where $\mu_T(f, y) = n_0 \neq 0$. Hence $\mu_T(f, z) = n_0$ for all $z \in S$. It follows now from Theorem 1.9 that $S \subseteq f(H)$.

Let S be the unbounded component of $K-f(C)$ and suppose $S \cap f(\overline{U}) \neq \emptyset$. Then from the above argument, $S \subseteq f(\overline{U})$. But $f(\overline{U})$ is compact. Thus

$$\sup_{x\in\overline{U}}|f(x)| \leqslant \sup_{t\in C}|f(t)|.$$

Suppose that f is light. We shall show that f is an open map on U. Let $x_0 \in U$ and R be a circular region with center x_0 lying in U. From the Zoretti Theorem, there exists a simple closed curve $T \subseteq R$, such that $x_0 \in I(T)$ and $T \cap f^{-1}f(x_0) = \emptyset$. Let S denote the component of $K-f(T)$ containing $f(x_0)$ and lying in $f(I(T)) \subseteq f(R)$. Thus if V is an open set in U, $f(V)$ must be open. In particular $f(U)$ is open, and hence $|f(x)| < \sup_{t\in C}|f(t)|$ for all $x \in U$.

Suppose that $f^{-1}(p) \cap I(T)$ contains more than m elements. Let $x_1, x_2, \ldots, x_{m+1}$ be distinct points of $f^{-1}(p) \cap I(T)$, and $R_1, R_2, \ldots, R_{m+1}$ be a collection of mutually disjoint circular regions in U, such that R_i has center x_i for $i = 1, 2, \ldots, m+1$. Then $f(R_i)$ is open in K for $i = 1, 2, \ldots, m+1$, and hence $V = S \cap \bigcap_1^\infty f(R_i)$ is a non-empty open set in K lying in $f(U)$, and containing p, where S is the component of $K-f(C)$ containing p. Suppose that $q \in V-f(A)-G$. Paralleling Whyburn, as above, we find that $\mu_C(f, q) \geqslant m+1$. But $\mu_C(f, q) = \mu_C(f, p) = m$. Thus $f^{-1}(p)$ has m or fewer elements.

THEOREM 1.12. *Let* T *be a simple closed curve,* $R = I(T)$, S *a set containing* $\overline{R}$, G *a finite subset of* R, $x_0 \in R-G$, *and* f *a map of* S *such that* f *is differentiable on* $R-G$. *For* $x \in S$, *let* $Q_{f,x_0}(x) = [f(x)-f(x_0)]\cdot[x-x_0]^{-1}$ *for* $x \neq x_0$, *and* $Q_{f,x_0}(x) = f'(x_0)$ *for* $x = x_0$. *Set* $Q = Q_{f,x_0}$. *Then* Q *is continuous on* $\overline{R}$, *differentiable on* $R-G-\{x_0\}$, *and such that* $|Q(x)| < \sup_{t\in T}|Q(T)|$ *for all* $x \in R$, *if* Q *is non-constant.*

Proof. By definition of derivative, Q is continuous on S. Suppose that Q is non-constant. Then, since Q is differentiable on $R-G-\{x_0\}$, and G is finite, $Q|R$ is light. Hence, from Theorem 1.11, Q is an open map on R, and $|Q(x)| < \sup_{t\in T}|Q(t)|$ for all $x \in R$.

THEOREM 1.13. *Let* A *be a finite subset of* U *and* $0 < r < 1$. *If* f *is a map of* U *such that* f *is differentiable on* $U-A$, *then there exists* $p > 0$ *such that the function* $g(z) = p\cdot f(z)+z$ *for* $z \in U$ *is one-to-one on* $\overline{U}_r$.

Proof. Let $r < r_0 < 1$, $M = \sup_{z\in\overline{U}_{r_0}}|f(z)|$, and $p = (r_0-r)/2(M+1)$. For $t \in I$, set $v(t) = r_0E(2\pi ti)$ and for $(s, t) \in Q$, set $h(s, t) = sp\cdot fv(t)+v(t)$. Then h is continuous, $h_0(t) = v(t)$, and $h_1(t) = p\cdot fv(t)$ for all $t \in I$. Now for $(s, t) \in Q$,

$$|h(s, t)| = |v(t)+sp\cdot fv(t)| \geqslant r_0 - sp|fv(t)| \geqslant r_0-(r_0-r)/2 = (r_0+r)/2 .$$

Thus $0 \notin h(Q)$, and hence from Theorems 1.2 and 1.9,

$$1 = \mu(v, 0) = \mu(h_0, 0) = \mu(h_1, 0) = \mu(p\cdot fv+v, 0)$$

Now g is continuous and differentiable on $U-A$. Since g is not constant and A is finite, g must be light. Suppose that S is the component of $K-g(C_r)$ containing 0. Now for $z \in C_{r_0}$, $g(z) = |z+p\cdot f(z)| \geqslant r_0-(r_0-r)/2 = (r_0+r)/2$, and thus $U_{(r_0+r)/2} \subseteq S$. For $z \in \overline{U}_r$, $|g(z)| = |p\cdot f(z)+z| < (r_0-r)/2+r = (r_0+r)/2$, so that $f(\overline{U}_r) \subseteq S$, and hence $\overline{U}_r \subseteq f^{-1}(S)$. For $x \in S$, $\mu(gv, x) = \mu(gv, 0) = 1$, and hence, from Theorem 1.11, $f^{-1}(x)$ has exactly one element. Consequently g is one-to-one on $f^{-1}(S) \supseteq \overline{U}_r$.

THEOREM 1.14. *Let* A *be an arc,* f *a map of* K *such that* f *is non-constant and differentiable on* $K-A$, *and* $s = \lim_{x\to\infty} f(x)$ *exists. Then*

$$f(A) = f(K).$$

Proof. Suppose that $p \in f(K)-f(A)$ and $p \neq s$. Let P be a circle with center s such that $p \in E(P)$. There exists $r_0 > 0$ such that $A \cup f^{-1}(p) \subseteq U_r$ for $r > r_0$. Since $\lim_{x\to\infty} f(x) = s$, there exists $r_1 > 0$ such that $r > r_1$ implies $f(C_r) \subseteq I(P)$. Let $r > \sup[r_0, r_1]$ and B be an arc in $\overline{U}_r$ with endpoints a and b, such that $B \cap C_r = \{a\} \cup \{b\}$, and $A \subseteq B$. Let D_1 and D_2 be the components of $U-B$, and $T_i = \overline{D}_i-D_i$ for $i = 1, 2$. Then from Theorem 1.11, $\mu_{C_r}(f, p) = \mu_{T_1}(f, p)+\mu_{T_2}(f, p) \geqslant n_0$, where $n_0 \neq 0$ is the number of elements of $f^{-1}(p)$. But $f(C_r) \subseteq I(P)$, and hence, from Theorem 1.9, $\mu_{C_r}(f, p) = 0$. Thus $f(K)-\{s\} \subseteq f(A)$ and $f(A) = f(K)$.

2. Polynomial approximations. This section shall center around the problem of obtaining polynomial approximations to differentiable functions; in particular power series expansions.

LEMMA 2.1. *Let f and g be polynomials such that $g(z) \neq 0$ for all $z \in \bar{U}$, S an open set in U, and $h(z) = f(z)/g(z)$ for $z \in \bar{U}$. Then if $P_1, P_2, \ldots$ is a sequence of polynomials converging uniformly on compact subsets of $h(S)$ to a limit function F, there exists a sequence of polynomials $Q_1, Q_2, \ldots$ converging uniformly on compact subsets of S to Fh.*

Proof. Clearly for $n \in \omega$ there exist polynomials f_n and g_n such that $g_n(z) \neq 0$ for all $z \in \bar{U}$, and $P_n h(z) = f_n(z)/g_n(z)$ for all $z \in \bar{U}$. Let $n \in \omega$. There exists a finite collection of numbers $a_0, z_1, z_2, \ldots, z_p$ such that $g_n(z) = a_0(z_1 - z)(z_2 - z) \ldots (z_p - z)$ for all $z \in K$. (Note: for the Fundamental Theorem of Algebra, cf. Whyburn [8], p. 77). Since $g(z) \neq 0$ for all $z \in \bar{U}$, we have $|z_k| > 1$ for $k = 1, 2, \ldots, p$. Hence for $k = 1, 2, \ldots, p$ the sequence of polynomials $\{T_{km}\}_{m=1}^{\infty}$ converges uniformly on $\bar{U}$ to $(z_k - z)^{-1} = z_k^{-1}(1 - z/z_k)^{-1}$, where $T_{km}(z) = \sum_{j=0}^{m} (z/z_k)^j$ for $k = 1, 2, \ldots, p$, $m \in \omega$, and $z \in K$. For $m \in \omega$, let $Q_{nm}(z) = a_0^{-1} f_n(z) T_{1m}(z) T_{2m}(z) \ldots T_{pm}(z)$ for all $z \in K$. Then $\{Q_{nm}\}_{m=1}^{\infty}$ is a sequence of polynomials converging uniformly on $\bar{U}$ to f_n/g_n.

Suppose that $p_1 < p_2 < \ldots$ is an increasing sequence in ω such that $|Q_{np_n}(z) - P_n h(z)| < 1/n$ for $n \in \omega$, $z \in \bar{U}$, M a compact subset of S, and $\varepsilon > 0$. Since h is continuous on S, $h(M)$ is compact, and there exists $N > 0$ such that $n > N$, $n \in \omega$, implies $1/n < \varepsilon/2$ and $|F(x) - P_n(x)| < \varepsilon/2$ for all $x \in h(M)$. Thus for $z \in M$ and $n > N$, $n \in \omega$, we have

$$|Fh(z) - Q_{np_n}(z)| \leqslant |Fh(z) - P_n h(z)| + |P_n h(z) - Q_{np_n}(z)| \leqslant 1/n + \varepsilon/2 < \varepsilon.$$

LEMMA 2.2. *Let F be a uniformly bounded collection of differentiable functions on an open set S. Then F is an equicontinuous family of functions.*

Proof. Let $p \in S$ and T be a circle with radius r and center p such that $H = T \cup I(T) \subseteq S$. There exists $M > 0$ such that $|f(z)| < M$ for all $z \in S$ and $f \in F$. If $f \in F$ and $z \in H$, then from Theorem 1.12, $|Q_{f,p}(z)| \leqslant \sup_{t \in T} |Q_{f,p}(t)| = \sup_{t \in T} |f(t) - f(p)| r^{-1} \leqslant 2M/r$ and thus $|f(z) - f(p)| \leqslant |z - p| 2Mr^{-1}$. Consequently F must be an equicontinuous family of functions.

Remark. Lemma 2.2 immediately enables us to obtain the Vitali-Porter-Stieltjes Theorem. Hurwitz's Theorem and the standard Maximum Modulus Theorem follow immediately from the open mapping theorem for differentiable functions. (Cf. Whyburn [8], pp. 72-76.)

LEMMA 2.3. *Suppose that $f_1, f_2, \ldots$ is a sequence of differentiable function defined on an open set S, converging uniformly on compact subsets of S to a limit function F. Then F is differentiable and $\{f'_p(z)\}_{p=1}^{\infty}$ converges to $F'(z)$ for all $z \in S$.*

Proof. The proof of this lemma is due to Porcelli and Connell (cf. [1] and [5]). Let $p \in S$ and let T be a circle with center p and radius r such that $D = T \cup I(T) \subseteq S$. Set $Q(z) = [f(z) - f(p)](z-p)^{-1}$ for $z \in K - \{p\}$, and for $n \in \omega$, let $Q_n = Q_{f_n,n}$. Then for $z \in D - \{p\}$, $\{Q_n(z)\}_{n=1}^{\infty}$ converges to $Q(z)$. From Theorem 1.12, we have

$$|Q_n(z) - Q_m(z)| \leqslant \sup_{t \in T} |Q_n(t) - Q_m(t)| \leqslant \sup_{t \in T} |f_n(t) - f_m(t)| r^{-1}$$

for $z \in D$. Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly on T, we see that $\{Q_n\}_{n=1}^{\infty}$ converges uniformly on D to a limit function Q_0. Clearly $Q_0(z) = Q(z)$ for all $z \in D - \{p\}$. Hence F is differentiable at p and $F'(p) = Q_0(p)$. Moreover $F'(p) = \lim_{n \to \infty} Q_n(p) = \lim_{n \to \infty} f'_n(p)$.

LEMMA 2.4. *Suppose that $f_1, f_2, \ldots$ is a sequence of one-to-one differentiable functions on a simply connected bounded open set S into U, converging uniformly on compact subsets of S to a limit function F non-constant on each component of S. Then F is a one-to-one differentiable function such that $F(S) \subseteq U$. Moreover if M is a compact subset of $f(S)$, there exists $N > 0$ such that $n \geqslant N$, $n \in \omega$, implies $M \subseteq f_n(S)$. Furthermore $\{f_n^{-1}\}_{n=N}^{\infty}$ converges uniformly to F^{-1} on M.*

Proof. Let V be a simple closed curve in S, and let W be the component of $K - F(V)$ containing $H = F[I(V)]$. Since S is simply connected, $I(V) \subseteq S$. By Lemma 2.3, F is differentiable and hence an open map, and consequently H is open in W. Now $H = H \cap W = [H \cup F(V)] \cap W = F[V \cap I(V)] \cap W$, and hence H is closed in W. Since W is connected, $H = W$. Then since H is bounded, $H = I[F(V)]$. Thus $F(S)$ is simply connected.

Since F is non-constant on each component of S, by Hurwitz's Theorem, F must be one-to-one. The differentiability of F^{-1} now follows readily from the fact that $F'(z) \neq 0$ for all $z \in S$. Let $z_0 \in S$ and T be a circle with center z_0 lying in S. Suppose $F'(z_0) = 0$ and set $w_0 = F(z_0)$ and $Q = Q_{F,z_0}$. Then $Q(z_0) = 0$, and hence from Theorem 1.11, $\mu_T(Q, 0) > 0$. Now $F(z) - w_0 = (z - z_0)Q(z)$ for $z \in S$, and hence from the remark preceeding Theorem 1.11, $\mu_T(F, w_0) = \mu_T(z, z_0) + \mu_T(Q, 0) = 1 + \mu_T(Q, 0) > 1$. But since F is a homeomorphism, $\mu_T(F, w_0) = \pm 1$ (cf. [8], pp. 74-75 and 84-85).

Let $y \in F(S)$, $\varepsilon > 0$, and R_ε a circular region with center $F^{-1}(y)$ and radius ε. Let T be a circle in $F(S)$ with center y and radius r such that $F^{-1}(T) \subseteq R_\varepsilon$. For $n \in \omega$, let $y_n = f_n F^{-1}(y)$ and $J_n = f_n F^{-1}(T)$. Since the sequence $f_1, f_2, \ldots$ converges uniformly on $F^{-1}[T \cup I(T)]$, there exists $N_y > 0$, such that for $n > N_y$, $n \in \omega$, $|f_n(x) - y| > r/2$ for all $x \in T \cup I(T)$, and thus $|f_n(x) - y| > r/2$ for all $x \in T$, and $\delta(L_n) < r/2$, where L_n is the line segment with endpoints y and y_n. Then for $n > N$, $n \in \omega$, $L_n \cap J_n = \emptyset$, and hence L_n lies in the same component, namely $I(J_n)$, of $K - J_n$

that contains y_n. Thus for $n > N_y$, $n \in \omega$, y, and indeed a circular region T_y of radius $r/2$ and center y lie in

$$I(J_n) = I[f_n F^{-1}(T_n)] = f_n F^{-1}[I(T)] \subseteq f_n F^{-1} F(S) = f_n(S).$$

Since $y \in I(J_n)$,

$$f_n^{-1}(y) \in f_n^{-1}[I(J_n)] = f_n^{-1} f_n F^{-1}[I(J_n)] = F^{-1}[I(J_n)] \subseteq R_\varepsilon$$

for $n > N_y$, $n \in \omega$. Since ε is arbitrary, $f_n^{-1}(y) \to F^{-1}(y)$, as $n \to \infty$.

Let M' be an open set containing M such that its closure $M_0 \subseteq S$. Then $M_0 \subseteq \bigcup_{y \in M} T_y$, and hence there exists a finite collection of points $y_1, y_2, \ldots, y_p$ of M_0 such that $M_0 \subseteq \bigcup_1^p T_{y_i}$. Thus for $n > N = \sum_1^p N_{y_i}$, $n \in \omega$, $M_0 \subseteq \bigcup_1^p T_{y_i} \subseteq f_n(S)$. Since S is bounded, by the Vitali-Porter-Stieltjes theorem, $\{f_n^{-1}\}_{n=1}^{\infty}$ converges uniformly on compact subsets of M_0, in particular M.

LEMMA 2.5. (Cf. [6], pp. 225-230.) *Let S be a connected and simply connected open set in U, such that $0 \in S$ and $S \neq U$. Then there exist polynomials f and g, such that $g(z) \neq 0$ for all $z \in \overline{U}$, and one-to-one differentiable function h on S into U such that $h'(0) > 1$, $h(0) = 0$, and $fh(z)/gh(z) = z$ for all $z \in S$.*

Proof. If $t \in U - S$, then for $z \in \overline{U}$, $|1 - \bar{t}z| \geqslant 1 - |\bar{t}z| \geqslant 1 - |t| > 0$. For $z \in \overline{U}$, let $A(z) = [t - z][1 - \bar{t}z]^{-1}$. Then by direct computation $AA(z) = z$ for all $z \in \overline{U}$. Thus A is one-to-one and $A(\overline{U}) = \overline{U}$. Trivially $A(0) = t$ and $A(t) = 0$, and hence $0 \notin A(S)$. Since A is differentiable on U, A is an open map on S, and consequently $A(S)$ is a connected and simply connected open set in U not containing 0.

From Theorem 1.8, there exists a one-to-one differentiable function H on $A(S)$ into K such that $H(z)^2 = z$ for $z \in A(S)$. Then $|H(t)^2| = |t| < 1$, and hence $|H(t)| < 1$. Then for $z \in \overline{U}$, $1 - \overline{H(t)}z \neq 0$. Define

$$B(z) = [H(t) - z][1 - \overline{H(t)}z]^{-1}$$

for all $z \in \overline{U}$.

For $z \in S$, let $P(z) = BHA(z)$. P is one-to-one and differentiable and $P(S) \subset U$. Now $P(0) = BHA(0) = BH(t) = 0$. For $z \in \overline{U}$, let $K(z) = z^2$. Then for $z \in \overline{U}$, let $Q(z) = AKB(z)$; Q is differentiable and $QP(z) = AKBBHA(z) = AKHA(z) = AA(z) = z$ for $z \in \overline{U}$. Then $P'(0)Q'(0) = 1$.

For $z \in \overline{U}$, set $Q_0(z) = Q(z)/z$ for $z \neq 0$ and $Q_0(z) = Q'(z)$ for $z = 0$. Then Q_0 is continuous on $\overline{U}$ and Q is differentiable on $U - \{0\}$. Q is not one-to-one on U and hence Q_0 can not be constant on U. From Theorem 1.11 (cf. also Theorem 1.12), $|Q_0(z)| < \sup_{t \in C} |Q_0(t)|$ for all $z \in U$. In particular, $|Q'(0)| = |Q_0(0)| < 1$. Thus $|P'(0)| > 1$, and if $s = \overline{P'(0)}|P'(0)|^{-1}$, then $sP(z)$ is the desired function.

The proof of the following theorem is adapted from a proof of the Riemann Mapping Theorem given by Saks and Zygmund (cf. [6], pp. 225-230).

THEOREM 2.1. *Let S be a bounded connected and simply connected open set and $z_0 \in S$. Then there exists a one-to-one differentiable map F of S onto U such that $F(z_0) = 0$, $F'(z_0) > 0$, and a sequence of polynomials $P_1, P_2, \ldots$ converging uniformly on compact subsets of U to F^{-1}.*

Proof. Let K be the set of all one-to-one differentiable maps f of S into U such that $f(z_0) = 0$, $f'(z_0) > 0$ and such that there exists a sequence of polynomials $Q_1, Q_2, \ldots$ converging uniformly on compact subsets of $f(S)$ to f^{-1}. K is a non-empty set.

If $s = \sup_{f \in K} f'(z_0)$, then there exists a sequence $f_1, f_2, \ldots$ in K such that $\lim_{n\to\infty} f_n'(z_0) = s$. From the Vitali-Porter-Stieltjes Theorem there exist $p_1 < p_2 < \ldots$ in ω, such that $\{f_{p_n}\}_{n=1}^{\infty}$ converges uniformly on compact subsets of S to a limit function F. From Lemma 2.3, F is differentiable and $F'(z_0) = s$, and hence F is non-constant. From the Hurwitz Theorem, F is one-to-one.

For $n \in \omega$, let $C_n = \{z \in F(S) \mid \delta(z, K - F(S)) \geqslant 1/n\}$. C_n is closed and hence compact for $n \in \omega$, and $F(S) = \bigcup_1^{\infty} C_n$. From Lemma 2.4, there exist $p_1 < p_2 < \ldots$ in ω, such that if $n \in \omega$, then $m \geqslant p_n$, $m \in \omega$, implies that $C_n \subseteq f_m(S)$. If $n \in \omega$, there exist a sequence of polynomials $P_{n1}, P_{n2}, \ldots$ converging uniformly on compact subsets of $f_{p_n}(S)$ to $f_{p_n}^{-1}$.

There exist $q_1 < q_2 < \ldots$ in ω, such that for $n \in \omega$, $|P_{nq_n}(z) - f_{p_n}(z)| < 1/n$ for $z \in C_n$. If D is a compact subset of $F(S)$, then there exists an integer n_0, such that $D \subseteq C_{n_0}$. Let $\varepsilon > 0$. Then from Lemma 2.4, there exists $M > n_0$, such that $m > M$, $m \in \omega$, implies $|f_{p_m}^{-1}(z) - F^{-1}(z)| < \varepsilon$ for all $z \in C_{n_0}$. Hence for $z \in D \subseteq C_{n_0}$ and $m > M$, $m \in \omega$, we have

$$|F^{-1}(z) - P_{mq_m}(z)| \leqslant |F^{-1}(z) - f_{p_m}^{-1}(z)| + |f_{p_m}^{-1}(z) - P_{mq_m}(z)| < \varepsilon + 1/m.$$

From Lemma 2.4, $F(S) \subseteq U$ and we have $F \in K$.

Suppose $F(S) \neq U$. Then clearly $F(S)$ is a connected and simply connected open set. From Lemma 2.5, there exist polynomials f and g such that $g(z) \neq 0$ for all $z \in \overline{U}$, and a one-to-one differentiable function h on $F(S)$ into U such that $fh(z)/gh(z) = z$ for all $z \in F(S)$, and such that $h'(0) > 1$. Then $(hf)'(z_0) = h'F(z_0) \cdot F'(z_0) = h'(0)s > s$. Now $W = hF(S)$ is an open set in U. Then $F(S) = h^{-1}(W)$. Since $F \in K$, there exists a sequence of polynomials $P_1, P_2, \ldots$ converging uniformly on compact subsets of $h^{-1}(W)$ to F^{-1}. From Lemma 2.1, there exists a sequence of polynomials $Q_1, Q_2, \ldots$ converging uniformly on compact subsets of W to $F^{-1}h^{-1}$. Thus $hF \in K$, which is a contradiction. Hence $F(S) = U$.

THEOREM 2.2. *Let S be a bounded connected and simply connected open set in K, $z_0 \epsilon S$, and $x_0 \epsilon U-\{0\}$. Then there exists a unique differentiable one-to-one function f on U onto S such that (1) f^{-1} is differentiable, $f(0) = z_0$, $f'(0) > 0$, and there exists a sequence of polynomials $P_1, P_2, \ldots$ converging uniformly on compact subsets of U to f, and such that (2) if g is a one-to-one map of U onto S, such that g is differentiable on $U-\{x_0\}$, $g(0) = z_0$, and $g'(0) > 0$, then $g = f$.*

Proof. The existence of at least one function f satisfying (1) is assured by Theorem 2.1. Let g be a function satisfying (2) and set $Q(z) = f^{-1}g(z)$ for all $z \epsilon U$. Then Q is a one-to-one map of U onto U, such that Q is differentiable on $U-\{x_0\}$, $Q(0) = 0$, and $Q'(0) > 0$. Let $T = Q_{Q,0}$. Then from Theorem 1.12,

$$\sup_{z \epsilon U} |T(z)| \leqslant \sup_{0<r<1} \sup_{t \epsilon C_r} |T(t)| \leqslant \sup_{0<r<1} 1/r = 1 .$$

Thus $|T(z)| \leqslant 1$ for all $z \epsilon U$. Since Q is one-to-one, $Q(z) \neq 0$ for $z \epsilon U-\{0\}$ and, since $Q'(0) > 0$, we have $T(z) \neq 0$ for all $z \epsilon U$. Thus we also have from Theorem 1.12, that $|T(z)| \geqslant 1$ for all $z \epsilon U$. Consequently $|T(z)| = 1$ for all $z \epsilon U$. From Theorem 1.11, since $Q'(0) > 0$, $T(z) = Q'(0) = 1$ for all $z \epsilon U$, and thus $f^{-1}g(z) = z$ for all $z \epsilon U$, and consequently $f = g$.

THEOREM 2.3. *Let $x_0 \epsilon U$, $x_0 \neq 0$, and f be a continuous function on U such that f is differentiable on $U-\{x_0\}$. Then f is differentiable and there exists a sequence of polynomials $P_1, P_2, \ldots$ which converges uniformly on compact subsets of U to f.*

Proof. If $\varepsilon > 0$ and $r = 1-\varepsilon$, then, from Theorem 1.13, there exists $p > 0$ such that the function $g(z) = f(z)+pz$ for $z \epsilon U$, is one-to-one on $\overline{U}_r$. From Theorem 2.2, g is differentiable and there exists a sequence of polynomials $P_1, P_2, \ldots$ which converges uniformly on compact subsets of U_r to g. Hence $\{P_i-pI_0\}_{i=1}^{\infty}$ converges uniformly on compact subsets of U_r and, in particular, $\overline{U}_{1-2\varepsilon}$, to f. By a diagonal process we obtain a sequence of polynomials $Q_1, Q_2, \ldots$, such that $|Q_n(z)-f(z)| < 1/2^n$ for $z \epsilon U_{1-1/n}$ and $n \epsilon \omega$. Clearly $Q_1, Q_2, \ldots$ is the desired sequence.

LEMMA 2.6. *If $P(z) = \sum_0^n a_p z^p$ for $z \epsilon K$ and $P(z) \leqslant 1$ for $z \epsilon \overline{U}$, then $|a_i| \leqslant 1$, for $i = 0, 1, \ldots, n$.*

Proof. This theorem and proof are due to Porcelli and Connell [2]. Trivially the theorem holds for polynomials of degree zero. Suppose that, for $n \epsilon \omega$, it holds for polynomials of degree n or less and $P(z) = \sum_0^{n+1} a_p z^p$ is a polynomial of degree $n+1$ such that $|P(z)| \leqslant 1$ for $z \epsilon \overline{U}$. Let $\theta \epsilon [0, 2\pi]$ and $Q(z) = 2^{-1}[P(z)-PE(i\theta)]$ for $z \epsilon K$. Then Q has no constant term and Q/I_0 is a polynomial of degree n or less. From Theorem 1.11, $|Q(z)/z| \leqslant 1$ for $z \epsilon \overline{U}-\{0\}$. By the inductive hypothesis, $|2^{-1}a_{p+1}[1-E((p+1)i\theta)]| \leqslant 1$, for $p = 0, 1, \ldots, n$. Upon setting $\theta = \pi/p$ we have $|a_p| \leqslant 1$, for $p = 1, 2, \ldots, n+1$. Finally $|a_0| = |P(0)| \leqslant 1$.

THEOREM 2.4. *If f is a differentiable function on U, then there exists a power series $\sum_0^{\infty} a_n z^n$ which converges uniformly on compact subsets of U to f.*

Proof. From Theorem 2.3, there exists a sequence of polynomials $P_1, P_2, \ldots$, such that for $n \epsilon \omega$, $|P_i(z)-P_j(z)| < 1/2^n$, for $z \epsilon U_{1-1/n}$, and $i, j \geqslant n$, $i, j \epsilon \omega$. Let $n \epsilon \omega$. Then from Lemma 2.6, $|a_{ip}-a_{jp}| < [2^n(1-1/n)^p]^{-1}$, for $i, j \geqslant n$, $i, j \epsilon \omega$, and $p \epsilon \omega$, where $\{a_{ij}\}_{i,j=1}^{\infty}$ is a sequence in K, such that $P_j(z) = \sum_0^{\infty} a_{jp} z^p$ for $j \epsilon \omega$. Thus for $p \epsilon \omega$, there exists $a_p \epsilon K$ such that for $n \epsilon \omega$, $|a_p - a_{ip}| < 2[2^n(1-1/n)^p]^{-1}$ for all $i > n$, $i \epsilon \omega$.

Let $n \epsilon \omega$, and let n_0 denote the degree of P_n. Then for $p > n_0$, $p \epsilon \omega$, $a_{np} = 0$, and hence $|a_p| = |a_p - a_{np}| < 2[2^n(1-1/n)^p]^{-1}$. Thus $\limsup_{p\to\infty} |a_p|^{1/p} \leqslant (1-1/n)^{-1}$. Since n is arbitrary, we have $\limsup_{p\to\infty} |a_p|^{1/p} \leqslant 1$. Thus the power series $T(z) = \sum_0^{\infty} a_p z^p$ converges uniformly on compact subsets of U. Let $z \epsilon U$. Then for $n \epsilon \omega$, such that $|z| < 1-2/n$, we have

$$|T(z)-P_n(z)| = \Big|\sum_{p=0}^{\infty} (a_p - a_{np}) z^p\Big| \leqslant \sum_{p=0}^{\infty} |a_p - a_{np}|\cdot |z|^p$$

$$\leqslant \sum_{p=0}^{\infty} 2|z|^p [2^n(1-1/n)^p]^{-1} \leqslant 2^{n-1}(1-r)^{-1} ,$$

where $r = (1-2/n)\cdot(1-1/n)^{-1}$. Thus $\lim_{n\to\infty} P_n(z) = T(z)$. Since by hypothesis $\lim_{n\to\infty} P(z) = f(z)$, we have $T(x) = f(x)$ for all $x \epsilon U$.

THEOREM 2.5. *Let S be a connected open set, $z_0 \epsilon S$, $f_1, f_2, \ldots$ a sequence of maps of S into K converging uniformly on compact subsets of S to a limit function f_0, and $g_1, g_2, \ldots$ a sequence of differentiable functions on S such that for $n \epsilon \omega$, $g_n' = f_n$ and $g_n(z_0) = 0$. Then $\{g_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of S to a limit function g_0, such that g_0 is differentiable and $g_0' = f_0$.*

Proof. Let H be a compact subset of S. Since S is connected, there exists a collection of squares $Q_1, Q_2, \ldots$ such that $H \subseteq Q \subseteq S$, $x_0 \epsilon Q$, and Q is connected, where $Q = \bigcup_1^n \overline{I(Q_i)}$. Then there exists $M > 0$ such that for $x \epsilon H$, there exists a polygonal arc P with endpoints x and x_0, having length $L(P)$ and lying in Q. By the mean value theorem for real valued functions, for $n, m \in \omega$,

$$|g_n(x)-g_m(x)| \leqslant 2M \sup_{t\in Q} |g'_n(t)-g'_m(t)| = 2M\varepsilon_{nm},$$

where $\varepsilon_{nm} = \sup_{t\in Q} |f_n(t)-f_m(t)|$. Since $\{f_n\}$ converges uniformly on Q to f_0, $\lim_{n,m\to\infty} \varepsilon_{nm} = 0$, and thus $\{g_n\}_{n=1}^{\infty}$ converges uniformly on Q. Hence $\{g_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of S to a limit function g_0. From Lemma 2.3,

$$f_0(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g'_n(x) = g'_0(x) \quad \text{for all} \quad x \in S.$$

DEFINITION 2.1. Let S be an open set, g a map of S into K, and F a collection of functions on subsets of K. Then the statement that F is an $I_{S,g}$ collection means that F is a C_S collection such that, for $f \in F$, $f'(x) = g(x)$ for all $x \in S_f$.

THEOREM 2.6. *Let S be an open set, h a map of I into S, f and g differentiable functions on S, and a and $b \in K$. Then there exists an $I_{S,f}$ collection. Moreover if S is connected and simply connected, there exists a differentiable function k on S, such that $k' = f$, and a unique number $I = I_0^1 f\,dh$, such that if F is an $I_{S,f}$ collection, then $I = I_0^1 F\,dh$. Finally*

$$I_0^1(af+bg)\,dh = a\cdot I_0^1 f\,dh + b\cdot I_0^1 g\,dh.$$

Proof. Let $x \in S$ and R be a circular region with center x lying in S. From Theorem 2.3, there exists a sequence of polynomials $P_1, P_2, \ldots$ converging uniformly on compact subsets of R_x to f. Trivially there exists a sequence of polynomials $Q_1, Q_2, \ldots$ such that, for $n \in \omega$, $Q_n(x) = 0$, and $Q'_n = P_n$. From Theorem 2.5, $\{Q_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of R_x to a differentiable limit function f_x, such that $f'_x(y) = f(y)$ for all $y \in R_x$.

Let $x, y \in S$ and $F(z) = f_x(z)-f_y(z)$ for all $z \in R_x \cap R_y$. Suppose that $F(u) \neq F(v)$ for some $u, v \in R_x \cap R_y$. By the mean value theorem for real valued functions, there exists w in the line segment with endpoints u and v, such that $F'(w) \neq 0$. But $F'(w) = f'_x(w)-f'_y(w) = f(w) = 0$. Thus $\{f_x\}_{x\in S}$ is an $I_{S,f}$ collection. Now if S is connected and simply connected, then the existence of a differentiable function h such that $h' = f$, follows from Theorem 1.5.

Let F and G be $I_{S,f}$ collections and $x \in S$. There exist $f_x \in F$ and $g_x \in G$ such that $x \in S_{f_x}$ and $x \in S_{g_x}$. Let R_x be a circular region with center x lying in $S_f \cap S_g$, and let $\tilde{f}_x = f_x|R_x$ and $\tilde{g}_x = g_x|R_x$. Then $F_0 = \{\tilde{f}_x\}_{x\in S}$ and $G_0 = \{\tilde{g}_x\}_{x\in S}$ are $I_{S,f}$ collections such that $I_0^1 F\,dh = I_0^1 F_0\,dh$ and $I_0^1 G\,dh = I_0^1 G_0\,dh$. Clearly $H = F_0 \cup G_0$ is an $I_{S,f}$ collection, and, from

Theorem 1.1, $I_0^1 F_0\,dh = I_0^1 G_0\,dh$. Then $I_0^1 F\,dh = I_0^1 G\,dh$, and consequently $I_0^1 f\,dh$ is uniquely defined.

We can obtain $I_{S,f}$ collections (vide supra) M and N such that $\{S_k\}_{k\in M} = \{S_k\}_{k\in N}$. It follows readily that

$$a\cdot I_0^1 f\,dh + b\cdot I_0^1 g\,dh = I_0^1(af+bg)\,dh.$$

THEOREM 2.7. *Let S be a simply connected open set, T a simple closed curve in S, $z_0 \in I(T)$, and f a differentiable function on S. Then*

$$f(z_0) = (2\pi i)^{-1} I_T f(z)/(z-z_0)\,dz \quad (1).$$

Proof. From Theorem 2.3, if $Q = Q_{f,z_0}$, then Q is differentiable on S. From Theorem 2.6, $I_T f(z)/(z-z_0)\,dz - I_T f(z_0)/(z-z_0)\,dz = I_T Q(z)\,dz = 0$, and thus $I_T f(z)/(z-z_0)\,dz = f(z_0) I(z-z_0)^{-1} dz$.

If L is an L_{S,z_0} collection, then for $f \in L$, $Ef(z) = z-z_0$ for $z \in S_f$, and hence $1 = Ef(z)\cdot f'(z)$ for $z \in S_f$. Thus L is an $I_{S-\{z_0\},1/(z-z_0)}$ collection and therefore from Theorem 1.10, $I_T(z-z_0)^{-1} = I_T L\,dz = 2\pi i$. Hence $I_T f(z)/(z-z_0)\,dz = 2\pi i f(z_0)$.

Remark. We can readily show that if T is rectifiable, then $\int_T f(z)/(z-z_0)\,dz$ is defined and equal to $I_T f(z)/(z-z_0)\,dz$, and hence (1) reduces to the Cauchy Integral Formula.

THEOREM 2.8. *Let $0 < r_0 < 1$ and f a differentiable function on $S = U-\overline{U}_{r_0}$. Then there exists a power series $\sum_{-\infty}^{+\infty} c_n z^n$ converging uniformly on compact subsets of S to f.*

Proof. For $x \in K$, set $F_x(z) = (2\pi i)^{-1} f(z)(z-x)^{-1}$ for all $z \in S-\{x\}$. From Theorem 1.3, for $x \in U$, $y \in K-\overline{U}_{r_0}$, there exist unique numbers $g(x)$ and $h(y)$, such that $g(x) = I_{C_r} F_x\,dz$ for all $1 > r > |x|$, and $h(y) = I_{C_r} F_y\,dz$ for all $r_0 < r < |y|$. Then from Theorem 2.7, $f(x) = g(x)-h(x)$ for all $x \in S$.

Suppose $x \in S$, $1 > r > |x|$, and $x_1, x_2 \ldots$ a sequence of points in U_r, distinct from x, converging to x. For $z \in M_r = U-\overline{U}_r$, $n \in \omega$, set $v_n(z) = [F_{x_n}(z)-F_x(z)](x_n-x)^{-1}$ and $w(z) = F_x(z)(z-x)^{-1}$. Then $\{v_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of M_r to w. Expressing M_r as the union of two connected and simply connected open sets and applying Theorems 2.5 and 2.6, we see that $[g(x_n)-g(x)](x_n-x)^{-1} = I_{C_s}(F_{x_n}-F_x)(x_n-x)^{-1} dz = I_{C_s} v_n\,dz$ converges to $I_{C_s} w\,dz$ as $n\to\infty$, for $1 > s > r$. Similarly h is differentiable on $K-\overline{U}_{r_0}$. Clearly $h(x)\to 0$ as $x\to\infty$.

For $z \in U_{1/r_0}$, set $k(z) = h(1/z)$ if $z \neq 0$, and $k(z) = 0$ if $z = 0$. Then k is continuous on U_{1/r_0} and differentiable on $U_{1/r_0}-\{0\}$. Then from Theorem 2.3, k is differentiable on U_{1/r_0}. Then from Theorem 2.4, there exist power series $\sum_0^\infty a_n z^n$ and $\sum_0^\infty b_n z^n$ converging uniformly on compact subsets of U and U_{1/r_0} to f and k respectively. Setting $c_n = a_n$ if $n \geqslant 0$, and $c_n = b_{-n}$ if $n < 0$, for $n \in \omega$, we have that $\sum_{-\infty}^{+\infty} c_n z^n$ converges uniformly on compact subsets of S to $f = g - h$.

3. Removable singularities. In this section we are concerned with the removable singularity problem. Our approach is motivated by the argument for the case when the singularity is a single point, given by Porcelli and Connell [1, 5], using differences of difference quotients.

DEFINITION 3.1. Let $A \subseteq K$, and let $T = \{T_i\}_{i=1}^\infty$ be a sequence of subsets of A. Then T is called a *partition* of A, if $A = \bigcup_1^\infty T_i$ and $T_j \cap T_i = \emptyset$ for $i \neq j$, $i, j \in \omega$. If $V = \{V_i\}_{i=1}^\infty$ is a partition of A, then V is called a *refinement* of T, if for every $i \in \omega$, there exists $j \in \omega$ such that $V_i \subseteq T_j$. A collection Σ of partitions of A is called a *family*, if for every T and V in Σ, there exists a common refinement W of T and V in Σ. We shall call Σ an *M family*, if for every $\varepsilon > 0$, there exists $T \in \Sigma$ such that if $V = \{V_i\}_{i=1}^\infty$ is a refinement of T in Σ, then $\sum_1^\infty \delta(V_i)^2 < \varepsilon$.

Let f be a function on a subset B of K containing A. Then f shall be called a *Σ function*, if for every $\varepsilon > 0$, there exists $T \in \Sigma$, such that if $V = \{V_i\}_{i=1}^\infty$ is a refinement of $T \in \Sigma$, then $\sum_1^\infty \delta[f(V_i)]^2 < \varepsilon$. We shall call f a *P_A function*, if for each $x \in A$, there exists $M_{f,x} > 0$ such that $|f(y) - f(x)| \leqslant M_{f,x}|y - x|$ for all $y \in A$. If there exists $M > 0$, such that $|f(y) - f(x)| \leqslant M|y - x|$ for all $x, y \in A$, then we shall call f a *P_A^* function*.

THEOREM 3.1. *Let $A \subseteq B \subseteq K$, and let Σ be a family of partitions of A. If f and g are Σ functions defined on B, such that $f|A$ and $g|A$ are bounded, then $f+g$ and $f \cdot g$ are Σ functions. If h is a P_A^* function on B and Σ is an M family, then h is a Σ function.*

Proof. There exists $M > 0$ such that $|f(x)| + |g(x)| < M$ for all $x \in A$. Let $\varepsilon > 0$. There exists $T \in \Sigma$ such that if $V = \{V_i\}_{i=1}^\infty$ is a refinement of T in Σ, then $\sum_1^\infty \delta[k(V_i)]^2 < \inf\{\varepsilon/4M^2, \varepsilon/4\}$ for $k = g$ and $k = h$. Let $W = \{W_i\}_{i=1}^\infty$ be a refinement of T in Σ, $i \in \omega$, and $x, y \in W_i$. Then

$$|(f+g)(y) - (f+g)(x)|^2 \leqslant [|(f(y) - f(x)| + |g(y) - g(x)|]^2$$
$$\leqslant 2|f(y) - f(x)|^2 + 2|g(y) - g(x)|^2.$$

Thus

$$\delta[(f+g)(W_i)]^2 \leqslant 2\delta[f(W_i)]^2 + 2\delta[g(W_i)]^2,$$

and hence

$$\sum_1^\infty \delta[(f+g)(W_i)]^2 < 2(\varepsilon/4) + 2(\varepsilon/4) = \varepsilon.$$

Thus $f+g$ is a Σ function.

Similarly for $i \in \omega$, and $x, y \in W_i$,

$$|(f \cdot g)(y) - (f \cdot g)(x)|^2 = |[f(y)g(y) - f(y)g(x)] + [f(y)g(x) - f(x)g(x)]|^2$$
$$\leqslant 2M^2[|f(y) - f(x)|^2 + |g(y) - g(x)|^2].$$

Thus

$$\sum_1^\infty \delta[(f \cdot g)(W_i)]^2 \leqslant 2M^2 \sum_1^\infty \delta[f(W_i)]^2 + \delta[g(W_i)]^2 < 2M^2 \cdot 2 \cdot [\varepsilon/4M^2] = \varepsilon.$$

Thus $f \cdot g$ is a Σ function.

Now there exists $M > 0$ such that $|h(x) - h(y)| \leqslant M|x - y|$ for all $x, y \in A$. Suppose $\varepsilon > 0$. Then there exists $T \in \Sigma$ such that for every refinement $V = \{V_i\}_{i=1}^\infty$ of T in Σ, $\sum_1^\infty \delta(V_i)^2 < \varepsilon/M$, and hence $\sum_1^\infty \delta[f(V_i)]^2 \leqslant \sum_1^\infty M \cdot \delta[(V_i)]^2 < M(\varepsilon/M) = \varepsilon$. Thus h is a Σ function.

THEOREM 3.2. *If $A \subseteq B \subseteq K$ and f and g are P_A functions on B that are bounded on A, then $f+g$ and $f \cdot g$ are P_A functions. If h is a P_A function on B and $m(A) = 0$, then $m[h(A)] = 0$. If S is an open set, H a compact subset of S, and f is a differentiable function on S, then f is a P_H^* function.*

Proof. There exists $M > 0$ such that $|f(x)| + |g(x)| < M$ for all $x \in A$. Let $x \in A$. Then for all $y \in A$,

$$|(f+g)(y) - (f+g)(x)| \leqslant |f(y) - f(x)| + |g(y) - g(x)|$$
$$\leqslant M_{f,x}|x - y| + M_{g,x}|x - y| = [M_{f,x} + M_{g,x}] \cdot |x - y|;$$

also,

$$|(f \cdot g)(y) - (f \cdot g)(x)| = |[f(y)g(y) - g(y)f(x)] + [g(y)f(x) - f(x)g(x)]|$$
$$\leqslant |g(y)| \cdot |f(y) - f(x)| + |f(x)| \cdot |g(y) - g(x)|$$
$$\leqslant [M_{f,x} + M_{g,x}] \cdot |x - y| \cdot M.$$

Thus $f+g$ and $f \cdot g$ are P_A functions.

Suppose $m(A) = 0$ and $\varepsilon > 0$. For $n \in \omega$, set $A_n = \{x \in A \mid M_{h,x} < n\}$. Then $\bigcup_1^\infty A_n = A$, and $m(A_n) = 0$. There exists a collection $\{R_{ni}\}_{n,i=1}^\infty$ of circular regions such that for $n \in \omega$, $A_n \subseteq U_{i=1}^\infty R_{ni}$, and $\sum_{i=1}^\infty \delta(R_{ni})^2 < \varepsilon/n\pi 2^n$. Then

$$f(A) \subseteq U_{n,i=1}^\infty f(R_{ni}),$$

and

$$\sum_{n,i=1}^{\infty} \pi\delta[f(R_{ni})]^2 \leqslant \sum_{n=1}^{\infty} \pi n \cdot \sum_{i=1}^{\infty} \delta(R_{ni})^2 < \sum_{n=1}^{\infty} n \cdot \varepsilon/2^n n = \varepsilon .$$

Thus $m[f(A)] < \varepsilon$. Since ε is arbitrary, $m[f(A)] = 0$.

Let $r = \delta(K-S, H)/2$, $M = \sup\{|f(t)| \,|\, t \in K, \delta(t, H) \leqslant r\}$, and $x, y \in H$. Then as in the proof of Lemma 2.2,

$$(1)\qquad |f(y)-f(x)| \leqslant 2Mr^{-1}|y-x| ,$$

if $|y-x| \leqslant r$. Trivially (1) holds if $|y-x| > r$.

THEOREM 3.3. *If A is a closed subset of $\overline{U}$ and f a map of $\overline{U}$ such that f is differentiable on $U-A$, then f is differentiable on U, if either:*

1. *$A_0 = A-C$ $(C = \overline{U}-U)$ possesses an M family of partitions Σ and f is a Σ function; or*

2. *$m(A_0) = 0$, and f is a P_{A_0} function.*

Proof. For $x \in U-A$, let $h_x = Q_{f,x}$. For $x, y \in U-A$, let $h_{xy} = h_x - h_y$, and C be the set of all numbers c, such that there exists a component S of $U-B$, where $B = A \cup \{x, y\}$, such that $h_{xy}(z) = a + cz$ for $z \in S$, for some $a \in K$. Clearly C is empty or countable. Hence there exists a sequence of positive numbers $c_1 > c_2 > \ldots$ converging to 0, each lying in $K-C$. For $z \in \overline{U}$, $i \in \omega$, set $w_i(z) = h_{xy}(z) - c_i z$. From our construction w_i is non-constant on each component of $U-B$, and thus w_i is light on $U-B$, for $i \in \omega$.

Suppose that 1 holds. Let $0 < r < 1$, and set $A_r = A \cap \overline{U}_r$ and $B_0 = B - C$. For $z \in U-\{x\}$, let $p(z) = (z-x)^{-1}$. Then p is differentiable, and hence from Theorem 3.2, p is a $P^*_{A_r}$ function. Since Σ_r is a M family, from Theorem 3.1, p must be a Σ_r function, where Σ_r is the collection of all partitions of A_r of the form $\{A_r \cap T_i \,|\, i \in \omega\} \cup \{x\} \cup \{y\}$ for some $T \in \Sigma$. Then since f is a bounded Σ_r function, and p is bounded on A_r, we have from Theorem 3.1, that $f_0' p$ is a Σ_r function, where $f_0 = f|\overline{U}-\{x\}$. Continuing in this manner, we deduce that W_i is a Σ_r function for $i \in \omega$. Then by Definition 3.1, we have $m[w_i(A_r)] = 0$, and hence $m[w_i(B_0)] \leqslant \sum_{0<r<1} m[w_i(A_r)] = 0$, and thus $w_i(B_0)$ is nowhere dense in K. Then from Theorem 1.11, for $z \in \overline{U}$,

$$|h_{xy}(z)| - |c_i| \leqslant |h_{xy}(z) + c_i z| = |w_i(z)| \leqslant \sup_{t \in C} |w_i(t)| \leqslant T_{xy} + c_i ,$$

where $T_{xy} = \sup_{t \in C} |h_{xy}(t)|$, and hence

$$|h_{xy}(z)| \leqslant T_{xy} + 2|c_i| .$$

Since $\lim_{i\to\infty} c_i = 0$, we have $|h_{xy}(z)| \leqslant T_{xy}$ for $z \in \overline{U}$.

Suppose that 2 holds. Then $m(B_0) = 0$. Since f and p are P_{A_0} functions, from Theorem 3.2, $f_0 \cdot p$ is a P_{A_0} function. Continuing in this manner we deduce that w_i is a P_{A_0} function for $i \in \omega$. Then from Theorem 3.2, since $m(A_0) = 0$, we have $m[w_i(B_0)] = 0$. Then, as above, we have $|h_{xy}(z)| \leqslant T_{xy}$ for $z \in \overline{U}$.

Hence in the case where 1 or 2 holds, letting $x_0 \in A-C$, and $x_1, x_2, \ldots$ be a sequence of points of $U-A$ converging to x_0, we have $|h_{x_i}(x_0) - h_{x_j}(x_0)| = |h_{x_i x_j}(x_0)| \leqslant T_{x_i x_j}$ for $i, j \in \omega$. Clearly $\lim_{i,j\to\infty} T_{x_i x_j} = 0$ and hence $\{h_{x_i}(x_0)\}_{i=1}^{\infty}$ is a Cauchy sequence. If $y_1, y_2, \ldots$ is a sequence in $U-A$ converging to x_0, then $\{h_{y_i}(x_0)\}_{i=1}^{\infty}$ is a Cauchy sequence. Now $x_1, y_1, x_2, y_2, \ldots$ is a sequence in $U-A$ converging to x_0, and hence $h_{x_1}(x_0), h_{y_1}(x_0), h_{x_2}(x_0), h_{y_2}(x_0), \ldots$ is also a Cauchy sequence. Thus $\lim_{i\to\infty} h_{x_i}(x_0)$ must equal $\lim_{i\to\infty} h_{y_i}(x_0)$. Since

$$h_{x_i}(x_0) = [f(x_i) - f(x_0)] \cdot [x_i - x_0]^{-1} \quad \text{for} \quad i \in \omega .$$

f must be differentiable at x_0. Thus f is differentiable on U.

LEMMA 3.1. *If T is a rectifiable simple closed curve, H a compact subset of $K-T$, A a subarc of T with endpoints x and y, and $s > 0$, then there exists an arc B in $T \cup I(T)$ and an arc $T_0 \subseteq T$, with common endpoints u and v such that $A \subseteq T_0$, $B \cap T = \{u\} \cup \{v\}$, $|u-x|, |v-y| < s$, $L(B) < 8L(A)$, where $L(A)$ denotes the length of A, and $H \subseteq E(B \cup T_0)$.*

Proof. Let W be a subarc of T with endpoints p and q and $f(t) = L([p, t])$ for $t \in W$, where $[p, t]$ denotes the subarc of W with endpoints p and t. Then clearly f is one-to-one and continuous, and hence f is a homeomorphism.

Let A_0 be a subarc of T with endpoints a and b such that $A \subseteq A_0$, $x \notin [y, b]$, and such that $L([a, x]), L([y, b]) < \inf[s, |x-y|/6]$. Let $\varepsilon > 0$, and $x = x_0 < x_1 < \ldots < x_{n+1} = y$ be a subdivision of A, such that $L([x_i, x_{i+1}]) < 2^{-1}\inf[\delta[A, T-(a, b)], \varepsilon, |x-y|/6]$. For $i = 0, 1, \ldots, n$ let Q_i be the square with side $2L([x_i, x_{i+1}])$ and center x_i with sides parallel to the x and y axis. If $Q = \bigcup_0^n \overline{I(Q_i)}$, then $Q \cap T \subseteq A_0$. If $P = B(Q)$, then P is a simple closed curve such that $A \subseteq I(P)$, and $\delta(p, A) < \varepsilon$, for all $p \in P$. From the proof of Theorem 1.6, we may choose ε, so that $H \cup [T-(a, b)] \subseteq E(P)$. Then since $x, y \in I(P)$, we have $(a, x) \cap P \neq \emptyset$ and $(y, b) \cap P \neq \emptyset$.

There exists an arc $M \subseteq P$, with endpoints x_1 and y_1, such that $A_0 \cap M = \{x_1\} \cup \{y_1\}$, $x_1 \in [a, x]$ and $y_1 \in [y, b]$. Thus $P_0 = P - [M - A_0]$ is a subarc of P, intersecting $[a, x]$ and $[y, b]$, and hence there exists an arc $N \subseteq P_0$, with endpoints x_2 and y_2, such that $N \cap A_0 = \{x_2\} \cup \{y_2\}$, $x_2 \in [a, x]$, and $y_2 \in [y, b]$.

Suppose that both of $M - A_0$ and $N - A_0$ lie in the same component D of $K-A$. Let $M_0 = M \cup [x_1, y_1]$ and $N_0 = N \cup [x_2, y_2]$. Since

$M \cap N - A_0 = \emptyset$, $M - A_0 \subseteq E(M_0)$ (cf. [8], p. 31). $N - A_0 \subseteq E(M_0)$, $|x_2 - x| < |x-y|/3$, $|y_2 - y| < |x-y|/3$, and hence there exists $w \in N$, such that $|w-x| > |x-y|/3$ and $|w-y| > |x-y|/3$. By our construction of P, there exists a polygonal arc $W \subseteq Q$ with endpoints w and z, such that $W \cap P = \{w\}$, and $W \cap A_0 = \{z\}$, and such that $L(W) \leqslant |x-y|/6$. If $z \in [x, y]$, then since $w \in E(M_0)$, $W - \{z\} \subseteq D$, and we must have $W \cap M \neq \emptyset$. But then $[W - \{w\}] \cap P \neq \emptyset$. Thus $z \in [a, x]$ or $z \in [y, b]$, and thus $|w-x| < |x-y|/3$ or $|w-y| < |x-y|/3$, which is impossible. Hence $M \subseteq \overline{I(T)}$ or $N \subseteq \overline{I(T)}$, $|x_i - x| < s$, and $|y_i - x| < s$, for $i = 1, 2$. Finally $L(M)$ and $L(N) \leqslant \sum_{1}^{n} 8L([x_i, x_{i+1}]) = 8L(A)$.

We now give a new proof of a classical theorem (cf. Titus and Young [7]).

THEOREM 3.4. *If A is a rectifiable arc in $\overline{U}$ with endpoints a and b such that $A \cap C = \{a\} \cup \{b\}$, and f a map of $\overline{U}$ such that f is differentiable on $U - A$, then f is differentiable on U.*

Proof. Let D_1 and D_2 denote the components of $U - A$. Then for $i = 1, 2$, there exists from Theorem 2.6, a differentiable function g_i on D_i, such that $g_i'(z) = f(z)$ for $z \in D_i$. For $i = 1, 2$ we set $T_i = \overline{D}_i - D_i$. Suppose $p \in T_i$ and $\delta > 0$. Let B be a subarc of T_i with endpoints a and b, such that $p \in (a, b)$ and $L(B) < \delta/24$, and R be a circular region with center p and radius less than $\delta/6$, such that $T_i \cap R \subseteq (a, b)$. Then there exist line segments P and Q lying in R with endpoints respectively p_1 and p_2, and q_1 and q_2, such that $P \cap T_i = \{p_2\}$, $Q \cap T_i = \{q_2\}$. Hence each of $L(P)$ and $L(Q)$ is less than $\delta/3$.

From Lemma 3.1, there exists a polygonal arc W, with endpoints x and y such that $[p_2, q_2] \subseteq (x, y)$ and $p_1, q_1 \notin I(W_0)$, where $W_0 = W \cup [x, y]$ and such that $L([x, y]) < 8L(B)$. Then $W \cap P \neq \emptyset$ and $W \cap Q \neq \emptyset$, and there exists an arc $W_1 \subseteq P \cup Q \cup W$ with endpoints p_1 and q_1. Hence $L(W_1) < \delta/3 + \delta/3 + 8L(B) < \delta$ and

$$|g_i(p_1) - g_i(q_1)| \leqslant 2L(W_1) \cdot \sup_{t \in W_1} |g_i'(t)| \leqslant 2\delta \sup_{t \in \overline{U}} |f(t)| .$$

Thus g_i may be continuous extended to D_i for $i = 1, 2$.

For $\varepsilon > 0$, there exists $\delta > 0$, such that $x, y \in \overline{U}$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let p and q be points of A such that $L([p, q]) < \delta/8$, and set $\varepsilon_0 = \varepsilon L([p, q])/8$. Let $i \in \{1, 2\}$. Then from Lemma 3.1, there exists a polygonal arc $P_i \subseteq \overline{D}_i$ with endpoints $p_i, q_i \in A$, such that $L(P_i) < 8L([p, q]) < \delta$, $P_i \cap T_i = \{p_i, q_i\}$, and $|f(p)| \cdot |p - p_i|$, $|f(p)| \cdot |q - q_i|$, $|g_i(p_i) - g_i(p)|$, $|g_i(q_i) - g_i(q)| < \varepsilon_0$. For $z \in K$, let $h_i(z) = g_i(z) - f(p)z$. Then $M = [g_1(p) - g_1(q)] - [g_2(p) - g_2(q)] = [h_1(p) - h_1(q)] - [h_2(p) - h_2(q)]$, and $|h_i(p) - h_i(p_i)| \leqslant |g_i(p_i) - g_i(p)| + |f(p)| \cdot |p - p_i| < \varepsilon_0 + \varepsilon_0 = 2\varepsilon_0$. Similarly $|h_i(q_i) - h_i(q)| \leqslant 2\varepsilon_0$.

Let $N = [h_1(p_1) - h_1(q_1)] - [h_2(p_2) - h_2(q_2)]$. Then $|M - N| < 8\varepsilon_0 = \varepsilon L([p, q])$. Now for $i = 1, 2$,

$$|h_i(p_i) - h_i(q_i)| \leqslant 2L(P_i) \cdot \sup_{t \in P_i} |h_i'(t)|$$

$$\leqslant 16 L([p, q]) \cdot \sup_{t \in P_i} |f(t) - f(p)| < 16\varepsilon L([p, q]) .$$

Thus

$$|N| < 32\varepsilon L([p, q]) ,$$

$$|M| \leqslant |M - N| + |N| < \varepsilon L([p, q]) + 32 L([p, q]) = 33\varepsilon L([p, q]) .$$

Let $u, v \in A$. Then taking a suitable subdivision of $[u, v]$, we see that

$$M_1 = [g_1(u) - g_1(v)] - [g_2(u) - g_2(v)] < 33\varepsilon L([u, v]) \leqslant 33\varepsilon L(A) .$$

Since ε is arbitrary, $M_1 = 0$. Thus there exists $c \in K$, such that $g_1(z) + c = g_2(z)$ for $z \in A$. Let $g(z) = g_1(z) + c$, for $z \in \overline{D}_1$ and $g(z) = g_2(z)$ for $z \in \overline{D}_2 - A$.

Let Σ be the family of all partitions of A, consisting of finitely many connected sets. Since A is rectifiable, Σ is an M family. Using above methods, we deduce that for $u, v \in A$, $|g(u) - g(v)| < 8L([u, v])$. Thus g is of bounded variation on A and hence g is a Σ function. Then from Theorem 3.3, g is differentiable on U. Since $g'(z) = f(z)$ for $z \in U - A$, and g' is continuous, we have $f(z) = g'(z)$ for all $z \in U$. Thus f is differentiable on U.

Remark 3.1. Theorem 3.4 gives a necessary but not sufficient condition for f to be differentiable on U. The following example, due to Denjoy (cf. [3], p. 33), is analogous to functions met in potential theory. Let H be a compact set such that $m(H) > 0$. For $x \in K$, let $F_x(z) = (z - x)^{-1}$ for $z \in K - \{x\}$, and let $F_x(z) = 0$ for $z = x$. Let R be a circular region containing H. Then for $x \in R$,

$$\int_H |F_x|\, dm \leqslant \int_{R_x} |F_x|\, dm = \int_{R_x} \delta(x, z)^{-1} dm(z) = \int_0^{\delta(R)} r \Big[\int_0^{2\pi} 1/r\, d\theta\Big] dr = 2\pi\delta(R) .$$

Thus $\int_H F_x dm$ exists for $x \in H$. For $x \in K - H$, the existence of $\int_H F_x dm$ is obvious. For $x \in K$, let $f(x) = \int_H F_x dm$.

Suppose $x_0 \in K - H$, and $x_1, x_2, \ldots$ is a sequence of points distinct from x_0, converging to x_0. Then the sequence of functions $g_1, g_2, \ldots$, where $g_n(z) = (x_n - z)/(x_0 - z)$ for $n \in \omega$, $z \in H$, converges uniformly to the function g_0, where $g_0(z) = (x_0 - z)^{-2}$ for $z \in H$. Then $\lim_{n \to \infty} \int_H g_n dm = \int_H g_0 dm$. Now

$$[f(x_n) - f(x_0)] \cdot [x_n - x_0]^{-1} = (x_n - x_0)^{-1} \int_H [(x_n - z)^{-1} - (x_0 - z)^{-1}] dm = \int_H h_n dm .$$

Thus

$$\lim_{n\to\infty}[f(x_n)-f(x_0)](x_n-x_0)^{-1}=\int_H g_0\,dm\,.$$

Thus f is differentiable at x_0, and hence f is differentiable on $K-H$.

Let $x\in H$, $\varepsilon>0$, R the circular region with center x and radius $\varepsilon/16\pi$, $f_1(y)=\int_{R\cap H}F_y\,dm$, and $f_2(y)=\int_{H-\bar R}F_y\,dm$ for $y\in K$. Then $f=f_1+f_2$, and $|f_1(y)|\leqslant 2\pi\delta(R)=\varepsilon/4$ for all $y\in R$. Since f_2 is differentiable on R and hence continuous on R, there exists an open set $S\subseteq R$, such that $x\in S$ and $|f(y)-f(x)|<\varepsilon/2$ for $y\in S$. Then for $y\in S$,

$$|f(y)-f(x)|=|f_1(y)+f_2(y)-f_1(x)-f_2(x)|\leqslant|f_1(y)|+|f_1(x)|+|f_2(y)-f_2(x)|$$
$$<\varepsilon/4+\varepsilon/4+\varepsilon/2=\varepsilon\,.$$

Thus f is continuous on K.

Now for $x\in K-H$, $|f(x)|\leqslant m(H)/\delta(x,H)$, and hence $\lim_{n\to\infty}f(x)=0$. For $x\in K$, $xf(x)=\int_H x/(x-z)\,dm=\int_H(1-z/x)^{-1}dm$, and hence $\lim_{n\to\infty}xf(x)=m(H)>0$. Thus f is non-constant on $K-H$. Since A can be readily taken so that $m(A)>0$, we have the desired example.

We note that Denjoy (cf. [3], p. 60) has also given an example of such a non-differentiable function, in the case when $m(A)=0$.

THEOREM 3.5. *Let A be an arc in $\bar U$ with endpoints a and b such that $A\cap C=\{a,b\}$, and f a map of $\bar U$ such that f is differentiable on $U-A$. Then if f is of bounded variation on A, f is differentiable on U, and there exists a discrete set $G\subseteq U$, such that $A-G-C$ is the union of a countable collection of rectifiable open subarcs of A.*

Proof. Since f is of bounded variation on A, f is an M_A function Hence $m[f(A)]=0$, and $f(A)$ is nowhere dense in K. Omitting the trivial case when f is constant, we have that f is light (cf. [8], pp. 93-95) so that, from Theorem 1.11, f is an open map on U. Let $p\in U$. Then from the proof of Theorem 1.11, there exists a simple closed curve $T\subseteq U$ such that $f^{-1}f(p)\cap T=\emptyset$, and $f^{-1}f(p)\cap I(T)$ is finite. Thus $f|U$ has the "scattered inverse property" (cf. [8], p. 83). Making use of results of Stoilow (cf. [8], pp. 86-88) concerning light open maps, we see that f is "locally equivalent to a power mapping" on U, and hence there exists a discrete set $G\subseteq U$, such that if $x\in U-G$, there exists a circular region $R_x\subseteq U$, with center x, such that $f|R_x$ is one-to-one.

Let D_1 and D_2 be the components of $U-A$. The lightness and openness of $f|U$ can also be deduced from a theorem of Titus and Young (cf. [7]) which makes use only of the fact that $f|D_i$ is light and open for $i=1,2$, and that $f(A)$ is nowhere dense in K.

Let $x\in A-C-G$. Then since $\bar D_i$ is homeomorphic to $\bar U$ for $i=1,2$, there exists a simple closed curve $T\subseteq U$, such that setting $S=I(T)$, we have that $B=\bar S\cap A$ is an arc in $\bar S$ with endpoints a and b, such that $B\cap T=\{a,b\}$, and that $f_0=f|S$ is a homeomorphism. Then from Theorem 2.2, f_0^{-1} is differentiable on $f(S)-f(B)$. Since clearly $f(B)$ is a rectifiable arc, from Theorem 3.4, f_0^{-1} is differentiable on $f(S)$, and hence from Theorem 2.2, f_0 is differentiable on S. Thus f is differentiable on $U-G$. Then from Theorem 2.3, f is differentiable on U.

Let $x\in A-C-G$, and define T,S,B as above. Then from Theorem 3.2, f_0^{-1} is a P_S function. Paralleling the arguments in the proof of Theorem 3.1, we see that each subarc of $B-T$ is rectifiable. In particular, there exists an open subarc A_x containing x such that $A_x\subseteq B$, and A_x is rectifiable. Thus $\{A_x\}_{x\in A-C-G}$ is the desired collection of open subarcs. We note that if f is a continuous function on U, and f is differentiable on $U-A$, then from Theorem 3.4, f is differentiable on $U-G$, and, from Theorem 2.3, f is differentiable on U.

Remark 3.2. Let A be an arc and f a map of K such that f is non-constant and differentiable on $K-A$, and $s=\lim_{n\to\infty}f(x)$ exists. Then from Theorem 1.14, $f(A)=f(K)$. Now $f|K-A$ is an open map and hence $f(K-A)$ is open in K. Thus $f(A)$ is not nowhere dense in K. If instead of the hypothesis of Theorem 3.5, we require that $f(A)$ be nowhere dense, then as in the proof of Theorem 3.5, we are reduced to the case where f is a homeomorphism. It is not yet known whether this latter condition is sufficient to insure that f is differentiable on U (cf. [7]).

Remark 3.3. Functions of the form discussed in Remark 3.1 may be obtained in a natural manner from consideration of functions discussed in Theorems 3.4 and 3.5. Suppose that A is a compact subset of U, and f is a map of $\bar U$, such that f is differentiable on $U-A$. Now there exists $0<r_0<1$, such that $A\subseteq U_{r_0}$. From the proof of Theorem 2.8, there exist differentiable functions f on U and h on $K-\bar U_{r_0}$, such that $f(z)=g(z)+h(z)$ for $z\in U-\bar U_{r_0}$, and $\lim_{x\to\infty}h(x)=0$. Let $f_0(z)=h(z)$ for $z\in K-\bar U_{r_0}$, and $f_0(z)=f(z)-g(z)$ for $z\in\bar U_{r_0}$. Then f_0 is continuous on K, $\lim_{x\to\infty}f_0(x)=0$, f_0 is differentiable on $K-A$, and $f(z)=f_0(z)+g(z)$ for $z\in U$.

We now take A to be an arc. If f_0 is differentiable on K, then, from the proof of Theorem 3.5, $f_0(A)$ is the union of a countable collection of arcs and points, and hence $f_0(A)$ is a first category set in K, and thus is nowhere dense in K. From Theorem 3.2, f_0 must be constant, and hence $f_0\equiv 0$. Thus a necessary and sufficient condition that f be differentiable on U, is that $f_0\equiv 0$.

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Reçu par la Rédaction le 17. 8. 1963

Sur les demi-groupes compacts et connexes

par

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Nous nsioorodnsé certains aspects de la structure des demi-groupes compacts et connexes, c'est-à-dire, nous étudions la question de l'existence de divers types de sous continus. Nous avons recours à la notion suivante (voir [2]). On dit qu'un tel demi-groupe S est algébriquement irréductible entre les points a et b lorsque ces deux points ne se laissent unir par aucun sous-demi-groupe compact et connexe qui soit différent de S. Il s'ensuit d'après la manière classique qu'un demi-groupe compact et connexe contient un sous-demi-groupe compact, connexe et algébriquement irréductible entre chaque couple de ses points. La terminologie suit généralement cela de [13].

THÉORÈME 1. *Soit S un demi-groupe compact et connexe algébriquement irréductible entre son identité e et son zéro z. Alors H , le sous-groupe maximal contenant e, est un continu.*

Démonstration. Soit C le composant de H_e qui contient e. Nous supposons, au contraire, que $C \neq H_e$. Puisque le quotient H_e/C est un groupe compact et $\dim(H_e/C) = 0$, il existe des petits sous-groupes de H contenant C qui sont ouverts et fermés. Donc, il existe des ouverts V et O tels que $V \cap H_e \neq \emptyset$, $C \subset O$, $z \notin V^* \cup O^*$, $T = H_e \cap O$ est un sous-groupe, $V^* \cap O^* = \emptyset$, et $H_e \subset V \cup O$. Puisque $TT = T$, il existe un ouvert W tel que $T \subset W \subset O$ et $WW \subset O$. Soit S_1 l'idéal bilatère engendré par l'ensemble fermé $S - V - W$. Donc S_1 est un idéal fermé, et puisque $H_e \subset V \cup W$, l'on a aussi, $S_1 \cap H_e = \emptyset$. Il découle que $S - S_1 = V' \cup W'$ où V' et W' sont des ouverts contenus dans V et W respectivement. En particulier, $S_1 = S - V' - W'$ est un idéal et $W'W' \cap V' = \emptyset$. On sait, [3], qu'il existe un sous-continu M tel que $e \in H_e \cap M$, $M \subset W'$ et $M - H_e \neq \emptyset$. Soit X le demi-groupe compact et connexe engendré par M.

$$X = (M \cup M^2 \cup M^3 \cup \ldots)^* .$$

Nous démontrons d'adorq que$V' \cap X = \emptyset$.

* Avec l'appui de The National Science Foundation NSF GP 237 et NSF GP 610.