Everywhere oscillating functions, extension of the uniformization and homogeneity of the pseudo-arc

by

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The purpose of this paper (¹) is a further study of the pseudo-arc, i.e. the heredetarily indecomposable snake-like continuum (see Knaster [5], Moise [10] and Bing [2]) by using the method of inverse limits. In my previous paper [7] the pseudo-arc was defined as the universal object for all snake-like continua (it was called therefore USC, the *universal snake-like continuum*). This paper is devoted to the study of the homogeneity of the pseudo-arc and of the connexion of this property with inverse limit expansions of USC. The functions appearing in such expansions (called EO-*functions*) are investigated in Part I. Part II contains a more detailed study of uniformizations of functions. The homogeneity theorem is the object of Part III and presents a strengthened form of theorems of Bing [2] and Lehner [6].

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I. Approximate symmetry of everywhere oscillating functions

Everywhere oscillating functions, briefly EO-functions, form a class of mappings of the closed interval onto itself which occur in the description of the pseudo-arc in terms of inverse limits (see [7]). These functions have been defined without topological notions (see also the definition below) and in this part they will be investigated independently of their topological applications. What is called here the oscillation property (OP) of EO-functions corresponds to the crookedness of chains in Bing's description of the pseudoarc. The purpose of this part of the paper is to show that EO-functions are at each point, only approximately of course, symmetric in the sense here defined. This approximate symmetry of EO-functions is, as will be seen in Part III, the reason for the homogeneity of the pseudo-arc.

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1. Definition of EO-functions. Let $\{X_n\}$, $n = 1, 2, \ldots$, be a sequence of closed intervals. Consider for every X_n a sequence of its subdivisions $\{X_{n,r}\}$, $r = n-1, n, \ldots$, such that for every n

(1) *all the simplices* [2] *of* $X_{n,n-1}$ *are congruent,*

(2) *subdivision* $X_{n,r+1}$ *is, for every* $r \geqslant n-1$, *a subdivision of* $X_{n,r}$ *such that each simplex of* $X_{n,r+1}$ *is a half of a simplex of* $X_{n,r}$.

Consider for every $r = 0, 1, \ldots$ a category K_r (for the notion of category, see Godement [3]) whose objects are $X_{n,r}$ and whose morphisms are simplicial mappings of $X_{n,r}$ onto $X_{m,r}$, where $m, n \leqslant r+1$. These simplicial mappings being onto, $\mathrm{Map}(X_{n,r}, X_{m,r}) \neq \emptyset$ is equivalent to $\nu(X_{m,r}) \leqslant \nu(X_{n,r})$, where ν is the number of simplices in the subdivision.

We shall assume that

$$\nu(X_{m,r}) \leqslant \nu(X_{n,r}) \quad \text{for} \quad m \leqslant n \leqslant r+1, \text{ i.e. } m \leqslant n \leqslant r+1 \Longleftrightarrow \Longleftrightarrow \mathrm{Map}(X_{n,r}, X_{m,r}) \neq \emptyset. \tag{3}$$

If K is a category and A, B are objects of K, then a morphism $\pi\colon X \to A$ is said to be a *majorant* for $\mathrm{Map}(A, B)$ if for each pair $f', f'' \in \mathrm{Map}(A, B)$ there exists a morphism $\alpha\colon X \to A$ such that

$$f' \circ \pi = f'' \circ \alpha.$$

Let X', X'' be closed intervals equipped with subdivisions. Let $\mathrm{Map}(X'', X')$ be the set consisting of all simplicial mappings of X'' onto X'. By definition, EO-*functions* are simplicial mappings $\pi\colon X''' \xrightarrow{\text{onto}} X''$ being majorants for $\mathrm{Map}(X'', X')$, where X''' is a closed interval equipped with a subdivision, whenever the set $\mathrm{Map}(X'', X')$ is non-empty and contains a non-isomorphic mapping.

We shall assume that categories K_r, $r = 1, 2, \ldots$, have the following property:

(4) *the set* $\mathrm{Map}(X_{r+1,r}, X_{r,r})$ *contains an* EO-*function* π_r^{r+1} *which is a majorant for all sets* $\mathrm{Map}(X_{r,r}, X_{n,r})$, $n \leqslant r$.

It is convenient to form a matrix consisting of objects of all K_r:

$$\begin{array}{l} X_{1,0} \\ X_{1,1} \leftarrow X_{2,1} \\ X_{1,2} \leftarrow X_{2,2} \leftarrow X_{3,2} \\ \cdots\cdots\cdots\cdots \\ X_{1,r} \leftarrow X_{2,r} \leftarrow X_{3,r} \leftarrow \ldots \leftarrow X_{r,r} \leftarrow X_{r+1,r} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

[2] 1-dimensional simplices of course.

The presence of an arrow between objects means that the corresponding set of morphisms is non-empty. This matrix contains a sequence of EO-functions $\pi_r^{r+1}\colon X_{r+1,r} \to X_{r,r}$ the inverse limit of which is the pseudo-arc. The last fact and the existence of a sequence of categories K_r satisfying conditions (1)-(4) was shown in [7].

2. Vector chains associated to a function. We shall consider vectors $\boldsymbol{P}$ lying on a given closed interval $I = \langle 0, 1\rangle$, which will occasionally be regarded as a vector $\boldsymbol{I} = \overline{01}$. Given $\boldsymbol{P} = \overline{p'p''}$, we set

$$[\boldsymbol{P}] = \{x\colon p' \leqslant x \leqslant p''\}$$

(for brevity we consider here only the case $p' < p''$). When no confusion is likely to result, we shall write briefly P for $[\boldsymbol{P}]$. Given subsets A, B of I, we write

$$A \leqslant B \quad (\text{or } A < B)$$

if $a \leqslant b$ (or $a < b$) for all $a \in A$, $b \in B$. If $\boldsymbol{P}, \boldsymbol{Q}$ are vectors having the same orientation, we write

$$\boldsymbol{P} \leqslant \boldsymbol{Q} \Leftrightarrow P \leqslant Q \quad \text{and} \quad \boldsymbol{P} \subset \boldsymbol{Q} \Leftrightarrow P \subset Q.$$

An ordered collection

$$\mathfrak{P} = \{\ldots, \boldsymbol{P}_i, \boldsymbol{P}_{i+1}, \ldots\}$$

of vectors is said to be a *vector chain* if for each i the origin of $\boldsymbol{P}_{i+1}$ coincides with the end of $\boldsymbol{P}_i$. If for each i we have $\boldsymbol{P}_i \leqslant \boldsymbol{P}_{i+1}$, then we say that $\mathfrak{P}$ forms a *decomposition* of $\boldsymbol{P} = \sum_i \boldsymbol{P}_i$. A vector chain is said to be *regular* if

$$\bigcup_i P_i = \Big[\sum_i \boldsymbol{P}_i\Big].$$

A vector chain is said to be *irreducible* if it contains no regular vector subchain consisting of more than one vector.

Let $f\colon I'' \to I'$ be a function, where I', I'' are closed intervals. Assume $I', I'' = \langle 0, 1\rangle$. Given $\boldsymbol{A} = \overline{a'a''}$, $f(\boldsymbol{A})$ denotes the vector $\overline{f(a')f(a'')}$. We have for every $\boldsymbol{A}$

$$[f(\boldsymbol{A})] \subset f(A).$$

If equality holds, vector $\boldsymbol{A}$ will be called f-*admissible*. An f-admissible vector $\boldsymbol{A}$ is said to be *saturated* if there exist no vectors $\boldsymbol{B}$ such that $\boldsymbol{A} \subset \boldsymbol{B}$, $\boldsymbol{A} \neq \boldsymbol{B}$ and $f(A) = f(B)$.

Each decomposition $\mathcal{A} = [\ldots, \boldsymbol{A}_i, \boldsymbol{A}_{i+1}, \ldots\}$ of I'' into f-admissible vectors induces a vector chain $\{\ldots, f(\boldsymbol{A}_i), f(\boldsymbol{A}_{i+1}), \ldots\}$. The following procedure associates with f and the right end 1 of I'' an irreducible vector chain, called the *right chain for* f, which is uniquely determined by f.

Namely, we define a suitable vector chain $\{A_0, A_1, \ldots\}$ which induces it. At first we define A_0 as the maximal f-admissible vector for which $f(A_0) = f(I'')$ and which is oriented from 0 to 1. Having A_i we define A_{i+1} as the maximal f-admissible vector whose origin coincides with the end of A_i and which is adjacent to A_i.

In an analogous way we define the *left chain for* f.

We have the following invariantness of chains for f, which will be formulated for right chains only.

(5) *Let* $\alpha\colon I''' \xrightarrow{\text{onto}} I''$ *be given, where* $I''' = \langle 0,1\rangle$. *If* $\alpha(1) \in A_r$ *then* f *and* $f\circ\alpha$ *have the same right chains up to vectors which are contained in* $f(A_r)$.

To prove this, we define a suitable decomposition of I''' inducing the right chain for $f\circ\alpha$. Take A_0', the maximal vector on I''' such that $\alpha(A_0') = A_0$. Having A_i' we define A_{i+1}' the maximal vector adjacent to A_i' and such that $\alpha(A_{i+1}') = A_{i+1}$. In this way we define a vector chain $\{A_0', A_1', \ldots, A_{r-1}'\}$ inducing that part of the right chain for $f\circ\alpha$ which coincides with the part of the right chain for f induced by $\{A_0, A_1, \ldots, A_{r-1}\}$. The remaining vectors of these chains are contained in $f(A_r)$.

In the case of simplicial functions $f\colon I''\to I'$ (here I', I'' denote intervals as well as their subdivisions; by I', I'' will be denoted, as before, the corresponding vectors) the irreducible vector chains for f are finite.

3. Oscillation property.

We consider simplicial functions $f\colon I''\to I'$ and the following property of these functions.

(OP) For each f-admissible and saturated vector $C\subset I''$ not containing any end of I'' there exists a decomposition

$$C = C_1 + C_2 + C_3$$

into f-admissible vectors such that

$$f(C) = f(C_1) = -f(C_2) = f(C_3)\,.$$

We shall prove (this, in fact, was proved in [7] but for another definition of oscillation property) that

(6) *Every* EO-*function* π *being a majorant of* $\mathrm{Map}(I'', I')$, *where* I', I'' *are closed intervals equipped with subdivisions, has* (OP) *whenever* $\nu(I'') \geqslant \nu(I')+2 \geqslant 7$.

To prove this, let $\pi\colon I'''\to I''$, where I''' is a closed interval equipped with a subdivision, be given EO-function and let C be a π-admissible and saturated vector on I''' not containing any end of I'''. Hence there exists a vector J on I'' such that C is a component of $\pi^{-1}(J)$. Consider $f'\in\mathrm{Map}(I'', I')$ such that $L = f'(J)$ is a saturated and f'-admissible vector and $f'^{-1}(\mathrm{Int}L) = \mathrm{Int}J$. Such a function exists whenever $\nu(I')\geqslant 5$. Consider $f''\in\mathrm{Map}(I'', I')$ having the following properties: there exists a saturated and f''-admissible vector M for which $f''(M) = L$, $f''^{-1}(L) = M$ and which has a decomposition $M = M_1+M_2+M_3$ into f''-admissible vectors such that $f''(M) = f''(M_1) = -f''(M_2) = f''(M_3)$. Such a function exists whenever $\nu(I'')\geqslant\nu(I')+2$. Function π is a majorant for $\mathrm{Map}(I'', I')$, whence there exists $\alpha\colon I'''\to I''$ such that $f'\circ\pi = f''\circ\alpha$. By the definition of f' and f'', C is a component of $\pi^{-1}f'^{-1}(L) = \alpha^{-1}f''^{-1}(L) = \alpha^{-1}(M)$. Furthermore the ends of C are mapped onto the ends of J and onto the ends of M and then onto the ends of L. It is easy to see that the decomposition of M induces a decomposition of C, $C = C_1+C_2+C_3$ into $f''\circ\alpha$-admissible vectors such that $f''\circ\alpha(C) = f''\circ\alpha(C_1) = -f''\circ\alpha(C_2) = f''\circ\alpha(C_3)$. This decomposition of C has the last property also for function $f'\circ\pi$ and in consequence also for π. Thus (6) is proved.

Let us observe that EO-functions π_r^{r+1} from the sequence defined in section 1 satisfy the condition of (6) for $r\geqslant 2$ (see [7], p. 187).

From (OP) one can easily deduce the following estimation.

(7) *If* $\varrho\colon A''\to A'$ *is a partial function of an* EO-*function satisfying* (OP) *and* $\varrho(A'') = A'$; *then* $\nu(A'')\geqslant 3^{\nu(A')/2}$.

4. Symmetry of functions.

Let $f\colon I''\to I'$ be a function (not necessarily simplicial) and let $a\in I''$. Consider the partial functions

$$f_{-a} = f|\langle 0, a\rangle \quad\text{and}\quad f_{+a} = f|\langle a, 1\rangle\,.$$

and the following vector chains:

(8) the right chain for f_{-a} and the left chain for f_{+a}

Function f is said to be *symmetric around* a if one of the vector chains (8) is a residual part of the other, excluding perhaps the first vector, which is assumed to be contained in the first vector of this residual part. The greatest of chains (8) will be called the *oscillation pattern* of f *around* a.

Examples. Function $f\colon I''\to I'$ is symmetric around the ends of I'' and around each point a such that $f(a)$ is an end of I'. In the first case one of chains (8) is empty, in the second one each of these chains consists of at most one vector.

In Fig. 1 an example of a symmetric function is shown.

We say that f is *strongly symmetric around a closed interval* $A = \langle a^-, a^+\rangle \subset I''$ if f is symmetric around each end of A. We say that f is *symmetric around* A if the right chain for f_{-a^-} and the left chain for f_{+a^+} are such that after the removal from those chains of all vectors which are contained in $f(A)$ one of the resulting chains is a residual part

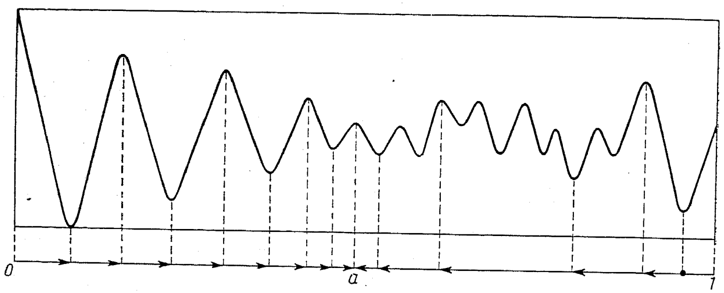


Fig. 1

of the other, excluding perhaps the first vector, which is assumed to be contained in the first vector of this residual part. According to the definitions, each strongly symmetric function around A is at the same time a symmetric function around A. If f is symmetric around A, the greater of two chains:

$$\text{the right chain for } f_{-a^-} \quad \text{and} \quad \text{the left chain for } f_{+a^+}$$

reduced mod $f(A)$ in the way just mentioned, will be called the *oscillation pattern* of f *around* A.

5. Zones of points and intervals. We now introduce some notions of a rather technical character. Let $f\colon I''\to I'$, $a \in I''$ and $\varepsilon \geqslant 0$ be given, where I', I'' are closed intervals. We define the *right* ε*-zone* of a relatively to f (briefly rel. f) as the closed interval $Z^{\varepsilon+}(a) = \langle a, z_a^{\varepsilon+}\rangle$, where $z_a^{\varepsilon+}$ is the first point such that $z_a^{\varepsilon+} \geqslant a$, and such that the distance from $f(z_a^{\varepsilon+})$ to the ends of I' is not greather than ε. Similarly we define the *left* ε*-zone* of a rel. f. The sum of the right and the left ε-zones of a will be called the ε*-zone* of a.

The *once enlarged* right ε-zone of a is, by definition, the right ε-zone of a augmented by a minimal closed interval Z adjacent to $Z^{\varepsilon+}(a)$ on the right side and such that the Hausdorff distance between $f(Z)$ and I' is not greater than ε. The *twice enlarged* right ε-zone of a arises from the once enlarged right ε-zone of a by an augmentation just described. We shall denote these zones by $Z_1^{\varepsilon+}(a)$ and $Z_2^{\varepsilon+}(a)$. If $\varepsilon = 0$ we omit ε in these symbols and we write simply zone instead of 0-zone. Similarly we define once and twice enlarged left ε-zones.

Now let $A \subset I''$ be a closed interval. The ε-zone of A rel. f is, by definition, the sum of the left ε-zone of the left end of A, the interval A and the right ε-zone of the right end of A. Similarly the once and twice enlarged ε-zones of A are defined.

Let $A, B \subset I''$ be closed intervals. We say that A, B are ε*-separated* by f if their twice enlarged ε-zones rel. f are disjoint.

6. Properties of irreducible vector chains for EO-functions. Let $\pi\colon I''\to I'$ be an EO-function and let a belong to the complex I''. There is a dependence between the the right chain for π_{-a} and the left chain for π_{+a}. Namely, as will be seen in Theorem 1, the behaviour of π in the ε-zone of a is approximately the same as the behaviour of a symmetric function around a, for some $\varepsilon \geqslant 0$.

Let a chain consisting of vectors $A_i = \overline{a_i a_{i+1}}$, $i = 0, \ldots, k-1$, and $A_k = \overline{a_k a}$ induce the right chain

$$\{P_0, \ldots, P_k\}, \qquad P_i = \overline{p_i p_{i+1}}, \qquad p_i = \pi(a_i), \qquad p_{k+1} = \pi(a),$$

for π_{-a}.

Denote by e the diameter of the simplices in I'' (by assumption, they are congruent). Given $p \in I'$, $p \neq \pi(a)$, we denote by $p+e$ (or $p-e$) the point whose distance from $\pi(a)$ is that from $\pi(a)$ to p enlarged (diminished) by e and whose distance from p is e. Let r be the smallest index such that $p_r \in \pi(\langle a, 1\rangle)$. By this assumption we have

THEOREM 1. *There exists a subsequence* $s_0 < s_1 < \ldots < s_l$ *of the sequence* $r < r+1 < \ldots < k$ *and a sequence*

$$a_{s_0}^* > a_{s_1}^* > \ldots > a_{s_l}^* > a$$

such that [3] $s_l = k-n$, *where* n *can be equal to* 0, 1, 2, 3 or 4, *and*

$$\pi(a_i^*) = p_i - e, \tag{9}$$

$$[\overline{p_i - e\, p_{i+1} - e}] \subset \pi([\overline{a_i^* a}]) \subset [\overline{p_i + 3e\, p_{i+1} + e}] \tag{10}$$

for $i = s_0, \ldots, s_1$, *and*

$$(*) \qquad s_{j+1} = s_j + 1 \qquad \textit{or} \qquad (**) \qquad s_{j+1} = s_j + 3$$

for each $j = 0, \ldots, l$. *Moreover, in case* (**) *we have*

$$|p_{s_{j+1}} - p_{s_j+1}| \leqslant 2e\,.$$

(3) $p_{k+1} + e = \pi(a) + e$ denotes the point whose distance from $\pi(a)$ is e and which lies on the opposite side of $\pi(a)$ to p_k.

Proof. The first step of induction. In order to define $a^*_{s_0}$, we consider two cases.

Case I. $|p_r - p_{r+2}| > e$. Denote by a'_r the first point on $\langle a, 1\rangle$ such that $\pi(a'_r) = p_r$. Such a point exists according to the assumption concerning r. Consider two subcases.

1. Function π assumes the value $p_{r+1}+2e$ between a and a'_r. Denote by $\tilde{a}_{r+1}$ the first point at which π assumes this value and $a < \tilde{a}_{r+1} < a'_r$. By definition of a'_r and $\tilde{a}_{r+1}$, there exists a saturated and π-admissible vector C such that $a \in C$, $C < \tilde{a}_{r+1}$ and $\pi(C) = \overline{p_r - e\; p_{r+1}+e}$. Take a decomposition $C = C_1 + C_2 + C_3$ in virtue of (OP). We see that $\pi|C$ does not assume the value $p_{r+1}+e$ on the left of a. Hence vector C_2 lies on the right of a and the end of C_2 satisfies (9) and (10) if we substitute it for a_i and r for i.

2. Function π does not assume the value $p_{r+1}+2e$ between a and a'_r. In this case we shall show the existence of points lying between a and a'_r at which value of π is $p_{r+1}+e$, p_{r+1} or $p_{r+1}-e$. In fact, suppose that π does not assume the values $p_{r+1}+e$ and p_{r+1} between a and a'_r. Then there exists a saturated and π-admissible vector C such that $a \in C$ and $\pi(C) = \overline{p_{r+1} - e\; p_r - e}$. Let $C = C_1 + C_2 + C_3$ be a decomposition of C in virtue of (OP). Since $|p_r - p_{r+2}| > e$, vector C_2 lies on the right of a. The value of π at the and of C_2 is $p_{r+1}-e$.

Take the last point lying before a'_r at which the value of π is $p_r - e$. This point satisfies (9) and (10) for $i = r$.

Thus we have proved that in Case I there exist points lying on $\langle a, 1\rangle$ and satisfying (9) and (10) for $i = r$. The last of these points we take as $a^*_{s_0} = a^*_{s_r}$.

Case II. $p_{r+2} = p_r - e$. Consider three essential subcases.

II′. There exists an a'_{r+1} such that $a < a'_{r+1}$, $\pi(a'_{r+1}) = p_{r+1}-e$ and $a'_{r+1} < a''_r$, where a''_r is a point satisfying (9) and (10) for $i = r$.

Since a''_r satisfies (9) and (10), we take the last such point as $a^*_{s_0} = a^*_r$.

II″. There exists an a'_{r+1} such that $a < a'_{r+1}$ and $\pi(a'_{r+1}) = p_{r+1}-e$ satisfying (9) and (10) for $i = r+1$ and there exist such a point for index r.

The last of such points, a'_{r+1}, is the desired point $a^*_{s_0} = a^*_{r+1}$.

II‴. There do not exist points satisfying (9) and (10) for indices r and $r+1$. In this case we define $a^*_{s_0} = a^*_{r+2}$ as follows.

Let a'_{r+2} be the first point on $\langle a, 1\rangle$ such that $\pi(a'_{r+2}) = p_{r+2} = p_r - e$. Consider two subcases, which are analogous to the subcases of the Case I.

1. Function π admits the value $p_{r+3}+2e$ between a and a'_{r+2}. Let $\tilde{a}_{r+3}$ be the first point at which π admits this value and such that $a < \tilde{a}_{r+3} < a'_{r+2}$. In virtue of the definition of a'_{r+2} and $\tilde{a}_{r+3}$ there exists a saturated and π-admissible vector C such that $a \in C$, $C < \tilde{a}_{r+1}$ and $\pi(C) = \overline{p_{r+2} - e\; p_{r+3}+e}$. Take the decomposition of C according to (OP). Since $\pi|C$ does not assume the value $p_{r+3}+e$ on the left of a, vector C_2 of this decomposition lies on the right of a. We see that the end of C_2 satisfies (9) and (10) for $i = r+2$.

2. Function π does not assume the value $p_{r+3}+2e$ between a and a'_{r+2}. Let a'_r be the first point on $\langle a, 1\rangle$ such that $\pi(a'_r) = p_r$. We have

$$a < a'_{r+2} < a'_r .$$

If π admits the value $p_{r+3}+2e$ between a'_{r+2} and a_r, then there exists a saturated and π-admissible vector C such that

$$a'_{r+2} < C < \tilde{a}_{r+3} < a'_r \quad \text{and} \quad \pi(C) = \overline{p_{r+2} - e\; p_{r+3}+e} ,$$

where $\tilde{a}_{r+3}$ is (as before) the first point on $\langle a, 1\rangle$ such that $\pi(\tilde{a}_{r+3}) = p_{r+3}+2e$. Let $C = C_1 + C_2 + C_3$ be a decomposition of C in virtue of (OP). We see that the end of C_2 satisfies (9) and (10) for $i = r+2$.

If π does not admit the value $p_{r+3}+2e$ before a'_3, we proceed as follows (compare subcase 2 of Case I). Namely, we shall show that there exist points lying between a and a'_r at which the value of π is $p_{r+3}+e$, p_{r+3} or $p_{r+3}-e$. In fact, suppose that π does not admit the values $p_{r+3}+e$ and p_{r+3} between a and a'_r. Then there exists a saturated and π-admissible vector C such that $a \in C$, $C < a'_r$ and $\pi(C) = \overline{p_{r+3} - e\; p_r - e}$. Let $C = C_1 + C_2 + C_3$ be a decomposition of C in virtue of (OP). Since $\pi|C$ does not assume the value $p_r - e = p_{r+2}$ on the left of a, vector C_2 lies on the right of a. The value of π at the end of C_2 is $p_{r+3}-e$. Thus we have shown that π admits the value $p_{r+3}-e$ between a and a'_{r+2}. Hence the last point lying before a'_r at which is $p_{r+2}-e$ the value of π satisfies (9) and (10) for $i = r+2$.

Summing up, we have proved that in Case II‴ there exist points lying on $\langle a, 1\rangle$ satisfying (9) and (10) for $i = r+2$. We take the last of such points as $a^*_{s_0} = a^*_{r+2}$.

The second step of the induction. Suppose that points

$$a^*_{s_0} > \ldots > a^*_{s_{m-1}} > a, \qquad s_{m-1} = t-1 ,$$

satisfying (9) and (10) are already defined. We shall define $a^*_{s_m}$.

Consider two cases which are analogous to those of the first step of the induction.

Case I. $|p_t - p_{t+2}| > 2e$. Denote by a''_t the first point lying on $\langle a, 1\rangle$ such that $\pi(a''_t) = p_t - e$. Such a point exists according to the first inclusion in (10) for $i = t-1$. We have

$$a < a''_t < a^*_{t-1} \ (= a^*_{s_{m-1}}) .$$

Consider two subcases which are analogous to those of Case I in the first step of the induction.

1. Function π assumes the value $p_{t+1}+2e$ between a and a_t''. Let $\tilde{a}_{t+1}$ be the first point on $\langle a, 1\rangle$ such that $\pi(\tilde{a}_{t+1}) = p_{t+1}+2e$. By the definition of a_t'' and $\tilde{a}_{t+1}$, there exists a saturated and π-admissible vector C such that $a \in C$, $C < \tilde{a}_{t+1}$ and $\pi(C) = \overline{p_t - e\, p_{t+1}+e}$. Let $C = C_1 + C_2 + C_3$ be a decomposition of C in virtue of (OP). Since $\pi|C$ does not assume the value $p_{t+1}+e$ on the left of a, vector C_2 lies on the right of a and the end of C_2 satisfies conditions (9) and (10) for $i = t$.

2. Function π does not assume the value $p_{t+1}+2e$ in the interval between a and a_t''. We shall show that if assumes there one of the values $p_{t+1}+e$, p_{t+1} or $p_{t=1}-e$.

In fact, suppose that π does not admit the values $p_{t+1}+e$ and p_{t+1} between a and a_t''. Then there exists a saturated and π-admissible vector C such that $a \in C$, $C < a_t''$ and $\pi(C) = \overline{p_{t+1} - e\, p_t - 2e}$. Taking the decomposition $C = C_1 + C_2 + C_3$ in virtue of (OP), we see that π assumes the value $p_{t+1}-e$ at the end of C_2, the vector lying on the right of a (the last assertion follows from the fact that $\pi|C$ does not assume the value $p_t - 2e$ on the left of a, because $p_t - 2e$ is greater than p_{t+2} in the case considered here).

Thus a_t'' satisfies (9) and (10) for $i = t$.

Summing up subcases 1 and 2 we see that there exist points lying between a and a_{t-1}^* for which conditions (9) and (10) are satisfied for $i = t$. The last of such points we take as $a_{s_m}^* = a_t^*$.

Case II. $|p_t - p_{t+2}| \leqslant 2e$. In this case, in virtue of the induction hypothesis contained in the first inclusion of (10) for $i = t$, there exists a point lying between a and a_{t-1}^* at which the value of π is p_{t+2}. Denote by a_{t+2}' the first such point. In order to define $a_{s_m}^* = a_{t+2}^*$, consider two subcases.

1. Function π assumes the value $p_{t+3}+2e$ between a and a_{t+2}'; let $\tilde{a}_{t+3}$ be the first point on $\langle a, 1\rangle$ where it assumes this value. In virtue of the definition of a_{t+2}' and $\tilde{a}_{t+3}$ there exists a saturated and π-admissible vector C such that $a \in C$, $C < \tilde{a}_{t+3}$ and $\pi(C) = \overline{p_{t+2} - e\, p_{t+3}+e}$. Since $\pi|C$ does not assume the value $p_{t+3}+e$ on the left of a, vector C_2, of the decomposition of C according to (OP), lies on the right of a. The end of C_2 satisfies conditions (9) and (10) for $i = t+2$.

2. Function π does not assume the value $p_{t+3}+2e$ in the interval between a and a_{t+2}'. Then, if it assumes there the value $p_{t+3}+e$, p_{t+3} or $p_{t+3}-e$, the last point in the interval $\langle a, a_{t+2}'\rangle$ at which the value of π is $p_{t+1}-e$ satisfies conditions (9) and (10) for $i = t+2$. If that point does not satisfy conditions (9) and (10) examine the behaviour of π in the interval between a_{t+2}' and a_{t-1}^*. It may happen that (a) π assumes first the value $p_{t+2}+e$ and then the value p_{t+3} or (b) π assumes first the value p_{t+3} and then (perhaps) the value $p_{t+2}+e$.

(a) In this case there exists a saturated and π-admissible vector C such that $a \in C$, $C < a_{t-1}^*$ and $\pi(C) = \overline{p_{t+3} - e\, p_{t+2}}$. Take a decomposition $C = C_1 + C_2 + C_3$ according to (OP). Since $\pi|C$ does not assume the value p_{t+2} on the left of a, the vectors C_2 and C_3 lie on the right of a. The last point lying on C_3 at which the value of π is $p_{t+2}-e$ satisfies conditions (9) and (10) for $i = t+2$.

(b) Since $|p_{t+1} - p_{t+3}| > e$, there exists a saturated and π-admissible vector C such that $a_{t+2}' < C < p_{t-1}^*$ and $\pi(C) = \overline{p_{t+2} - e\, p_{t+3} - e}$. In addition, let C be the first such vector. Let $C = C_1 + C_2 + C_3$ be a decomposition of C in virtue of (OP). We see that the end of C_2 satisfies conditions (9) and (10) for $i = t+2$.

Summing up, in all subcasese of Case II there exist points lying between a and $a_{s_{m-1}}^* = a_{t-1}^*$ and satisfying (9) and (10) for $i = t+2$. We take the last such point as $a_{s_m}^* = a_{t+2}^*$.

The end of the induction. The procedure described in the second step of the induction, which leads to the definition of the next point $a_{s_m}^*$, fails if

(i) $p_t - \pi(a) = e$ or

(ii) $|p_t - p_{t+2}| \leqslant 2e$ and $p_{t+2} - \pi(a) = e$.

In case (i) we have $t = k$ or $t = k-1$ and we regard the induction as finished: the last member in the sequence $\{a_{s_j}^*\}$ is $a_{s_1}^* = a_{t-1}^* = a_{k-1}^*$ or $a_{s_1}^* = a_{t-1}^* = a_{k-2}^*$.

In case (ii) we have $t = k - n$, where n may be equal to 2 or 3. In this case we also regard the induction as finished: the last member in the desired sequence $\{a_{s_j}^*\}$ is a_{k-3}^* or a_{k-4}^*.

7. Approximation theorem for EO-functions. We shall deduce from Theorem 1 the existence, for any EO-function $\pi\colon I'' \to I'$ and for any closed interval $A \subset I''$, of an approximation of π^* by a function $\pi^*\colon I'' \to I'$ symmetric around A, the accurancy ε of this approximation depending only upon mesh I'. The last fact will be expressed shortly by $\pi =_\varepsilon \pi^*$, and this notation will be used in the sequel. The statement formulated above is an immediate consequence of the following theorem.

THEOREM 2. *Let an EO-function $\pi\colon I'' \to I'$, a point $a \in I''$ and $\varepsilon \geqslant 6e$ be given. Then there exists a function $\pi^*\colon I'' \to I'$ symmetric around a, equal to π at point a as well as outside the $\varepsilon/2$-zone of a and outside some intervals U^- and U^+ containing the ends of this $\varepsilon/2$-zone with $\operatorname{diam}\pi(U^-)$, $\operatorname{diam}\pi(U^+) \leqslant \varepsilon$, a function which is furthermore such that*

(i) $\pi^* =_{\varepsilon} \pi$,

(ii) *the ends of the zone of a* rel.π^* *lie in* U^- *and* U^+,

(iii) π^* *is constant in some neighbourhood of* a.

Proof. We define π^* by a modification of π. We shall do this in four steps.

1st step. Consider $\pi|\langle a_{s_j}, a^*_{s_j}\rangle$, $j = 0, \ldots, l$. We modify this function by cutting with the value $p_{s_{j+1}}$; this means that we replace π on the interval $\langle a_{s_j}, a^*_{s_j}\rangle$ by a simplicial function π_j defined in the division points of I'' by

$$\pi_j(x) = \begin{cases} p_{s_{j+1}} & \text{if} \quad \pi(x) = p_{s_{j+1}} + ne, \quad n = 1, 2 \text{ or } 3, \\ \pi(x) & \text{for other } x. \end{cases}$$

In virtue of Theorem 1 we have $\pi_j =_{3e} \pi|\langle a_{s_j}, a^*_{s_j}\rangle$.

We also make analogous modification of π in the interval between a_{s_0-1} and a_{s_0}. The function which arises is denoted by π_{I}.

2nd step. Let j be such that a_{s_j} lies in the interior of the $\varepsilon/2$-zone of a and

(11) $$|p_{s_j} - \pi(a)| > 3e.$$

We now modify π_{I} in some neighbourhoods of $a^*_{s_j}$. Namely, consider the component of $\pi_{\mathrm{I}}^{-1}(\langle p_{s_j}, p_{s_j} - 3e\rangle)$ containing point $a^*_{s_j}$. Denote this component by C_j. If function π_{I} assumes on C_j the value p_{s_j}, we do not change π_{I} on C_j. If π_{I} does not assume that value we construct on C_j a function α_j which is equal to π_{I} at the ends of C_j and assumes on C_j the value p_{s_j}. Furthermore α_j is simplicial with respect to subdivisions I' and I''. Such a function exists if C_j consists of at least six simplices. An easy evaluation by making use of (OP), based on the fact that the set of values of $\pi_{\mathrm{I}}|C_j$ consists of at least two simplices, shows that C_j consists of at least eight simplices.

According to (11), these improvements of π_{I} at several a_{s_j} satisfying this condition are independent. The resulting function will be denoted by π_{II}.

3rd step. Let j_0 be the greatest index such that (11) holds. Consider $\pi_{\mathrm{II}}|\langle a_{s_{j_0}}, a^*_{s_{j_0}}\rangle$ and cut this function with the value $\pi(a)$, i.e. form a new function π'_{II} by the formula

$$\pi'_{\mathrm{III}}(x) = \begin{cases} \pi(a) & \text{for } x \text{ such that } \pi(x) \text{ lies on the opposite} \\ & \text{side of } \pi(a) \text{ to } p_{s_{j_0}}, \\ \pi_{\mathrm{II}}(x) & \text{for other } x. \end{cases}$$

We then improve the function in a small neighbourhood of a in such a way that (iii) is satisfied (the possibility of the last improvement is obvious). For the resulting function π_{III} we have $\pi_{\mathrm{III}} =_{6e} \pi$, according to the second inclusion of (10).

4th step. Let z^-, z^+ be the ends of $\varepsilon/2$-zone of a rel. π. If the value of π at z^- (which is equal to the value of π_{III} at z^-) is equal 0 or 1, we leave π_{III} unchanged. Otherwise, we proceed as follows. E.g. let $\pi(z^-) = p$ and $1 - p = ne \leqslant \varepsilon/2$. Consider an interval $J = \langle 1 - 2ne\rangle \subset I'$ and the component U^- of $\pi_{\mathrm{III}}^{-1}(J)$ containing z^-. If U^- contains the end of I'', we leave π_{III} unchanged on U^-. If U^- does not contain the end of I'', we replace $\pi_{\mathrm{III}}|U^-$ by a simplicial function β: $U^- \to I'$ whose values at the ends of U^- are $1 - 2ne$ and which assume on U^- the

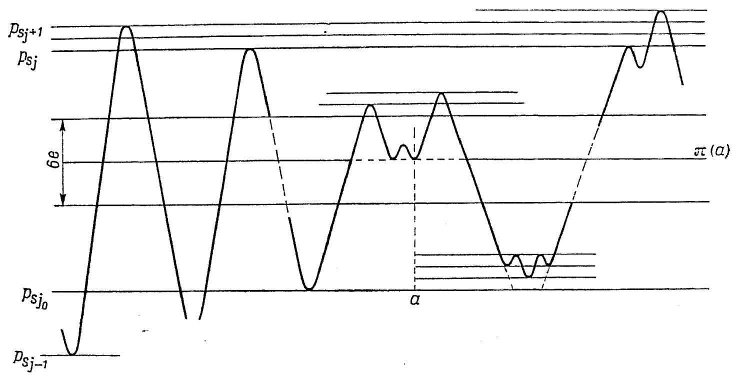


Fig. 2

value 1. Such a function p exists whenever the number of simplices in U^- is at least $4n$. From (7) it follows that this number is at least 3^n, and therefore the construction is possible if $n \geqslant 3$. Then we do the same at the end z^+.

The resulting function is the desired function π^*. It is symmetric around a with pattern $p_0 = 0$ or $1, \ldots, p_{s_0}, \ldots, p_{s_{j_0}}, \pi(a)$. Figure 2 illustrates the construction of π^*.

II. Uniformization of symmetric functions

Let K be a category and f, g morphisms of K. A pair α, β of morphisms of K is said to be a *uniformization* of f, g if

$$f \circ \alpha = g \circ \beta.$$

We shall consider the category S of the simplicial mappings of the closed interval I onto itself and its subcategory S_0 consisting of mappings which map ends onto ends. The possibility of uniformizing any pair of mappings was proved for category S_0 in the papers of Homma [4] and of Sikorski and Zarankiewicz [11], and then the result was extended to category S in [8]. The purpose of this part is a further study of uniformization, namely connected with the following question: given a uniformization α, β of partial functions $f|A, g|B$, where A, B are closed intervals, what are the conditions under which the uniformization α, β admits an extension to a uniformization of f, g. We shall see that the required conditions are closely related to the symmetry of f, g around A, B.

In the construction of uniformizations we shall utilize the so-called *bigraphs* of two functions (see [11] and [8]) which are the subsets of $I\times I$ given by

$$[f, g] = \{(x, y)\colon f(x) = g(y)\}.$$

In order to find a uniformization of f, g it is sufficient to find a curve $x = \alpha(t)$, $y = \beta(t)$, $t \in I$, with projections into x and y axes being onto, which lies in the bigraph. We shall call such a curve a *uniformization curve*. Therefore the problem of the extension of uniformizations is equivalent to that of prolongations of uniformization curves.

Let us observe that the bigraph $[f, g]$ is, in fact, the inverse limit of the system consisting of two mappings f, g having the same range.

1. Similarity of symmetric functions. Let $f, g\colon I'', II''' \to I'$ be simplicial functions symmetric around closed intervals $A, B \subset I'', I'''$. We say that f, g are *similar around* A, B if the oscillation patterns of f around A and of g around B are the same.

It follows from (5) that

(12) *If f, g are symmetric and similar around A, B, then for each pair of functions $\alpha, \beta\colon I^{\mathrm{IV}}, I^{\mathrm{V}} \to I'', I'''$ and each pair of closed intervals $C, D \subset I^{\mathrm{IV}}, I^{\mathrm{V}}$ such that $\alpha(C) = A$, $\beta(D) = B$ functions $f\circ\alpha$, $g\circ\beta$ are symmetric and similar around C, D.*

2. Extension of uniformizations to the zones of points and intervals. At first we consider the case when A, B are single points a, b. The ends of the once enlarged zones $Z_1(a), Z_1(b)$ of a, b will be denoted briefly by z_a^-, z_a^+, z_b^- and z_b^+.

THEOREM 3. *Let $f, g\colon I'', I''' \to I'$ be symmetric and similar around a, b. Let z_b' be an end of $Z_1(b)$ such that $f(z_a^+) = g(z_b')$. Then there exists a pair of functions*

$$\alpha, \beta\colon T = \langle t, t'\rangle \to \langle a, z_a^+\rangle, I'''$$

for which $f\circ\alpha = g\circ\beta$ and which satisfy the following initial conditions

$$\alpha(t) = a, \quad \beta(t) = b, \quad \alpha(t') = z_a^+, \quad \beta(t') = z_b'.$$

If, in addition, f is constant in some neighbourhood of a, then α may be chosen in such a way that $\alpha^{-1}\alpha(t) = t$.

Proof. Let $\{A_i^-\}$ and $\{A_i^+\}$, $i = 0, \ldots, k$, be vector chains inducing the oscillation pattern of f around a. Let $A_i^- = \overline{a_i^- a_{i+1}^-}$, where $a_0^- = z_a^-$ and $a_{k+1}^- = a$. Similarly, $A_i^+ = \overline{a_i^+ a_{i+1}^+}$, where $a_0^+ = z_a^+$ and $a_{k+1}^+ = a$. Note that some first members of sequences $\{A_i^-\}$ or $\{A_i^+\}$ may be empty. Note that vectors A_i^-, A_i^+ are f-admissible. Assume analogous notation for function g. By symmetry and similarity, $f(A_i^-) = f(A_i^+) = g(B_i^-) = g(B_i^+)$ for each i, whenever the vectors exist.

According to the uniformization theorem for S_0, there exist uniformizations $\alpha_i, \beta_i\colon \overline{t_i t_{i+1}} \to A_i^+, B_i'$ of $f|A_i^+, g|B_i'$ (the sign $'$ should be changed into $+$ or $-$ according to whether $z_b' = z_b^+$ or $z_b' = z_b^-$; t_{k+1} will be denoted also by t). Note that if there exists a neighbourhood of a on which f is constant, function α_k may be chosen in such a way that $\alpha_k^{-1}\alpha_k(t) = t$. The last fact is a consequence of the existence in the bigraph of $f|A_k^+, g|B_k'$ of a horizontal segment having one of its ends at points a, b.

These partial uniformizations α_i, β_i having common values at corresponding ends may be joined to a pair of functions α, β having the desired properties.

THEOREM 4. *Let $f, g\colon I'', I''' \to I'$ be symmetric and similar around closed intervals A, B and therefore let f be strongly symmetric around A. Let $\alpha, \beta\colon T = \langle t^-, t^+\rangle \to A, B$ be a uniformization of $f|A, g|B$ such that $\alpha(t^+) = a^+$ (the right end of A). Let z_b' be an end of $Z_1(B)$ such that $f(z_a^+) = g(z_b')$. Then there exists a pair of functions*

$$\alpha^+, \beta^+\colon T^+ = \langle t^+, t'\rangle \to \langle a, z_a^+\rangle, I'''$$

for which $f\circ\alpha^+ = g\circ\beta^+$ and which satisfy the initial conditions

$$\alpha^+(t^+) = a^+, \quad \beta^+(t^+) = \beta(t^+), \quad \alpha^+(t') = z_a^+, \quad \beta^+(t') = z_b'.$$

If, in addition, there exists a neighbourhood of a on which f is constant, then α^+ may be chosen in such a way that $(\alpha^+)^{-1}\alpha^+(t^+) = t^+$.

Proof. Assume without loss of generality that $z_b' = z_b^+$. As before for the point a, let $\{A_0^+, \ldots, A_k^+\}$ be the left chain for f_{+a^+} inducing the irreducible vector chain for f at point a^+. Assume similar notation for function g. Note that the origin a_0^+ of A_0^+ coincides with z_a^+ and the origin b_0^+ of B_0^+ with z_b^+. Let k_0 be the first index such that $f(A_{k_0}) \subset f(A)$. Denote the origin $a_{k_0}^+$ of $A_{k_0}^+$ by x. Consider a point $y \in I'''$ such that $g(y) = f(x)$ and an interval $\langle t, t^+\rangle \subset T$ such that $\beta(t) = y$. By (5), and the fact that f is symmetric around a^+, functions $f\circ\alpha|\langle t, t^+\rangle$ and $f|\langle a^+, x\rangle$ are similar around a^+, t^+. We have, by assumption, $f\circ\alpha|\langle t, t^+\rangle = g\circ\beta|\langle t, t^+\rangle$. Hence functions $f|\langle a^+, x\rangle$, $g\circ\beta|\langle t, t^+\rangle$ are similar around

a^+, t^+. According to Theorem 3, we may uniformize the last pair of functions by functions

$$\gamma,\ \delta\colon\ \langle t^+, \tilde{t}\rangle \to \langle a^+, x\rangle,\ \langle t, t^+\rangle,$$

which satisfy the initial conditions: $\gamma(t^+) = a^+$, $\delta(t^+) = t^+$, $\gamma(\tilde{t}) = x$, $\delta(\tilde{t}) = t$. Hence we have a pair of functions

$$\tilde{a},\ \tilde{\beta} = \gamma, \quad \beta\circ\delta\colon\ \langle t^+, \tilde{t}\rangle \to \langle a^+, x\rangle,\ I'''$$

such that $f\circ\tilde{a} = g\circ\tilde{\beta}$ and satisfying the initial conditions:

$$\tilde{a}(t^+) = a^+, \quad \tilde{a}(\tilde{t}) = x, \quad \tilde{\beta}(t^+) = \beta(t^+), \quad \tilde{\beta}(\tilde{t}) = y\,.$$

Then we take a uniformization $a^+_{k_0-1}$, $\beta^+_{k_0-1}$ of the pair $f|\langle x, a^+_{k_0-1}\rangle$, $g|\langle y, b^+_{k_0-1}\rangle$, according to the uniformization theorem for S_0, and next, in virtue of the same theorem, the uniformizations a^+_i, β^+_i of pair $f|A^+_i$, $g|B^+_i$ for all $i\leqslant k_0$.

All the partial uniformizations defined above have the same values at corresponding ends, and they may be joined to a pair of functions a^+, β^+ having the desired properties including $(a^+)^{-1}a^+(t^+) = t^+$ if f is constant in some neighbourhood of a^+.

3. Global extension of uniformizations. We shall extend the results of the preceding section to the case where A, B consists of a finite number of components which are separated from each other in the sense of section 5 of Part I. We start with two lemmas.

Let $a, c\in I''$, $a < c$, be separated by f. Consider the uniformization curve for $f|\langle z^+_a, z^-_c\rangle, g$, where z^+_a and z^-_c denote the suitable ends of the once enlarged zones of a and c.

LEMMA 1. *Let b and d be such that $f(z^+_a) = g(b)$ and $f(z^-_c) = g(d)$. There exists a uniformization curve C for $f|\langle z^+_a, z^-_c\rangle$, g joining points (z^+_a, b) and (z^-_c, d).*

Proof. At first we construct a curve $C'\subset[f|\langle z^+_a, z^-_c\rangle, g]$ joining the points mentioned above (with projection into y-axis being not necessarily onto the whole of I'''). We distinguish two cases in the construction of C'.

1. $f(z^+_a)\neq f(z^-_c)$. In this case functions $f|\langle z^+_a, z^-_c\rangle$ and $g|\langle b, d\rangle$ belong to S_0 and the uniformization curve for this pair is the desired curve C'.

2. $f(z^+_a) = f(z^-_c)$. In this case take a point z, $z^+_a < z < z^-_c$, such that $f(z)$ is the end of I' opposite to $f(z^+_a)$. Then take $b^*\in I'''$ such that $g(b^*) = f(z)$. Consider the uniformization curves C_1 and C_2 for pairs $f|\langle z^+_a, z\rangle, g|\langle b, b^*\rangle$ and $f|\langle z, z^-_c\rangle, g|\langle b^*, d\rangle$ resulting from the uniformization theorem for S_0. The curve $C_1\cup C_2$ is the desired curve C'.

Now we take an arbitrary uniformization curve C'' for the pair $f|\langle z^+_a, z^-_c\rangle, g$. The sum $C'\cup C''$ is the desired curve C.

LEMMA 2. *There exists a curve $C\subset[f|\langle z^+_a, 1\rangle, g]$ joining a given point (z^+_a, b), where b is such that $g(b) = f(z^+_a)$, with a point of the form $(1, d)$.*

Proof. Let $\{\boldsymbol{A}_0, \ldots, \boldsymbol{A}_k\}$ be a vector chain inducing the irreducible vector chain for $f|\langle z^+_a, 1\rangle$ at the end 1. Define the following vector chain $\{\boldsymbol{B}_0, \ldots, \boldsymbol{B}_k\}$ on I'''. Let b be the origin of $\boldsymbol{B}_0$. Suppose that the origin of $\boldsymbol{B}_i$ is already defined; we define the end of $\boldsymbol{B}_i$ in such a way as to get an admissible vector for which $g(\boldsymbol{B}_i) = f(\boldsymbol{A}_i)$. Thus the end of $\boldsymbol{B}_k$ is a point d such that $g(d) = f(1)$.

Then for each pair of partial functions $f|\boldsymbol{A}_i$, $g|\boldsymbol{B}_i$ we take uniformizations according to the uniformization theorem for S_0. These uniformizations may be joined to a pair of functions

$$a,\ \beta\colon\ \langle t', t''\rangle \to \langle z^+_a, 1\rangle,\ I'''$$

such that $f\circ a = g\circ\beta$ and satisfying the conditions: $a(t') = z^+_a$, $\beta(t') = b$, $a(t'') = 1$, $\beta(t'') = d$. The desired curve C is given by the equations: $x = a(t)$, $y = \beta(t)$, $t\in\langle t', t''\rangle$.

Combining Theorem 4 and Lemmas 1 and 2 we get the following

THEOREM 5. *Let $A^1 < \ldots < A^s$ and $B^1, \ldots, B^s$ be closed subintervals of I' and I'' respectively and let f, g be symmetric and similar around each pair A^i, B^i. Let all A^i be separated from each other by f and let f be strongly symmetric around each A^i. Let $a_i, \beta_i\colon T_i = \langle t^-_i, t^+_i\rangle \to A^i, B^i$ be uniformizations of $f|A^i, g|B^i$ such that a_i map ends onto ends. Then there exists a uniformization $a, \beta\colon T\to I'', I'''$ of f, g, where T is a closed interval containing each T_i in such a manner that $T_i < T_{i+1}$ for the i in question, a uniformization such that $a|T_i = a_i$, $\beta|T_i = \beta_i$.*

If, in addition, f is constant in some neighbourhoods of each end of A^i, then a may be taken in such a way that $a^{-1}a(T_i) = T_i$ for each i.

Proof. Extend uniformizations a_i, β_i of $f|A^i, g|B^i$ to uniformizations a^*_i, β^*_i of $f|Z_1(A^i), g|Z_1(B^i)$ according to Theorem 4. Let the corresponding uniformization curves C^*_i go through the given corners $(z^-_{a^i}, z'_{b^i})$ and $(z^+_{a^i}, z'_{a^{i\prime}})$ of rectangles $Z_1(A^i)\times Z_1(B^i)$. Assume also that the additional assertion of Theorem 4 holds for a_i if f is constant in neighbourhoods of the ends of A^i.

Consider also for each i the uniformization curves for pairs $f|\langle z^+_{a^i}, z^-_{a^{i+1}}\rangle, g$ resulting from Lemma 1 and going through the points $(z^+_{a^i}, z'_{b^i})$ and $(z^-_{a^{i+1}}, z'_{b^{i+1}})$. Consider finally the curves lying in $\langle 0, z^-_{a^1}\rangle\times I'''$ and in $\langle z^+_{a^s}, 1\rangle\times I'''$ resulting from Lemma 2.

The curve C which is the sum of all the curves mentioned above is the desired uniformization curve for f, g: it admits a parametrization $a, \beta\colon T\to C$ satisfying all the assertions of the Theorem including the last one if f is constant in neighbourhoods of the ends of A^i.

III. A homogeneity theorem for the pseudo-arc

We base ourselves on the definition of the pseudo-arc, according to which it is an inverse limit

(14) $$X = \varprojlim \{X_m, \pi_m^n\},$$

where X_m, $m = 1, 2, \ldots$, are closed intervals provided with subdivisions $X_{m,r}$, $r \geqslant m-1$, which are defined in section 1 of Part I, and where $\pi_n^{n+1}\colon X_{n+1,n} \to X_{n,n}$ are EO-functions from the sequence defined there (for details see [7]).

Bing showed in [2] that the pseudo-arc is n-homogeneous in the following sense: if $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are points of X such that X is irreducible between each pair of points p_i, p_j and q_i, q_j then each homeomorphism $(h', h'')\colon \{p_1, \ldots, p_n\} \rightleftarrows \{q_1, \ldots, q_n\}$ may be extended to a homeomorphism $(\widetilde{h}', \widetilde{h}'')\colon X \rightleftarrows X$. Recently, Lehner [6] stated a more general result, where instead of points p_i, q_j continua P_i, Q_j may be taken.

A homogeneity theorem for the pseudo-arc which will be proved here is a generalization of Lehner's result to infinitely many continua satisfying some conditions concerning their position in X. In the proof we shall use: general theorems on mappings of inverse limits analogous to that of Alexandroff [1] (see my paper [9]), the approximate symmetry of EO-functions and theorems on the extension of uniformization from the preceding parts of this paper.

1. Non-dense position of subsets of the pseudo-arc. Let A be a closed subset of pseudo-arc X given by (14). We have

$$A = \varprojlim \{A'_m, \pi'^n_m\},$$

where $A'_m = \pi_m(A)$ and $\pi'^n_m = \pi^n_m | A'_n$. Let A_n be the star of A'_n in the complex $X_{n,n}$. We have

$$A = \varprojlim \{A_m, \widetilde{\pi}^n_m\},$$

where $\widetilde{\pi}^n_m = \pi^n_m | A_n$.

We say that A is in a *non-dense position* in X if for each m and each $\varepsilon > 0$ there exists an n such that the components of A_n are ε-separated by π^n_m and all the functions π^n_m of (14) map components of A_n onto components of A_m. Note that

(15) *If the set A consists of finitely many components such that X is irreducible between each of them, then A is in a non-dense position in X.*

It is obvious that in this case π^n_m map components onto components for sufficiently large indices n. Therefore we must only prove

(16) *For each m and each $\varepsilon > 0$ there exists an n such that if C'_n and C''_n, $C'_n < C''_n$, are components of A_n then $\varrho[\pi^m_n(J_n), X_m] < \varepsilon$, where $J_n = \{x\colon C'_n \leqslant x \leqslant C''_n\}$, ϱ denoting the Hausdorff distance.*

Suppose that (16) is not true. Then there exists a continuum of the form $\varprojlim \{J_k, \pi^l_k\}$ contained in X and different form X (because of the fact that its projections into each X_k differs from X_k by a given $\varepsilon > 0$) joing two given components $C' = \varprojlim \{C'_k, \pi^l_k\}$ and $C'' = \varprojlim [C''_k, \pi^l_k\}$ of the set A. This contradicts the fact that X is irreducible between C' and C''.

The non-dense position of A follows from (16) in virtue of (OP) of EO-functions.

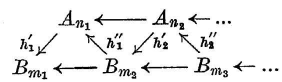
2. Homogeneity theorem. Let X, Y be pseudo-arcs given by their inverse expansions (14).

THEOREM 6. *If $A = \bar{A} \subset X$, $B = \bar{B} \subset Y$ are in non-dense positions in X, Y and $(h', h'')\colon A \rightleftarrows B$ is a homeomorphism, then there exists an extension of (h', h'') to a homeomorphism $(\widetilde{h}', \widetilde{h}'')\colon X \rightleftarrows Y$.*

We begin with some reductions of the proof of the theorem. Assume that the inverse expansion of Y is

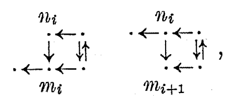
$$Y = \varprojlim \{Y_m, \sigma^n_m\}.$$

Let $(h', h'')\colon A \rightleftarrows B$ be given by a diagram

(17) 

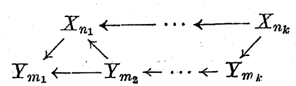
such that there exist sequence $\{\varepsilon'_i\}$, $\{\varepsilon''_i\}$, $i = 1, 2, \ldots$, of positive numbers tending to 0 and satisfying the following conditions:

(i) *each subdiagram of* (17) *consisting of finitely many arrows, i.e. each subdiagram of one of the following four forms*

(18) 

is ε'_i or ε''_i-commutative,

(ii) *if all the subdiagrams of the form* (18) *contained in the diagram*



are $\alpha'_k\cdot 4\varepsilon'_i$ *or* $\alpha'_k\cdot 4\varepsilon''_i$*-commutative, then the analogous subdiagrams of the augmented diagram*

$$\begin{array}{ccccccccc} & X_{n_1} & \longleftarrow & \cdots & \longleftarrow & & X_{n_k} & & \\ \swarrow & & \nwarrow & & & \swarrow & & \nwarrow & \\ Y_{m_1} & \longleftarrow & Y_{m_2} & \leftarrow \cdots \leftarrow & & Y_{m_k} & & \leftarrow & Y_{m_{k+1}} \end{array}$$

are $\alpha''_k\cdot 4\varepsilon'_i$ *or* $\alpha''_k\cdot 4\varepsilon''_i$*-commutative, where the adjoined triangle diagram is* $4\varepsilon'_k$*-commutative and*

$$1<\alpha'_1<\alpha''_1<\alpha'_2<\ldots$$

is a given sequence tending to 2.

The possibility of representing any homeomorphism by a diagram (17) which satisfies property (i) was proved in [9]. That (17) may be chosen in such a way that (ii) is satisfied is a consequence of Theorem 2 of [9], according to which a diagram (17) with property (i) may be made for each sequence $\varepsilon'_1, \varepsilon''_1, \varepsilon'_2, \ldots$ tending to 0. Hence it may be made, in particular, for a sequence satisfying recurrence relations (ii).

Let us remark that since diagram (17) represents a homeomorphism, then

(19) *for each* k *there exists an* r *such that if* D'_{m_k}, D''_{m_k} *are distinct component of* B_{m_k} *then for each component* C_{n_r} *of* A_{n_r} *we have*

$$C_{n_r}\subset h''_r[(\sigma^{m_r}_{m_k})^{-1}(D'_{m_k})]\Rightarrow h''_r[(\sigma^{m_r}_{m_k})^{-1}(D''_{m_k})]\cap C_{n_r}=\emptyset\,.$$

The proof of Theorem 6 reduces to the extension of diagram (17) to a diagrams of pairs

(20)
$$\begin{array}{ccccc} & X_{n'_1}, A_{n'_1} & \rightarrow & X_{n'_2}, A_{n'_2} & \rightarrow \cdots \\ \swarrow{\scriptstyle \tilde h'_1} & & \nwarrow{\scriptstyle \tilde h''_1} & \swarrow{\scriptstyle \tilde h'_2} & \\ Y_{m'_1}, B_{m'_1} & \leftarrow & Y_{m'_2}, B_{m'_2} & \leftarrow \cdots & \end{array}$$

where $\{m'_k\}, \{n'_k\}$ are subsequences of $\{m_k\}, \{n_k\}$, and which is nearly commutative with respect to some sequences $\{\varepsilon'_k\}, \{\varepsilon''_k\}$ tending to 0.

3. A lemma. We shall now prove a lemma which is of a rather technical character and which will be used in the construction of functions $\tilde h'_k$ and $\tilde h''_k$ in diagram (20).

Let $\pi\colon I''\to I'$ be an EO-function with $n=\nu(I')$ sufficiently large. Let $E_1<\ldots<E_k$ be closed subintervals of I'' separated from each other by π. Suppose $E_i=\langle e_i^-, e_i^+\rangle$, $i=1,\ldots,k$. Let $\mathfrak{I}_1^-, \mathfrak{I}_1^+, \ldots, \mathfrak{I}_k^-, \mathfrak{I}_k^+$ be given decreasing vector chains in I' (a vector chain is decreasing if each vector is contained as a set in the preceding one).

LEMMA 3. *Given simplicial functions* $\varphi_i\colon E_i\to I'$, *there exists a function* $\varphi\colon I''\to I'$, *such that* $\varphi|E_i=\varphi_i$, *whose oscillation patterns on the left of* e_i^- *and on the right of* e_i^+ *are* $\mathfrak{I}_i^-$ *and* $\mathfrak{I}_i^+$, *respectively.*

Proof. Let us remark that

(21) *if* $A\subset I''$ *is a closed interval and* $\nu(A)\geqslant n(n+1)$ *then there exists a function* $\psi\colon A\to I'$ *having the given oscillation pattern at one of the ends of* A.

Then we construct φ as follows.

Let H_j be the component of $I''-\bigcup_i E_i$ lying between E_j and E_{j+1}. Let $H'_j=Z_1(E_j)\cap H_j$ and $H''_j=Z_1(E_{j+1})\cap H_j$. Since E_j and E_{j+1} are separated, we have

$$\pi(H'_j)=\pi(H''_j)=I'\,.$$

Then, by (7), we have $\nu(H'_j), \nu(H''_j)\geqslant 3^{n/2}$. Assume that n is so large that $3^{n/2}\geqslant n(n+1)$. Hence, by (21), there exist functions

$$\varphi'_j\colon H'_j\to I', \quad \varphi''_j\colon H''_j\to I'$$

which coincide with φ_j, φ_{j+1} at the ends e_j^+, e_{j+1}^- of E_j, E_{j+1} and having at those ends the oscillation patterns $\mathfrak{I}_j^+, \mathfrak{I}_{j+1}^-$.

Denote by H_j^* the sets $H_j-H'_j-H''_j$. Since E_j and E_{j+1} have the twice enlarged zones disjoint, we have $\pi(H_j^*)=I'$. Therefore the number of simplices in H_j^* is sufficiently large to construct a function $\varphi_j^*\colon H_j^*\to I'$ having at the ends of H_j^* the values which coincide with those of φ'_j, φ''_j.

Consider for each j a function $\varkappa_j=\varphi'_j\cup\varphi_j^*\cup\varphi''_j$ and join all those functions and functions φ_j on E_j to a function defined on the whole of I''. This is the desired function φ.

4. Proof of the homogeneity theorem. We shall construct diagram (20) by induction. Let $m'_1=m_1$ and $\tilde\varepsilon'_1=4e'_1$. Choose an index m''_1, $m''_1>m'_1$, such that

(a) components of $B_{m''_1}$ are ε'_1-separated by $\sigma^{m''_1}_{m'_1}$,

(b) 6 mesh $Y_{m'_1,m''_1}\leqslant\varepsilon$.

Define $n'_1=n_{l_1}$ to be an index chosen for m''_1 in virtue of (19). Consider an auxiliary function

$$\sigma^{*m''_1}_{m'_1}\colon\ Y_{m''_1}, B_{m''_1}\to Y_{m'_1}, B_{m'_1}$$

which is a $2\varepsilon'_1$-approximation of $\sigma^{m''_1}_{m'_1}$ such that

(c) the zones as well the once and twice enlarged zones of components of $B_{m''_1}$ rel. $\sigma^{*m''_1}_{m'_1}$ coincide with the corresponding ε'_1-zones, of the three kinds mentioned above, rel. $\sigma^{m''_1}_{m'_1}$,

(d) $\sigma^{*m''_1}_{m'_1}$ is symmetric around each component of $B_{m''_1}$.

The existence is a consequence of Theorem 2 in virtue of (a) and (b). We define $\tilde{h}'_1$ as a function which is symmetric around each component $A_{n'_1}$ in the following manner:

(e) if $C_{n'_1}$ is a component of $A_{n'_1}$ such that the inclusion

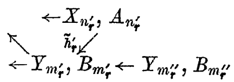
$$C_{n'_1} \subset h''_{l_1}[(\sigma^{m_{l_1+1}}_{m''_1})^{-1}(D_{m''_1})]$$

holds for a component $D_{m''_1}$ of $B_{m''_1}$ then the oscillation pattern of $\tilde{h}'_1$ around $C_{n'_1}$ is the same as that of $\sigma^{*m''_1}_{m'_1}$ around $D_{m''_1}$.

This pattern is well defined because n'_1 is chosen for m''_1 in virtue of (19).

Besides, function $\tilde{h}'_1$ is such that $\tilde{h}'_1|A_{n'_1}$ is equal to the function $A_{n'_1} \to B_{m'_1}$ resulting from diagram (17). The existence of h'_1 follows from Lemma 3 if n'_1 is sufficiently large.

Assume that the following part

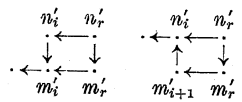


of the diagram (20) is already defined as well as the positive numbers $\tilde{\varepsilon}'_i, \tilde{\varepsilon}''_j$ for $i = 1, \ldots, r$, $j = 1, \ldots, r-1$ such that the following conditions are satisfied:

(22) $\qquad m'_i = m_{k_i} \in \{m_k\}, \quad n'_j = n_{l_j} \in \{n_l\}, \quad i, j \leqslant r,$

(23) $\qquad \tilde{\varepsilon}'_i = 4\varepsilon'_{k_i}, \ i \leqslant r, \quad \varepsilon''_j = 4\varepsilon''_{l_j}, \ j \leqslant r-1,$

(24) *subdiagrams of the form*



are $\alpha'_r \cdot 4\varepsilon'_{k_i}$ *or* $\alpha'_r \cdot 4\varepsilon''_{l_i}$*-commutative, where* α'_r, α''_r *are defined in* (ii) *in section* 2.

The remaining induction hypotheses are as follows:

(a′) components of $B_{m''_r}$ are ε'_{k_r}-separated by $\sigma^{m''_r}_{m'_r}$,

(b′) 6 mesh $Y_{m'_r, m''_r} \leqslant \varepsilon'_{k_r}$,

and n'_r is chosen for m''_r in virtue of (19).

Finally assume that $\tilde{h}'_r$ is defined as symmetric and similar to $\sigma^{m''_r}_{m'_r}$ around corresponding components of $A_{n'_r}$ and of $B_{m''_r}$ (in virtue of (19)), and coincides with the map $A_{n'_r} \to B_{m'_r}$ resulting from diagram (17).

Here $\sigma^{*m''_r}_{m'_r}\colon Y_{m''_r}, B_{m''_r} \to Y_{m'_r}, B_{m'_r}$ is an $2\varepsilon'_{k_r}$-approximation of $\sigma^{m''_r}_{m'_r}$ satisfying the conditions:

(c′) the zones as well as the once and twice enlarged zones of components of $B_{m'_r}$ rel. $\sigma^{*m''_r}_{m'_r}$ coincide with the corresponding ε'_{k_r}-zones, of the three kinds mentioned above, rel. $\sigma^{m''_r}_{m'_r}$,

(d′) $\sigma^{*m''_r}_{m'_r}$ is symmetric around each component of $B_{m''_r}$.

Now we shall define $m'_{r+1}, \tilde{\varepsilon}''_r, \tilde{h}''_r$ and n''_r and we shall reconstruct for them the induction hypotheses.

Let $\varepsilon''_r = 4\varepsilon''_{l_r}$. Choose an index n''_r, $n''_r > n'_r$, such that

(a″) components of $A_{n''_r}$ are ε''_{l_r}-separated by $\pi^{n''_r}_{n'_r}$,

(b″) 6 mesh $X_{n'_r} \leqslant \varepsilon''_{l_r}$.

Define $m'''_r = m_s$ as an index chosen for n''_r in virtue of (19). Assume $m'''_r \geqslant m_r$. Define an auxiliary function

$$\bar{h}''_r\colon Y_{m'''_r}, B_{m'''_r} \to X_{n'_r}, A_{n'_r}$$

as a function which is symmetric around each component of $B_{m'''_r}$ in the following manner:

(e″) if $D_{m'''_r}$ is a component of $B_{m'''_r}$ such that the inclusion

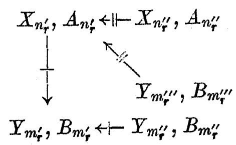
$$D_{m'''_r} \subset h''_s[(\pi^{n_s}_{n''_r})^{-1}(C_{n''_r})]$$

holds for a component $C_{n''_r}$ of $A_{n''_r}$, then the oscillation pattern of $\bar{h}''_r$ around $D_{m'''_r}$ is the same as that of $\pi^{*n''_r}_{n'_r}$ around $C_{n''_r}$, and, moreover, $\bar{h}''_r$ is on $B_{m'''_r}$ equal to the function resulting from diagram (17). Function $\pi^{*n''_r}_{n'_r}$ mentioned in (e″) is a $2\varepsilon''_{l_r}$-approximation of $\pi^{n''_r}_{n'_r}$ satisfying conditions (c″) and (d″) analogous to (c) and (d). The existence of this function follows from Theorem 2 in virtue of (a″) and (b″).

The existence of $\bar{h}''_r$ follows from Lemma 3 if m'''_r is sufficiently large.

In this way the following part of the next triangle of diagram (20) is defined:

(25)



The similar functions are denoted by similar arrows. We shall close the bottom part of (25). To this end, we uniformize the following functions:

(f) $\qquad Y_{m_r}, B_{m'_r} \leftarrow Y_{m''_r}, B_{m''_r} \leftarrow Y_{m'''_r}, B_{m'''_r},$

(g) $\qquad Y_{m'_r}, B_{m'_r} \leftarrow X_{n'_r}, A_{n'_r} \leftarrow Y_{m'''_r}, B_{m'''_r}$

where f, g will be only defined with the use of the corresponding functions of (25).

Function f is a modification of function

$$\sigma^{*m_r''}_{m_r'} \circ \sigma^{m_r'''}_{m_r''} . \tag{26}$$

Namely, we modify (26) in virtue of Theorem 2 on components $D^*_{m_r'''}$ of $(\sigma^{m_r''}_{m_r})^{-1}(B_{m_r'''})$ so as to get a function symmetric around the ends of $D_{m_r'''}$, the component of $B_{m_r'''}$ which is contained in $D^*_{m_r'''}$. The function which results is strongly symmetric around each component of $B_{m_r'''}$. In addition, take an f which is constant in neighbourhoods of the ends of components of $B_{m_r'''}$. These modifications are independent of the modification leading from $\sigma^{m_r'''}_{m_r'}$ to function (26). According to (b′), we have

$$f =_{2\varepsilon_{k_r}'} \sigma^{m_r'''}_{m_r'} . \tag{27}$$

Function g is a modification of $\widetilde{h}_r' \circ \overline{h}_r''$. Namely we define g as equal to function (26) on $B_{m_r'''}$. Then we modify $\widetilde{h}_r' \circ \overline{h}_r''$ outside $B_{m_r'''}$ so as to get a continuous simplicial function from $Y_{m_r'''}$ into $Y_{m_r'}$ being, in addition, such that

$$g =_{2\varepsilon_{k_r}'} \widetilde{h}_r' \circ \overline{h}_r'' . \tag{28}$$

The possibility of constructing such an approximation depends only on the quotient $\nu(Y_{m_r''',m_r''})/\nu(Y_{m_r',m_r''})$ which is sufficiently large if m_r'' is so.

Let α, β: $T \to B_{m_r'''}$, where T is a complex isomorphic to $B_{m_r'''}$, be canonical isomorphisms. They form a uniformization for $f|B_{m_r'''}$, $g|B_{m_r'''}$. By the definition of f, g all the hypotheses of Theorem 5 are satisfied for these functions. Hence there exists a uniformization

$$\alpha_*, \beta_*\colon\ T_* \to Y_{m_r'''},\ Y_{m_r'''}, \qquad T_* \text{ being a simplicial interval,}$$

as an extension of α, β such that

$$\alpha_*^{-1}\alpha_*(U) = U \quad \text{for each component } U \text{ of } T. \tag{29}$$

As usual, we suppose an imbedding $T \to T_*$.

By (27) and (28), we have

$$h_r' \circ h_r'' \circ \beta_* =_{4\varepsilon_{k_r}'} \sigma^{m_r'''}_{m_r'} \circ \alpha_* .$$

We may assume that T_* is isomorphic to a complex $Y_{m_r^{\mathrm{IV}}}$, where m_r^{IV} is an index greater than m_r'''. By the majorization property of EO-functions, there exists a function

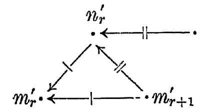
$$\gamma\colon\ Y_{m_r^{\mathrm{IV}}+1} + Y_{m_r^{\mathrm{IV}}}$$

such that $\alpha_* \circ \gamma = \sigma^{m_r^{\mathrm{IV}}}_{m_r'''}$. According to (29), $\alpha_*|\alpha_*^{-1}(B_{m_r'''})$ is a homeomorphism. Hence the composite mapping

$$\overline{h}_r'' \circ \beta_* \circ \gamma | B_{m_r^{\mathrm{IV}}+1}$$

is identical with the mapping $B_{m_r^{\mathrm{IV}}+1} \to A_{n_r'}$ resulting from diagram (17).

We take $m_{1+r}' = m_r^{\mathrm{IV}} + 1$ and $\widetilde{h}_r'' = \overline{h}_r'' \circ \beta_* \circ \gamma$. In this way the desired next triangle of (20), i.e. the diagram



is constructed. According to the construction of $\overline{h}_r''$ and by (5), function $\widetilde{h}_r''$ is similar to $\pi^{*n_r''}_{n_r'}$. The constructed triangle is $4\varepsilon_{k_r}'$-commutative. Therefore the adjoining of this triangle to the diagram given by the induction hypotheses does not destroy the approximate commutativity, in the sense of (24), of the augmented diagram. Function $\widetilde{h}_r''$, index n_r'' and $\tilde{\varepsilon}_r'' = 4\varepsilon_{l_r}''$ form a basis for the construction of the next triangle.

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