Il est à remarquer qu'il résulte tout de suite du théorème II que *si F est une famille indénombrable d'ensembles infinis distincts de nombres naturels, il existe un ensemble E de la famille F et une suite infinie E_i $(i = 1, 2, ...)$ d'ensembles de la famille F autres que E et tels que* $E \subset E_1 + E_2 + ...$

Il en résulte que dans le théorème I le terme finie ne peut pas être remplacé par dénombrable.

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Invariant Borel sets

by

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Let $N = 0, 1, 2, ...$, be the set of non-negative integers, and let Γ be a group of permutations of N. Each permutation of N induces in the obvious way a permutation of the product space 2^N which is continuous in the usual product topology on 2^N. With this topology 2^N is of course homeomorphic to Cantor's Discontinuum, and so the theory of Borel subsets of 2^N is well understood. We shall let the whole group Γ act on 2^N and ask which subsets of 2^N invariant under Γ are Borel. The main result presented in Section 1 shows that, under suitable conditions on Γ, *every minimal Γ-invariant subset of 2^N is a Borel set*. By a *minimal Γ-invariant set* we understand a non-empty set invariant under the action of Γ which includes no smaller such set. In other words, the orbits under Γ of single points of 2^N are exactly the minimal Γ-invariant sets.

In Section 2 we shall apply the main result to solve a problem of Kuratowski and will thereby show that several kinds of interesting subsets of the space of all sets of rational numbers are actually Borel sets. Previously it was only clear that these sets were analytic sets. The method applied to this problem is then generalized to give a useful, more abstract result.

In Section 3 it is shown, with the aid of the abstract result of Section 2, that certain subsets of the (complete metric) space of all closed subsets of Cantor's Discontinuum are likewise Borel sets. The interesting question (proposed to the author by David Freedman of the University of California, Berkeley) of whether these conclusions extend to the space of closed cubsets of an arbitrary compact metric space will be discussed, but unfortunately the methods of this paper do not seem to give the answer.

The author has to admit that metamathematical considerations originally led to the main result. Indeed the main theorem is a generalization of the author's result about the possibility of characterizing countable relational structures up to isomorphism using sentences from a certain infinitary first-order language. Further information about the metamathematical results can be found in [4]. For the sake of the author's mathematician friends who were kind enough to express an interest the conslusions about Borel sets but do not feel comfortable with metamathematical terminology, it was decided that purely mathematical versions of the proofs would be more useful.

The authors gratefully acknowledges his indebtedness both to Professor David Freedman and to Professor Czesław Ryll-Nardzewski for many stimulating conversations on Borel sets in general and the topics of this paper in particular.

1. The main result. To formulate the proper condition on the group Γ, we let $N!$ denote the group of *all* permutations of the set N. Further N^N denotes the Baire space which we regard as a product space with the product topology. Now $N! \subseteq N^N$, so we can treat $N!$ as a subspace of N^N with the relative topology. Finally the desired condition on Γ is that Γ be a *closed* subset of $N!$. The combinatorial meaning of this condition is very simple: namely, if $g \in N!$ and if every restriction of g to a finite subset of N agrees with the corresponding restriction of a permutation in Γ, then g itself must be a member of Γ.

There are many examples of closed subgroups of $N!$. To cite one we shall find useful later, let $\pi\colon N\times N \to N$ be a fixed one-one correspondence between ordered pairs of integers and integers. Let Γ_0 be a given closed subgroup of of $N!$ (e.g. let $\Gamma_0 = N!$). Then another closed subgroup Γ_1 can be obtained by taking the set of all permutations $g_1 \in N!$ for which there exists a (unique) permutation $g_0 \in \Gamma_0$ such that for all $x, y \in N$,

$$g_1\big(\pi(x, y)\big) = \pi\big(g_0(x), g_0(y)\big) .$$

In words we can say that Γ_1 is obtained from Γ_0 by transferring the coordinatewise action of Γ_0 on $N\times N$ back to the set N with the aid of the correspondence π. That Γ_1 is indeed closed is very easily checked.

Turning now to the proof of the main result, we let Γ be a fixed closed subgroup of $N!$. The action of a permutation $g \in \Gamma$ on the space 2^N is defined formally as the mapping that takes a function $f \in 2^N$ to the function $f\circ g^{-1}$. The symbol $\circ$ denotes the operation of composition of two functions. If we think of $f \in 2^N$ as the characteristic function of a subset of N, then $f\circ g^{-1}$ is the characteristic function of the *direct* image of the set under g; whereas $f\circ g$ would be the characteristic function of the *inverse* image. If f_0, f_1 are arbitrary functions with sets of integers as domains, we write

$$f_0 \equiv f_1\,(\Gamma)$$

to mean that $f_1 = f_0\circ g^{-1}$ for some $g \in \Gamma$. Clearly this relation is an equivalence relation between functions, and our problem is just to show that the equivalence classes under this relation of functions in 2^N are Borel subsets of 2^N.

Let N^∞ denote the set of all *finite* sequences of integers. A finite sequence $s = \langle n_0, n_1, \ldots, n_{k-1}\rangle$ is just a function with domain $\{0, 1, \ldots, k-1\}$ where $s(i) = n_i$ for all $i < k$. We write $|s| = k$ and call $|s|$, the *length* of s. The empty sequence of length 0 is denoted by $\langle\,\rangle$. If $s = \langle n_0, n_1, \ldots, n_{k-1}\rangle$ and $t = \langle m_0, m_1, \ldots, m_{l-1}\rangle$, then

$$s^\frown t = \langle n_0, n_1, \ldots, n_{k-1}, m_0, m_1, \ldots, m_{l-1}\rangle .$$

We shall write

$$s \approx t\,(\Gamma)$$

to mean that $t = h\circ s$ for some $h \in \Gamma$. Note that $s \approx t\,(\Gamma)$ implies that $|s| = |t|$.

With the above notation we can now state a fundamental lemma which might appropriately be called "Cantor's Lemma", since it slightly generalizes Cantor's well-know method of proof.

LEMMA. *Let Γ be a closed subgroup of $N!$ and let $f_0, f_1 \in 2^N$. Suppose that R is a binary relation on N^∞ such that $\langle\,\rangle R \langle\,\rangle$. Suppose further that for all $s, t \in N^\infty$, if sRt, then we have:*

(i) $s \approx t\ (\Gamma)$;

(ii) $f_0\circ s = f_1\circ t$;

(iii) *for all $n \in N$ there exists an $m \in N$ such that $s^\frown\langle n\rangle R t^\frown\langle m\rangle$;*

(iv) *for all $m \in N$ there exists an $n \in N$ such that $s^\frown\langle n\rangle R t^\frown\langle m\rangle$.*

Then it follows that $f_0 \equiv f_1\,(\Gamma)$.

Proof. We define two infinite sequences $n_0, n_1, \ldots$, and $m_0, m_1, \ldots$ simultaneously by recursion:

$$n_{2k} = \text{the least integer } n \notin \{n_i\colon i < 2k\};$$

$$m_{2k} = \text{the least integer } m \text{ such that } \langle n_0, n_1, \ldots, n_{2k}\rangle R \langle m_0, m_1, \ldots, m_{2k-1}, m\rangle;$$

$$m_{2k+1} = \text{the least integer } m \notin \{m_i\colon i < 2k+1\};$$

$$n_{2k+1} = \text{the least integer } n \text{ such that } \langle n_0, n_1, \ldots, n_{2k}, n\rangle R \langle m_0, m_1, \ldots, m_{2k+1}\rangle .$$

Since we assume that $\langle\,\rangle R \langle\,\rangle$ holds, it is clear from (iii) and (iv) that the two infinite sequences are well-defined and that every integer occurs in each (possibly with repetitions). Now if $n_i = n_j$, then by (i) there is a permutation $h \in \Gamma$ such that $m_i = h(n_i)$ and $m_j = h(n_j)$; whence $m_i = m_j$. Similarly $m_i = m_j$ implies $n_i = n_j$. Thus the correspondence

$$n_i \leftrightarrow m_i$$

defines a permutation $g \in N!$ such that $g(n_i) = m_i$ for all $i \in N$. Condition (i) shows at once that every finite restriction of g agrees with some $h \in \Gamma$; so $g \in \Gamma$, because Γ is closed. Finally condition (ii) shows that $f_0(g^{-1}(m_i)) = f_1(m_i)$ for all $i \in N$; that is, $f_1 = f_0 \circ g^{-1}$. Thus $f_0 \equiv f_1(\Gamma)$ is proved.

Taking the hint from the Lemma, we are now ready for the construction that is the major step in the argument. We let N^k denote the space of all k-termed sequences of integers with the discrete topology. A point in $2^N \times N^k$ will be written as $\langle f, s\rangle$ where $f \in 2^N$ and $s \in N^k$. The space $2^N \times N^k$ is given the product topology, and a subset $S \subseteq 2^N \times N^k$ is called *Γ-invariant* if whenever $\langle f, s\rangle \in S$, then for all $g \in \Gamma$, $\langle f \circ g^{-1}, g \circ s\rangle \in S$. We shall identify 2^N with $2^N \times N^0 = 2^N \times \{\langle\,\rangle\}$; so that the Γ-invariant subsets of $2^N \times N^0$ in the sense just defined are exactly the subsets of 2^N invariant under the action of Γ in our previous sense. We shall write

$$\langle f_0, s\rangle \equiv \langle f_1, t\rangle \ (\Gamma\text{-Borel})$$

to mean that for all Γ-invariant Borel subsets $S \subseteq 2^N \times N^k$, where $|s| = |t| = k$,

$$\langle f_0, s\rangle \in S \quad \textit{if and only if} \quad \langle f_1, t\rangle \in S\,.$$

If $f_0 \equiv f_1(\Gamma)$, then obviously $f_0 \equiv f_1$ (Γ-Borel) (i.e. $\langle f_0, \langle\,\rangle\rangle \equiv \langle f_1, \langle\,\rangle\rangle$ (Γ-Borel).) Our main result will show that the converse also holds; however the result is stronger than just this implication.

Let f_0 now be a fixed element of 2^N. We note that with the aid of the Axiom of Choice we can choose for each k and each $s \in N^k$ a Γ-invariant Borel set $\Psi_s \subseteq 2^N \times N^k$ such that for all $t \in N^k$

$$\langle f_0, t\rangle \in \Psi_s \quad \textit{if and only if} \quad \langle f_0, s\rangle \equiv \langle f_0, t\rangle \ (\Gamma\text{-Borel})\,.$$

This is possible because for each $s \in N^k$ there are at most a denumerable number of $t \in N^k$ such that $\langle f_0, s\rangle \not\equiv \langle f_0, t\rangle$ (Γ-Borel). Let these t be $t_0, t_1, t_2, \ldots, t_i, \ldots$ For each i choose a Γ-invariant Borel set $\Theta_i \subseteq 2^N \times N^k$ such that

$$\langle f_0, s\rangle \in \Theta_i \quad \text{and} \quad \langle f_0, t_i\rangle \notin \Theta_i\,.$$

Then set

$$\Psi_s = \bigcap \{\Theta_i\colon\ i \in N\}\,.$$

Clearly Φ_s is both Γ-invariant and Borel.

Next consider the following subsets of 2^N which are numbered to correspond to the conditions of the Lemma:

$$\Phi^{(i)} = \{f \in 2^N\colon \textit{ for all } k \in N, \textit{ all } s, t \in N^k, \textit{ if } \langle f, t\rangle \in \Psi_s, \textit{ then } s \approx t(\Gamma)\};$$

$$\Phi^{(ii)} = \{f \in 2^N\colon \textit{ for all } k \in N, \textit{ all } s, t \in N^k, \textit{ if } \langle f, t\rangle \in \Psi_s, \textit{ then } f_0 \circ s = f \circ t\};$$

$$\Phi^{(iii)} = \{f \in 2^N\colon \textit{ for all } k \in N, \textit{ all } s, t \in N^k, \textit{ if } \langle f, t\rangle \in \Psi_s, \textit{ then for all } n \in N \textit{ there exists an } m \in N \textit{ such that } \langle f, t^\frown\langle m\rangle\rangle \in \Psi_{s^\frown\langle n\rangle}\};$$

$$\Phi^{(iv)} = \{f \in 2^N\colon \textit{ for all } k \in N, \textit{ all } s, t \in N^k, \textit{ if } \langle f, t\rangle \in \Psi_s, \textit{ then for all } m \in N \textit{ there exists an } n \in N \textit{ such that } \langle f, t^\frown\langle m\rangle\rangle \in \Psi_{s^\frown\langle n\rangle}\}\,.$$

The sets $\Phi^{(i)}$ and $\Phi^{(ii)}$ are obviously countable intersections of Borel sets and so are Borel. The separate Borel sets in these intersections are not themselves necessarily Γ-invariant, but the final intersections are. For suppose $f \in \Phi^{(i)}$ and $g \in \Gamma$. Then if $k \in N$ and $s, t \in N^k$ and $\langle f \circ g^{-1}, t\rangle \in \Psi_s$, we can conclude $\langle f, g^{-1} \circ t\rangle \in \Psi_s$ because Ψ_s is chosen to be Γ-invariant. But since $f \in \Phi^{(i)}$, we have $s \approx g^{-1} \circ t(\Gamma)$ and thus $s \approx t(\Gamma)$. This shows that $f \circ g^{-1} \in \Phi^{(i)}$ and that $\Phi^{(i)}$ is Γ-invariant. The argument for $\Phi^{(ii)}$ is similar. Since every Γ-invariant subset of N^k is a Borel subset of N^k, it is clear that from the basic property of the sets Ψ_s that $f_0 \in \Phi^{(i)}$. Similarly, in view of the fact that for $i < k$ the set

$$\{\langle f, s\rangle \in 2^N \times N^k\colon\ f(s(i)) = 1\}$$

is a Γ-invariant Borel subset of $2^N \times N^k$, we see that $f_0 \in \Phi^{(ii)}$.

The sets $\Phi^{(iii)}$ and $\Phi^{(iv)}$ are countable intersections of countable unions of Borel sets, and by the very form of their definitions they are seen to be Γ-invariant. Let us check that $f_0 \in \Phi^{(iii)}$. Suppose that $k \in N$, $s, t \in N^k$, and $\langle f_0, t\rangle \in \Psi_s$. Then $\langle f_0, s\rangle \equiv \langle f_0, t\rangle$ (Γ-Borel). Suppose that $n \in N$; we wish to show that there is an $m \in N$ such that

$$\langle f_0, s^\frown\langle n\rangle\rangle \equiv \langle f_0, t^\frown\langle m\rangle\rangle \ (\Gamma\text{-Borel})\,.$$

Assume not; then for each $m \in N$ we can choose a Γ-invariant Borel subset Ξ_m of $2^N \times N^{k+1}$ such that

$$\langle f_0, s^\frown\langle n\rangle\rangle \in \Xi_m \quad \textit{but} \quad \langle f_0, t^\frown\langle m\rangle\rangle \notin \Xi_m\,.$$

Let

$$\Xi = \bigcap \{\Xi_m\colon\ m \in N\}\,,$$

Now Ξ is a Γ-invariant Borel subset of $2^N \times N^{k+1}$ and its projection Ξ^* on $2^N \times N^k$ is obviously also a Γ-invariant Borel subset of $2^N \times N^k$. Since $\langle f_0, s^\frown\langle n\rangle\rangle \in \Xi$, then $\langle f_0, s\rangle \in \Xi^*$; therefore $\langle f_0, t\rangle \in \Xi^*$. Thus there must exist an $m \in N$ such that $\langle f_0, t^\frown\langle m\rangle\rangle \in \Xi$. A contradiction is reached by noticing that $\Xi \subseteq \Xi_m$.

Finally set

$$\Phi = \{f \in 2^N\colon \langle f, \langle\,\rangle\rangle \in \Psi_{\langle\,\rangle}\} \cap \Phi^{(\mathrm{i})} \cap \Phi^{(\mathrm{ii})} \cap \Phi^{(\mathrm{iii})} \cap \Phi^{(\mathrm{iv})} .$$

We see by construction that $f_0 \in \Phi$ and Φ is a Γ-invariant Borel set. Suppose $f_1 \in \Phi$; we wish to show that $f_0 \equiv f_1(\Gamma)$. To this end define the binary relation R on N^∞ so that for $s, t \in N^k$,

$$sRt \quad \textit{if and only if} \quad \langle f_1, t\rangle \in \Psi_s .$$

We see at once the definition of Φ that R has all the properties demanded in the Lemma. Hence we have $f_0 \equiv f_1(\Gamma)$ as desired; in fact, we have proved that

$$\Phi = \{f_1 \in 2^N\colon f_0 \equiv f_1(\Gamma)\} .$$

Since Φ is Borel, the Γ-orbit of f_0 in 2^N is thus shown to be a Borel set. In summary we have the

THEOREM. *If Γ is a closed subgroup of $N!$, then the relation $f_0 \equiv f_1(\Gamma)$ partitions 2^N into the minimal Γ-invariant sets each of which is a Borel subset of 2^N.*

It is worthwhile to note that the Theorem *cannot* be strengthened to read: $\{\langle f_0, f_1\rangle \in 2^N \times 2^N\colon f_0 \equiv f_1(\Gamma)\}$ is a Borel subset of $2^N \times 2^N$. This set is analytic (assuming that Γ is closed), but for certain Γ it is not a Borel set. A counterexample follows from known results in [2] as will be indicated in the next section.

If Γ_0 is a closed subgroup of $N!$, then as already explained it induces another closed subgroup Γ_1 by use of a pairing function $\pi\colon N \times N \leftrightarrow N$. Indeed π induces a homeomorphism between $2^{N\times N}$ and 2^N. Instead of using all these round-about mappings, we can simply let Γ_0 act as a group of autohomeomorphisms on $2^{N\times N}$ in the direct and obvious way. It is then seen that our Theorem above holds with 2^N replaced by $2^{N\times N}$. Of course, $f_0 \equiv f_1(\Gamma_0)$ now means that for some $g \in \Gamma_0$ we have for all $x, y \in N$:

$$f_1(\langle x, y\rangle) = f_0(\langle g^{-1}(x), g^{-1}(y)\rangle) .$$

Note also that an arbitrary denumerable set could replace N in the theorem.

It should also be clear that the Theorem holds for $2^N \times N^k$ replacing 2^N. The easiest way to see this is to go back to the proof. The Lemma should be changed by replacing condition (i) by

$$s_0{}^\frown s \approx s_1{}^\frown t\ (\Gamma) ,$$

where s_0, s_1 are given elements of N^k. Then condition (ii) must be altered to read:

$$f_0 \circ (s_0{}^\frown s) = f_1 \circ (s_1{}^\frown t) .$$

Conditions (iii) and (iv) remain the same but the conclusion of the Lemma now reads:

$$\langle f_0, s_0\rangle \equiv \langle f_1, s_1\rangle\ (\Gamma) .$$

These changes necessarily dictate corresponding changes in the definitions of $\Phi^{(\mathrm{i})}$, $\Phi^{(\mathrm{ii})}$ and Φ.

The σ-fields of invariant Borel subsets of $2^N \times N^k$ and their relationships for different k are quite interesting. The author hopes to make them the topic of a subsequent paper.

2. The solution to Kuratowski's problem. Let Q be the set of rational numbers. The product space 2^Q is indeed homeomorphic to 2^N, but in this section we prefer to replace the set N by Q since we want to emphasize the natural ordering of the rationals. In fact, if $f_0, f_1 \in 2^Q$, we shall think of these functions as representing sets of rationals, and we write

$$f_0 \cong f_1$$

to means that the corresponding ordered sets are order isomorphic. More precisely, $f_0 \cong f_1$ means that there is a one-one function h mapping $\{q \in Q\colon f_0(q) = 1\}$ onto $\{q \in Q\colon f_1(q) = 1\}$ such that if $f_0(q) = f_1(r) = 1$, then $q \leqslant r$ if and only if $h(q) \leqslant h(r)$.

The problem which Prefossor Kuratowski states in his book ([3], p. 377, Remarque) then amounts to this: *for a given $f_0 \in 2^Q$, is the subset $\{f_1 \in 2^Q\colon f_0 \cong f_1\}$ a Borel subset of 2^Q?* In case the set $\{q \in Q\colon f_0(q) = 1\}$ is well-ordered by $\leqslant$, the result is established ([3], § 26, XII, 1) by transfinite induction on the order type of the well-ordered set. Invoking the Theorem of Section 1, we shall now answer the question affirmatively for general order types.

First let $T\colon 2^Q \to 2^{Q\times Q}$ be the transformation defined by the condition that for all $f \in 2^Q$ and all $q, r \in Q$:

$$T(f)(\langle q, r\rangle) = \begin{cases} 1, & \textit{if} \quad f(q) = f(r) = 1 \textit{ and } q \leqslant r , \\ 0, & \textit{otherwise} . \end{cases}$$

It is obvious that T is continuous.

Next let I be the subset of all $f \in 2^Q$ such that *both* of the sets $\{q \in Q\colon f(q) = 1\}$ and $\{q \in Q\colon f(q) = 0\}$ are infinite. The set $2^Q \sim I$ is denumerable, and hence all of its subsets are Borel.

Finally notice that for $f_0, f_1 \in I$,

$$(*) \qquad f_0 \cong f_1 \quad \textit{if and only if} \quad T(f_0) \equiv T(f_1)\ (Q!) ,$$

where $Q!$ is the group of all permutations of Q which we think of as acting on $2^{Q\times Q}$. This remark shows that for $f_0 \in 2^Q$

$$\{f_1 \in 2^Q\colon f_0 \cong f_1\} = I \cap T^{-1}\{f \in 2^{Q\times Q}\colon T(f_0) \equiv f(Q!)\} \cup D ,$$

where D is a suitable subset of the denumerable set $2^Q \sim I$. It at once follows from our Theorem that $\{f_1 \in 2^Q\colon f_0 \cong f_1\}$ is a Borel subset of 2^Q.

As is pointed out in [2], p. 173, note, $\{\langle f_0, f_1\rangle \in 2^Q \times 2^Q\colon f_0 \cong f_1\}$ is *not* a Borel set; hence we see that the set $\{\langle f_0, f_1\rangle \in 2^{Q\times Q} \times 2^{Q\times Q}\colon f_0 \equiv f_1(Q!)\}$ is also not a Borel set. This gives the specific example mentioned after the statement of the Theorem in Section 1.

Let Π be the group of all order preserving permutations of Q. It is quite easy to see that Π is a closed subgroup of $Q!$. Thus the sets of the form $\{f_1 \in 2^Q\colon f_0 \equiv f_1(\Pi)\}$ are all Borel. Note that

$$\{f_1 \in 2^Q\colon f_0 \equiv f_1(\Pi)\} \subseteq \{f_1 \in 2^Q\colon f_0 \cong f_1\};$$

however, these new sets are much less interesting and useful than the sets in Kuratowski's problem. The example does show how easy the method is to apply.

The reader should verify that the only special property of the relation $\leqslant$ that was used in the solution to Kuratowski's problem was the fact that $q \leqslant q$ for all $q \in Q$. This is needed to verify condition $(*)$, because the function $T(f_0) \in 2^{Q\times Q}$ must uniquely determine the set $\{q \in Q\colon f_0(q) = 1\}$. This special circumstance can be completely avoided as will now be shown when we give the abstract version of the solution.

Let us return to the use of the set N rather that Q. Instead of the product spaces 2^N of $2^{N\times N}$ it will be more convenient and natural to use the power-set spaces

$$P(N) = \{A\colon A \subseteq N\}, \quad \text{and} \quad P(N\times N) = \{R\colon R \subseteq N \times N\},$$

which are given their own topology induced by the obvious one-one correspondences with the product spaces.

We shall make special use of the space

$$\mathfrak{S}_2 = \{\langle A, R\rangle\colon A \subseteq N \text{ and } R \subseteq A \times A\}.$$

$\mathfrak{S}_2$ is a *closed* subspace of $P(N)\times P(N\times N)$. We refer to the elements of $\mathfrak{S}_2$ as *binary relational structures* (on subsets of N). We shall now use the symbol $\cong$ to denote the relationship of *isomorphism* between elements of $\mathfrak{S}_2$, so that

$$\langle A, R\rangle \cong \langle B, S\rangle$$

means that there is a one-one function $h\colon A \leftrightarrow B$ such that for all $x, y \in A$

$$\langle x, y\rangle \in R \quad \text{if and only if} \quad \langle h(x), h(y)\rangle \in S.$$

The equivalence classes of elements of $\mathfrak{S}_2$ under $\cong$ may be called *isomorphism types* of (countable) relational structures. The method of solution of Kuratowski's problem now leads to the following general result.

THEOREM. *Each isomorphism type of a structure in $\mathfrak{S}_2$ is a Borel subset of $\mathfrak{S}_2$.*

For the proof we may restrict attention to the subspace $\mathfrak{J}_2$ of all $\langle A, R\rangle \in \mathfrak{S}_2$ where A is infinite. This is a Borel subset of $\mathfrak{S}_2$ because $\mathfrak{S}_2 \sim \mathfrak{J}_2$ is denumerable. Corresponding to each infinite subset of N, let $\varepsilon_A\colon N \to A$ denote the function which enumerates the elements of A in order of magnitude; thus for all $i \in N$ we have

$$\varepsilon_A(i) < \varepsilon_A(i+1),$$

and

$$A = \{\varepsilon_A(i)\colon i \in N\}.$$

Define next a mapping $U\colon \mathfrak{J}_2 \to 2^{N\times N}$ such that for all $\langle A, R\rangle \in \mathfrak{J}_2$ and all $n, m \in N$,

$$U(\langle A, R\rangle)(\langle n, m\rangle) = 1 \text{ if and only if } \langle \varepsilon_A(n), \varepsilon_A(m)\rangle \in R.$$

In the solution of Kuratowski's problem, the mapping $T\colon 2^Q \to 2^{Q\times Q}$ was actually *continuous*; for the present argument it is quite enough to check that U is a *Borel* function. That is, for each $n, m \in N$ we need to show that the set

$$\mathfrak{B}_{nm} = \{\langle A, R\rangle \in \mathfrak{J}_2\colon U(\langle A, R\rangle)(\langle n, m\rangle) = 1\}$$

is a Borel subset of $\mathfrak{J}_2$. Let $k = \max(n, m)$ and notice that

$$\mathfrak{B}_{nm} = \bigcup_{x_0 < x_1 < \ldots < x_k \in N} \ \bigcap_{y \leqslant x_k} \big\{\langle A, R\rangle \in \mathfrak{J}_2\colon \langle x_n, x_m\rangle \in R, \text{ and } y \in A \text{ if and only if } y \in \{x_0, x_1, \ldots, x_k\}\big\}.$$

This formula makes it obvious that $\mathfrak{B}_{n,m}$ is a Borel set.

To complete our argument, we have only to remark that for $\langle A, R\rangle$, $\langle B, S\rangle \in \mathfrak{J}_2$,

$$\langle A, R\rangle \cong \langle B, S\rangle \quad \text{if and only if} \quad U(\langle A, R\rangle) \equiv U(\langle B, R\rangle)(N!),$$

and the desired conclusion follows at once.

This Theorem is the first step in the *geometrisation of countable relation types* (cf. [1] and [2] for results about order types). The above result extends easily to the space $\mathfrak{S}_k$ of k-ary relational structures $\langle A, R\rangle$ where $A \subseteq N$ and $R \subseteq A^k$. We could even go on to consider more complicated structures $\langle A, R, S\rangle$ where, say, $R \subseteq A^k$ and $S \subseteq A^l$. Nevertheless, the binary case is sufficient for the purposes of this paper.

Some readers may prefer the Baire space N^N to the spaces 2^N, $2^{N\times N}$ or $\mathfrak{S}_2$. Again the basic resultf ollows quite easily. If Γ is a closed subgroup of $N!$, then each $g \in \Gamma$ induces the autohomeomorphism of N^N that sends a function $f \in N^N$ to $g \circ f \circ g^{-1}$. Now when we write

$$f_0 \equiv f_1\ (\Gamma)$$

we mean that $f_1 = g \circ f_0 \circ g^{-1}$ for some $g \in \Gamma$. Does this relation partition N^N into Borel sets? Yes, for consider the mapping $V\colon N^N \to 2^{N\times N}$ such that for each $f \in N^N$ and $n, m \in N$

$$V(f)(\langle n, m\rangle) = 1 \quad \textit{if and only if} \quad m = f(n)\,.$$

The mapping V is continuous, and further for $f_0, f_1 \in N^N$,

$$f_0 \equiv f_1(\Gamma) \ \textit{if and only if}\ V(f_0) \equiv V(f_1)\,(\Gamma)\,,$$

where the symbol $\equiv$ is used in both its old and new senses. The answer to the question is now obvious.

3. The space of closed sets. If $\mathfrak{X}$ is any compact metric space, let $2^{\mathfrak{X}}$ denote the space of all closed subset of $\mathfrak{X}$. With a very natural metric, the space $2^{\mathfrak{X}}$ is a complete metric space ([3], § 15 VII, VIII; § 29 IV). Consider the group $\mathcal{H}$ of all autohomeomorphisms of the space $\mathfrak{X}$. This group also acts in a natural way as a group of autohomeomorphisms of $2^{\mathfrak{X}}$. David Freedman asked the author this interesting question: *Is the $\mathcal{H}$-orbit of a point of $2^{\mathfrak{X}}$ always a Borel subset of $2^{\mathfrak{X}}$?* Using the methods of this paper we can give an affirmative answer for the very special cases where $\mathfrak{X}$ is (homeomorphic to) a cloed subset of Cantor's Discontinuum 2^N. The author has been unable to see the answer, however, even in the case where $\mathfrak{X}$ is the unit interval $[0, 1]$.

For illustration, let us suppose that $\mathfrak{X} = 2^N$. For closed sets $F_0, F_1 \in 2^{\mathfrak{X}}$ we shall write

$$F_0 \equiv F_1\,(\mathcal{H})$$

to mean that F_1 is the image of F_0 under some autohomeomorphism in $\mathcal{H}$. Next let $\mathfrak{B}$ be the denumerable Boolean algebra of all *clopen* (i.e. closed-open) subsets of $\mathfrak{X}$. In this case of Cantor's Discontinuum we are very fortunate that $\mathfrak{B}$ completely determines the topology of $\mathfrak{X}$; indeed $\mathfrak{B}$ is a base for the open sets. Even better there is a natural isomorphism between the autohomeomorphism group $\mathcal{H}$ and the automorphism group $\mathfrak{A}$ of $\mathfrak{B}$ (every autohomeomorphism obviously induces an automorphism of $\mathfrak{B}$; but every automorphism induces an autohomeomorpshism of $\mathfrak{X}$, because the points of $\mathfrak{X}$ correspond exactly to the prime ideals of the Boolean algebra $\mathfrak{B}$). Let the reader verify that $\mathfrak{A}$ is a closed subgroup of $\mathfrak{B}!$. (The same conclusion holds for the automorphism group of any relational structure with finitary relations or operations.)

Define next the mapping $T\colon 2^{\mathfrak{X}} \to P(\mathfrak{B})$ such that for $F \in 2^{\mathfrak{X}}$

$$T(F) = \{B \in \mathfrak{B}\colon F \cap B = 0\}\,.$$

The powerset space $P(\mathfrak{B})$ is of course given its natural topology. Inasmuch as for each $B \in \mathfrak{B}$, the set $\{F \in 2^{\mathfrak{X}}\colon F \cap B = 0\}$ is clopen in $2^{\mathfrak{X}}$, we see that T is a continuous function. There is no difficulty in showing for $F_0, F_1 \in 2^{\mathfrak{X}}$ that

$$F_0 \equiv F_1(\mathcal{H}) \quad \textit{if and only if} \quad T(F_0) \equiv T(F_1)\,(\mathfrak{A})\,.$$

Thus the fact that $F_0 \equiv F_1(\mathcal{H})$ partitions $2^{\mathfrak{X}}$ into Borel sets follows at once from our main Theorem. The same argument applies word for word to all closed subspaces of 2^N; indeed the closed subspaces of 2^N correspond exactly to the Stone spaces (prime ideal spaces) of countable Boolean algebras.

We have just applied the Theorem of Section 1; now we shall apply the Theorem of Section 2. Again let $\mathfrak{X} = 2^N$. This time write, for $F_0, F_1 \in 2^{\mathfrak{X}}$,

$$F_0 \approx F_1$$

to mean that the closed subspaces F_0 and F_1 are simply homeomorphic. The relationship $F_0 \equiv F_1(\mathcal{H})$ implies $F_0 \approx F_1$, but not conversely. For $F_0 \in 2^{\mathfrak{X}}$, we may call the set $\{F_1 \in 2^{\mathfrak{X}}\colon F_0 \approx F_1\}$ the *homeomorphism type* of F_0 in $2^{\mathfrak{X}}$. We wish to prove that the homeomorphism types of elements of $2^{\mathfrak{X}}$ are Borel subsets of $2^{\mathfrak{X}}$.

To this end, let $B_0, B_1, \ldots, B_i, \ldots$ be an enumeration of all elements of the Boolean algebra $\mathfrak{B}$ of all clopen subsets of $\mathfrak{X}$. If $F \in 2^{\mathfrak{X}}$, let

$$\mathfrak{B}_F = \{F \cap B\colon B \in \mathfrak{B}\}\,.$$

$\mathfrak{B}_F$ is the Boolean algebra of all clopen subsets of F when F is considered as a topological space in its own right. We are going to make use of the fact that $F_0 \approx F_1$ if and only if the Boolean algebras $\mathfrak{B}_{F_0}$ and $\mathfrak{B}_{F_1}$ are isomorphic.

Define two mappings $A\colon 2^{\mathfrak{X}} \to P(N)$ and $R\colon 2^{\mathfrak{X}} \to P(N \times N)$ by the formulae

$$A(F) = \{n \in N\colon F \cap B_i \neq F \cap B_n \ \textit{all}\ i < n\}\,,$$
$$R(F) = \{\langle n, m\rangle\colon n, m \in A(F) \ \textit{and}\ F \cap B_n \subseteq B_m\}$$

for all $F \in 2^{\mathfrak{X}}$. Then let $U\colon 2^{\mathfrak{X}} \to \mathfrak{S}_2$ be such that for $F \in 2^{\mathfrak{X}}$,

$$U(F) = \langle A(F), R(F)\rangle\,.$$

There is no trouble in checking that A, R, and U are continuous functions. Furthermore, $\langle A(F), R(F)\rangle$ is constructed to be isomorphic to the the inclusion relation restricted to the Boolean algebra $\mathfrak{B}_F$. Now two Boolean algebras are isomorphic if and only if their inclusion relations are isomorphic; hence, for $F_0, F_1 \in 2^{\mathfrak{X}}$,

$$F_0 \cong F_1 \quad \textit{if and only if} \quad U(F_0) \cong U(F_1)\,.$$

Thus each homeomorphism type in $2^{\mathfrak{X}}$ is the inverse image under U of an isomorphism type in $\mathfrak{S}_2$; which implies that the homeomorphism types are Borel.

The major reason why the author cannot extend his method beyond the closed subspaces of 2^N is that for more general compact metric spaces (even the unit interval $[0,1]$), he does not know how to associate a canonical countable relational structure with the space in such a way that the isomorphism type of the structure determines the homeomorphism type of the space. Maybe a new method is required for this more general question.

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On Borel measurability of orbits

by

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The aim of this note is to extend, using an alternative method, the result of D. Scott [2] concerning groups of permutations to a large class of topological groups. This has also permited us to solve in the whole generality a problem of Freedman partially solved in [2].

Let $\{F_g\}$ be a decomposition of a topological space G, i.e. for every $g \in G$ we have $g \in F_g$ and $F_{g_1} = F_{g_2}$ or $F_{g_1} \cap F_{g_2} = 0$ and F_g is closed. We say that such a decomposition is *open* if $\{g: F_g \cap U \neq 0\}$ is an open set provided U is open. The distance function of G is denoted by ϱ.

LEMMA. *Every open decomposition $\{F_g\}$ of a complete and separable metric space G has a Borel selector, i.e. there exists a Borel set $S \subseteq G$ such that* card $(S \cap F_g) = 1$ *for* $g \in G$.

Proof. Let $r_0, r_1, r_2, \ldots$ be a fixed sequence dense in G. Now we define some functions: $\varphi_n(g) = r_i$ where $i = \min\{k: \varrho(r_k, F_g) < 1/2^n$ and, if $n > 0$, then $\varrho(r_k, \varphi_{n-1}(g)) < 1/2^{n-1}\}$. In view of our assumption all sets $\{g: \varrho(r_k, F_g) < 1/2^n\}$ are open and hence φ_n are Borel functions. Moreover we have the inequality $\varrho(\varphi_n(g), \varphi_{n+1}(g)) < 1/2^n$, and all φ_n are constant on each F_g and $\varrho(\varphi_n(g), F_g) < 1/2^n$. Consequently the limit function $\varphi(g) = \lim \varphi_n(g)$ is Borel, constant on F_g, and $\varphi(g) \in F_g$, which clearly implies that the set $\{g: \varphi(g) = g\}$ is a Borel selector.

THEOREM 1. *If a topological group G admits a complete and separable metrisation and G acts transitively on a metric space X in such a way that, for some $x_0 \in X$, $x_0 g$ is a continuous function of $g \in G$, then X is a Borel space (i.e. X is a Borel set in any metrisable extension of X).*

Proof. Put $F = \{g: x_0 g = x_0\}$. F is a closed subgroup of G. Obviously the decomposition of G given by $F_g = Fg$ $(g \in G)$ is open. Let S be a Borel selector given by Lemma. The continuous mapping of S into X defined by the formula $x_0 s$ $(s \in S)$ is one-to-one and onto (S is a selector!); hence X is a Borel space (cf. [1], p. 396).