Semigroups on continua ruled by arcs

by

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Introduction. In our paper [4], we show that an acyclic Peano continuum P which has a compact set of endpoints I admits a semigroup structure with zero and unit. Later, we were able to prove that *each* acyclic Peano continuum P admits such a structure, i.e., there is a continuous mapping $m\colon P\times P\to P$ such that m is associative and there are points 0 and 1 in P such that $m(x,0)=0$ and $m(x,1)=1$ for each x in P. We made the observation that the concept of being acyclic (i.e., contains no simple closed curve) was not essential to defining a semigroup structure on a Peano continuum P (i.e., locally connected, compact, and metric) provided that certain properties were possessed by P. One of these is a natural kind of ruling of P by simple arcs. A disk, for example, can be ruled with arcs.

It is known [2] that a one dimensional compact connected semigroup with zero and unit is a generalized tree [3], i.e., arcwise connected hereditarily unicoherent and satisfies an arc convergence property: for some point 0 (necessarily a point of local connectivity), if $x_n\to x$, then the arcs $[0,x_n]\to[0,x]$. A question which remains unanswered is the following. Does a one-dimensional generalized tree admit a semigroup structure with zero and unit? Although a "ruled continuum" may be of large dimension, the one-dimensional ones include certain generalized trees, and therefore, admit the desired semigroup structure.

A special case of a ruled continuum: trees (acyclic Peano continuum). We take up this special case separately to provide motivation for the rather complicated description of a ruled continuum. The concepts were suggested by this special case and an example due to Professor Haskell Cohen. He has shown that a Cantorian swastika admits the required semigroup structure. It remains unpublished. However, Cohen's example is a ruled continuum of a special kind which furnishes a technique for overcoming an obstacle which we encountered.

Suppose that S is an acyclic Peano continuum and that I is the set of all endpoints of S with the exception of one endpoint which we denote by 0. There exists u in S such that $\sup\{d(0,x): x \in S\} = d(0,u)$ where d is a convex metric for S. Here, we refer to R. H. Bing's result [1] that each Peano continuum admits a convex metric. This is an extremely powerful tool and our techniques depend heavily upon it.

For each point x in S, there is a unique arc $[0,x]$ in S. Furthermore, for each x in S, there is an arc $[0,e]$ such that $e \in I$ and $[0,e] \supset [0,x]$. That is, S is a union of a collection $\mathfrak{A}$ of arcs $[0,e]$ for the various points e in I. If $[0,e]$ and $[0,f]$ are two elements of $\mathfrak{A}$, then $[0,e] \cup [0,f]$ is a proper subarc of each if and only if $e \neq f$. Also, if $x_n \to x$, then the unique arcs $[0,x_n] \to [0,x]$. We can assume without loss of generality that $d(0,u) = 1$. Since d is a convex metric for S, it can be shown that for each x in S, the arc $[0,x]$ is isometric to a straight line interval. These are all properties which we shall use in our description of a "ruled continuum". We shall say that S is radially convex since our arcs emanate from the point 0.

A linear ordering of I. Our next step is to linearly order the points of I. Note that I may be dense in S.

Let B denote the set of branch points of S, i.e., cutpoints of order >2. It is known that if $b \in B$, then $S-b$ has at most countably many components. For $b \in B$ let $C_i(b)$ denote those components of $S-b$ which do not contain 0, and let $I_i(b) = I \cap C_i(b)^*$ (the set of endpoints of the tree $C_i(b)^*$, excluding b). Note that the $C_i(b)$ form a null collection, i.e. for $\varepsilon > 0$, $\{C_i(b): \operatorname{diam} C_i(b) \geqslant \varepsilon\}$ is countable. We order the collection $C_i(b)$ so that (i) $\operatorname{diam} C_i(b) \geqslant \operatorname{diam} C_{i+1}(b)$ and (ii) $u \in C_1(b)$ if 0 and u do not lie in the same component of $S-b$.

Define a relation $R(b)$ on $\bigcup_i I_i(b)$ by: $(x,y) \in R(b)$ iff $x \in I_i(b)$, $y \in I_j(b)$ and $i<j$. We note that $R(b)$ is transitive and has the property that $(x,y) \in R(b) \to (y,x) \notin R(b)$. Now let $R = \bigcup_{b \in B} R(b) \cup \Delta$, where Δ is the diagonal of $I \times I$. The proof that R is an ordering is similar to that given in [4]. We next coordinatize S as follows: to each $x \in S$ is assigned $e_x \in I$ subject to the conditions $x \in [0, e_x]$, and $e_x = u$ iff $x \in [0,u]$. Now assign to each $x \in S$ two coordinates (a_x, e_x), where $a_x = d(0,x)$ and e_x is given above. Now, (a_x, e_x) uniquely represents x since S is radially convex. Let $y = (a_y, e_y)$ and define

$$xy = (a_x, e_x)(a_y, e_y) = \big(\min(a_x, a_y), \min(e_x, e_y)\big).$$

Multiplication in S is easily seen to be well defined, associative, has the zero $(0, e_x)$, and unit $(1, u)$. It remains to be shown that multiplication is continuous.

Next, we state two lemmas which are not difficult to prove for trees. We shall prove them later for "ruled continua".

LEMMA 1. *Suppose that $x_n \to x$ with $x_n = (a_{x_n}, e_{x_n})$ and $x = (a_x, e_x)$. Let t_n be a sequence of real numbers such that $0 \leqslant t_n \leqslant 1$ with $t_n \to t$ and $t_n \leqslant a_{x_n}$; then $(t_n, e_{x_n}) \to (t, e_x)$.*

LEMMA 2. *Suppose that $x_n \to x$, $y_n \to y$, and $e_x \leqslant e_y$. If either $x \in [0,y]$ or $y \in [0,x]$, then $x_n y_n \to xy$.*

We recall the following result ([4], Lemma 5): The function $f\colon S \times S \to S$ defined by: $p, q \in S \Rightarrow [0,p] \cap [0,q] = [0, f(p,q)]$ is continuous. Let $\{x_n\} \to y$. By Lemma 2 we need only consider the case $x \notin [0,y]$ and $y \notin [0,x]$. For such x, y let $f(x,y) = b$. We have that $f(x_n, x) \to f(x,x) = x$ and $f(y_n, y) \to f(y,y) = y$; hence we may choose subsequences $\{x_n\}$ and $\{y_n\}$ with $f(x_n, x) \in C_i(b)$, $f(y_n, y) \in C_j(b)$, and either $a_{x_n} \leqslant a_{y_n}$ or $a_{y_n} \leqslant a_{x_n}$ for some i, j, all n. Note that x and y are distinct from b since $x \notin [0,y]$ and $y \notin [0,x]$; hence $e_x < e_y$. It follows from the definition of the ordering that $e_{x_n} < e_{y_n}$ all n.

If $a_{x_n} \leqslant a_{y_n}$ all n, then

$$x_n y_n = (a_{x_n}, e_{x_n})(a_{y_n}, e_{y_n}) = (a_{x_n}, e_{x_n}) = x_n \to x = xy.$$

If $a_{y_n} \leqslant a_{x_n}$ for each n, then

$$x_n y_n = (a_{x_n}, e_{x_n})(a_{y_n}, e_{y_n}) = (a_{y_n}, e_{x_n}) \to (a_y, e_x) \quad \text{(by Lemma 1)}$$

and

$$(a_y, e_x) = (a_x, e_x)(a_y, e_y) = xy.$$

Thus, we have proved the following theorem.

THEOREM A. *Suppose that P is an acyclic Peano continuum. Then P admits the structure of a topological semilattice with zero and unit.*

A description of ruled continua. The fact that trees admit a semigroup structure with 0 and 1 depends upon a number of properties which we list below. It is easily seen that these are taken from trees. However, a large class of continua including disks as well as pathological continua (non-locally connected) satisfy these conditions.

Let S be a compact metric continuum, and let $0 \in S$. Suppose $I \subset S$, and suppose $\mathfrak{A} = \{[0,e]: e \in I\}$ is a collection of arcs in S satisfying the following conditions (1)-(8).

(1) $S = \bigcup \mathfrak{A}$

(2) For each $e \in I$ there is a unique arc $[0,e]$ in $\mathfrak{A}$.

(3) If $e, f \in I$ with $e \neq f$, then $[0,e] \cap [0,f]$ is a proper subarc of each.

For $x \in S$ we denote by $[0,x]$ the subarc with endpoints $0, x$ of any member of $\mathfrak{A}$ which contains x. This is seen to be well defined by (3). We say that a metric d for S is *radially convex* if for each $e \in I$ and $x, y \in [0,e]$ with $x \neq y$, $d(0,x) \neq d(0,y)$. Suppose further

(4) If $x_n \to x$, then $[0, x_n] \to [0, x]$.

(5) S has a radially convex metric d.

Using (5) and (1) it can be seen that there exists $u \in I$ such that $d(0, u)$ is maximal among $\{d(0, e)\colon e \in I\}$, and without loss of generality we may assume that $d(0, u) = 1$.

(6) For $x \in S$, let $e_x \in I$ be chosen with $x \in [0, e_x]$ and satisfying $x \in [0, u]$ iff $e_x = u$.

Suppose next that I can be ordered, with maximal element u, subject to the restrictions.

(7) If $[x_n] \to x$ and $\{y_n\} \to y$ with $e_x < e_y < u$ and $x \notin [0, y]$, $y \notin [0, x]$, then there are subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $e_{x_{n_k}} < e_{y_{n_k}}$ for each k.

Denote by C the point 0 and all points y in $[0, u]$ such that there exists $a \notin [0, u]$ and sequences $\{y_n\} \to y$, $\{a_n\} \to a$ such that (a) $e_{y_n} < e_{a_n}$, (b) $y \notin [0, a]$, and (c) $[0, y_n] \cap [0, u] \subset [0, p]$ where $y \notin [0, p]$. Either C is the set consisting of only 0 or $C = [0, k] \subset [0, u]$ for some $k \neq 0$ in $[0, u]$.

(8) If $C = [0, k]$ $(k \neq 0)$, then S can be remetrized with a radially convex metric d so that $d(0, u) = 1$, $d(0, k) = \frac{1}{2}$, and if $e \in I$ with $[0, e] \cap [0, u] = [0, p]$, then $d(e, p) \leqslant \frac{1}{2}$.

It follows from (8), that $u \notin C$.

The class $\mathfrak{C}$ of all ruled continua is that collection of arcwise connected continua satisfying conditions (1)-(8).

Examples. In view of the somewhat formidable description of $\mathfrak{C}$ it seems appropriate to give some illustrative examples.

Example 1. *The Cantorian swastika S.* This pathological continuum consists of four copies of $[0, 1] \cup \{C \times [0, 1]\}$ where C is the Cantor set. These are put together as illustrated (Fig. 1). Here, I is the collection of all non weak-cutpoints and S is given the metric inherited from the plane. It is clear that S satisfies (1)-(5). The ordering of I is given clockwise. The continuum $C = [0, k]$ describes a "discontinuity" in the ordering $\leqslant$ on I in the sense that there are elements on low-indexed lines which are close to elements on the high indexed line $[0, u]$. This type of order discontinuity occurs only at the points of C and occurs with a single multiplicity. Condition (6) can be satisfied by choosing the largest possible e_x for points x on the coordinate axes.

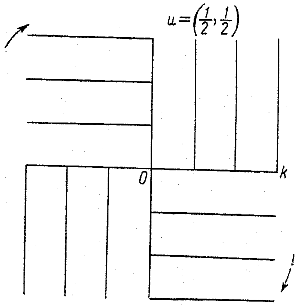

Fig. 1

Condition (7) follows easily, and expresses the continuity of the ordering away from $[0, u]$. Condition (8) is clear.

Example 2. *Two Cantor fans tangent along a segment* (Fig. 2). Here, I consists of the non weak-cutpoints. The ordering on I is clockwise. The continuum C is the point 0. We may describe this as the continuously ordered case. Conditions (1)-(8) are immediate.

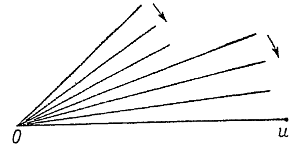

Fig. 2

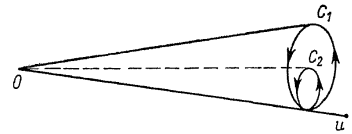

Fig. 3

Example 3. *Two cones tangent along a line segment* (Fig. 3). Here, I consists of $(C_1 \cup C_2) - k$ together with u where C_1 and C_2 are the boundaries of the cones (topological disks). The ordering is as indicated, with u the maximal element. This illustrates what we may call a *discontinuity in the ordering of multiplicity two.* An uncountable multiplicity may be obtained by filling in cones along a Cantor set.

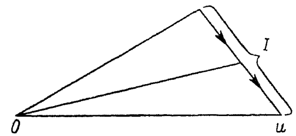

Fig. 4

Example 4. *The two cell* (Fig. 4). We may take I to be a side opposite a vertex, and $C = 0$. This is a continuous ordering, as motivated by Example 1. Some typical members of $\mathfrak{A}$ are indicated.

Here (Fig. 5), $I = (a, u] \cup (c, b] \cup [d, e) \cup [f, k)$ and $C = [0, k]$. Some typical members of $\mathfrak{A}$ are indicated.

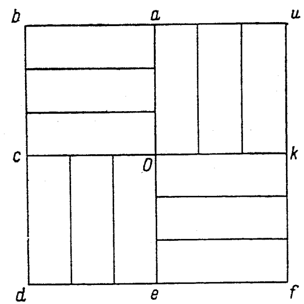

Fig. 5

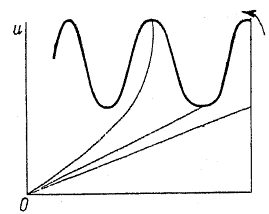

Fig. 6

Example 5 (Fig. 6). The "closed up" $\sin(1/x)$ curve, together with interior. Here I is u together with the graph of $y = \sin(1/x)$, $0 < x \leqslant 1$.

This is another example of the continuously ordered case. Some typical members of $\mathfrak{A}$ are indicated.

Definition of multiplication in a ruled continuum S. Let S be in $\mathfrak{C}$; if $C = 0$, then assign to each $x \in S$ two coordinates (a_x, e_x), where $a_x = d(0, x)$ and $e_x \in I$ is given by (6). If C is non-degenerate, we remetrize according to (8) and then assign coordinates as above. Note that (a_x, e_x) uniquely represents x since S is radially convex. Let $y = (a_y, e_y)$ and define $xy = (a_x, e_x)(a_y, e_y) = [(a_x \cdot a_y), \min(e_x, e_y)]$, where $a_x \cdot a_y = \max(a_x + b_y - 1, 0)$. We note that $\frac{1}{2} \cdot \frac{1}{2} = 0$, and $\frac{1}{2} \cdot (\frac{1}{2} + p) = p$ for $0 \leqslant p \leqslant \frac{1}{2}$. Multiplication in S is easily seen to be well defined, associative, has the zero $(0, e_x)$, and unit $(1, u)$. It remains to be shown that multiplication is continuous. We proceed to show this fact.

Proof of Lemma 1. Note that $(t_n, e_{x_n}) \in [0, x_n] \to [0, x]$ by (4). Hence there is a subsequence $(t_{n_k}, e_{x_{n_k}})$ converging to an element of $[0, x] \subset [0, e_x]$, i.e., $(t_{n_k}, e_{x_{n_k}}) \to (t, e_x)$. But each subsequence of (t_n, e_{x_n}) has the cluster point (t, e_x), and the result follows.

Proof of Lemma 2. Suppose $x \neq y$ and $x \in [0, y]$. We may assume, by choosing subsequences, that either $e_{x_n} \leqslant e_{y_n}$ for each n, or that $e_{y_n} \leqslant e_{x_n}$ for each n, and that $a_{x_n} < a_{y_n}$.

If $e_{x_n} \leqslant e_{y_n}$, then

$$x_n y_n = (a_{x_n}, e_{x_n})(a_{y_n}, e_{y_n}) = (a_{x_n}, e_{x_n}) = x_n \to x$$
$$= (a_x, e_x) = (a_x, e_x)(a_y, e_y)\, xy.$$

If $e_{y_n} \leqslant e_{x_n}$, then $x_n y_n = (a_{x_n}, e_{x_n})(a_{y_n}, e_{y_n}) = (a_{x_n}, e_{y_n})$. Since $a_{x_n} < a_{y_n}$, we conclude from Lemma 1 that $(a_{x_n}, e_{y_n}) \to (a_x, e_y)$. But $(a_x, e_y) = (a_x, e_x)$ since $x \in [0, y] \subset [0, e_y]$; hence, $x_n y_n \to (a_x, e_y) = (a_x, e_x) = (a_x, e_x)(a_y, e_y) = xy$. A similar argument applies in case $y \in [0, x]$.

Now, suppose $x = y$. Then, by choosing subsequences we have either (1) $a_{x_n} \leqslant a_{y_n}$ and $e_{x_n} \leqslant e_{y_n}$ for each n or (2) $a_{x_n} \leqslant a_{y_n}$ and $e_{y_n} \leqslant e_{x_n}$ for each n. If (1) holds, then $x_n y_n \to x = xy$ since each element of S is idempotent. If (2) holds, then $x_n y_n = (a_{x_n}, e_{y_n}) \to (a_x, e_y) = (a_y, e_y) = y = xy$. The proof is complete.

Thus, we may assume that $x_n \to x$, $y_n \to y$, $x \notin [0, y]$, and $y \notin [0, x]$. It follows that $e_x < e_y$, for otherwise $x, y \in [0, e_y]$; and hence, either $x \in [0, y]$ or $y \in [0, x]$.

Case 1. $e_x < e_y < u$. By (7) we may assume, by choosing subsequences, that $e_{x_n} < e_{y_n}$ for each n.

Then $x_n y_n = (a_{x_n}, e_{x_n})(a_{y_n}, e_{y_n}) = (a_{x_n} \cdot a_{y_n}, e_{x_n})$. Let $x_n = a_{x_n} \cdot a_{y_n}$ and note that $x_n \leqslant a_n$. By Lemma 1, we conclude that $(a_{x_n} \cdot a_{y_n}, e_{x_n}) \to (a_x \cdot a_y, e_x) = xy$.

Case 2. $e_x < e_y = u$.

I) We suppose first that $y \in C$. Now, if $y \neq 0$, then by remetrization (8) we have $C = [0, k]$ with $d(0, k) = \frac{1}{2}$. By taking subsequences, we may assume that either (1) $e_{x_n} \leqslant e_{y_n}$, all n or (2) $e_{y_n} \leqslant e_{x_n}$, all n.

If (1), then $x_n y_n = (a_{x_n} \cdot a_{y_n}, e_{x_n}) \to (a_x \cdot a_y, e_x) = xy$ (using Lemma 1). Now, suppose $e_{y_n} \leqslant e_{x_n}$ for all n, and let p denote the endpoint of $[0, p] = [0, e_x] \cap [0, u]$ (using (3)). Then $a_x = d(0, x) \leqslant d(0, e_x) \leqslant d(0, p) + d(p, e_x) \leqslant a_p + \frac{1}{2}$. Since $y \in C$, $a_y \leqslant \frac{1}{2}$; hence $a_x \cdot a_y \leqslant (a_p + \frac{1}{2}) \cdot \frac{1}{2} = p$. Therefore, $x_n y_n = (a_{x_n} \cdot a_{y_n}, e_{y_n}) \to (a_x \cdot a_y, u) = (a_x \cdot a_y, e_x) = xy$. The next to the last equality holds because $a_x \cdot a_y \leqslant p$; hence, $a_x \cdot a_y$ may be marked off along either $[0, u]$ or $[0, e_x]$ with the same final position.

Next, assume $y = 0$; then $x_n y_n = (a_{x_n} \cdot a_{y_n}, \min(e_{x_n}, e_{y_n})) \to 0$ by Lemma 1; hence, $x_n y_n \to xy$. This completes the Case 2 (I).

II) Suppose next that $y \in [0, u] - C$. We may assume, by choosing subsequences, that either $e_{x_n} \leqslant e_{y_n}$, all n, or $e_{y_n} < e_{x_n}$ all n. If $e_{x_n} \leqslant e_{y_n}$, all n, then $x_n y_n = (a_{x_n} \cdot a_{y_n}, e_{x_n}) \to (a_x \cdot a_y, e_x) = xy$ (using Lemma 1). If $e_{y_n} < e_{x_n}$, all n, we may further assume that $e_{x_n} \to a$, and note that $a \notin [0, u]$, and $y \notin [0, a]$. For if $a \in [0, u]$, then by (4), $x \in [0, a] \subset [0, u]$, a contradiction to (6) and the fact that $e_x < u$. If $[0, y_n] \cap [0, u] \to [0, y]$, then $x_n y_n \to xy$. On the other hand, if $[0, y_n] \cap [0, u] \subset [0, p]$ for each n and for some p in $[0, u]$ where $y \notin [0, p]$, then $y \in C$ contrary to the assumption that $y \in [0, u] - C$. Thus, II is completed.

We have proved the following theorem:

THEOREM B. *Suppose that S is a ruled continuum. Then S admits the structure of a topological semigroup with zero and unit.*

Remarks. Thus, each member of $\mathfrak{C}$ supports the structure of a topological semigroup with zero and unit. Note that the multiplication introduced is commutative. It was necessary to use the multiplication in $[0, 1]$ given by $a \cdot b = \max(a + b - 1, 0)$ in order to handle the first part of Case 2. In the case that C is degenerate, *any* continuous associative multiplication on $[0, 1]$ (for which 0 acts as a zero and 1 acts as a unit) can be used. In particular, we may use $a \cdot b = \min(a, b)$, and we conclude that if $S \in \mathfrak{C}$ with card $C = 1$, then S can be given the structure of a topological semilattice (i.e., idempotent commutative semigroup) with zero and unit.

We note also that if $S \in \mathfrak{C}$ then S is contractible, since an arcwise connected compact topological semigroup with zero and unit is contractible. (For $t \in [0, u]$ let $h(t, x) = tx$; then h contracts S to 0.) It is false, however, that the members of $\mathfrak{C}$ have the fixed point property.

It should be clear that a tree is a special case of a ruled continuum [satisfying conditions (1)-(8)]. Note that $C = 0$ in this case and that the ordering of I may have "discontinuities" along arcs other than $[0, u]$

but these discontinuities are nice in the sense that $C = 0$ for these arcs. It appears that our techniques would handle the cases of a finite numbers of arcs A_n like $[0, u]$ with bad discontinuities, that is, the sets C_n (like C) are nonempty and also $A_n - C_n$ is nonempty.

Finally, it is conjectured that C contains n-cells, and that C is closed under the operation of taking cones.

Question. Suppose that a compact metric continuum S contains a subset I such that (a) S satisfies Conditions (1)-(6) for a ruled continuum and (b) I admits a topological semigroup structure with zero z and unit u where I and u have the same meaning as in (1)-(6). Does S admit the structure of a topological semigroup with zero and unit u?

The continuum S above is a more general type of ruled continuum than that considered in Conditions (1)-(8).

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On the lexicographic dimension of linearly ordered sets

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1. Introduction. In earlier papers from the theory of representation of linearly ordered sets chief interest was concentrated on finding so-called universal sets. Under an $\mathfrak{m}$-universal linearly ordered set (where $\mathfrak{m}$ is cardinality) we understand a linearly ordered set which contains a subset isomorphic with every linearly ordered set of cardinality $\leqslant \mathfrak{m}$. It was been shown that such universal sets are ordinal powers (in Birkhoff's sense) in which the base in any chain containing at least two elements and the exponent is a well-ordered set. Thus Hausdorff proved [1], p. 181) that every linearly ordered set of cardinality $\leqslant \aleph_\xi$ where $\aleph_\xi$ is a regular cardinal number is isomorphic with a certain set of sequences of type ω_ξ formed from three cyphers $0, 1, 2$, and ordered lexicographically. In other words, he proved that an ordinal power of type ${}^{\omega_\xi}3$ is an $\aleph_\xi$-universal linearly ordered set if $\aleph_\xi$ is regular. Sierpiński ([2]) improved his result in the following way: An ordinal power of type ${}^{\omega_\xi}2$ is an $\aleph_\xi$-universal linearly ordered set for every cardinal number $\aleph_\xi$.

Now it is clear that the type of base cannot be reduced. Hence interest has been concentrated on the problem if it is possible to reduce the type of the exponent. It has been shown, however, that in general this type cannot be reduced. In some cases it is possible, however, to map a given linearly ordered set of cardinality $\aleph_\xi$ isomorphically onto a subset of a power with the exponent of a lower type than ω_ξ. Thus Novotný ([3]) proves that: Every $\aleph_\xi$-separable [1] linearly ordered set can be isomorphically mapped onto a subset of ordinal power of type ${}^{\omega_\xi}2$. This survey makes clear the effort to find the most economical representation, i.e. a representation in which both the base and the exponent are of the smallest possible types.

Now it is possible to pose this problem: Let the type of the base be constant. What is the smallest possible type of exponent such that the given linearly ordered set can be mapped isomorphically onto a subset of the corresponding ordinal power? This problem was partially

[1] A linearly ordered set G is called $\mathfrak{m}$-separable if it contains a dense subset H of minimal possible cardinality $\mathfrak{m}$.