Remarks on ω_μ-additive spaces

by

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§ 1. Preliminary notions. According to Sikorski [9], the set $\mathfrak{X}$ is called an *ω_μ-additive* (¹) *space* if there is defined (for every subset X) a closure operation $X \to \overline{X}$ satisfying the following axioms:

I. $\overline{\sum_{0 \leqslant \zeta < \alpha} X_\zeta} = \sum_{0 \leqslant \zeta < \alpha} \overline{X}_\zeta$, for every α-sequence of sets $\{X_\zeta\}$, $\alpha < \omega_\mu$;

II. $\overline{X} = X$ for every finite subset X;

III. $\overline{\overline{X}} = \overline{X}$.

If $\mu = 0$, the axiomatic system I-III coincides with the closure axiomatic system of Kuratowski, but for $\mu > 0$ it is stronger than that system. Similar spaces were also considered by Parovicenko [8], Cohen, Goffman [1], [2], and others. A regular ω_μ-additive space, for $\mu > 0$, must be 0-dimensional.

Let A be an ordered group (²), and if there exists a decreasing positive ω_μ-sequence $\{\varepsilon_\xi\}$, $\xi < \omega_\mu$ and $\varepsilon_\xi \in A$, satisfying the condition that for every positive element $\varepsilon \in A$ there exists an ordinal $\xi_0 < \omega_\mu$ such that $\varepsilon_\xi < \varepsilon$ for every $\xi > \xi_0$ $(\xi < \omega_\mu)$, then we say that A is *of character* ω_μ.

Suppose $\mathfrak{X}$ is a set and with every given pair of points $p, q \in \mathfrak{X}$, there is associated an element $\varrho(p,q) \in A$, where A is an ordered group of character ω_μ, such that

a) $\varrho(p,p) = 0$;

b) $\varrho(p,q) = \varrho(q,p) > 0$ for $p \neq q$;

c) $\varrho(p,q) \leqslant \varrho(p,r) + \varrho(r,q)$.

Then ϱ is called an *ω_μ-metric* on $\mathfrak{X}$, and $\mathfrak{X}$ is called an *ω_μ-metric space*.

(¹) ω_μ denotes a regular initial ordinal number.

(²) I.e. an ordered set in which with every $a, b \in A$ there is associated an element $c \in A$ called the *sum* of a and b: $c = a+b$ and such that: 1° $a+(b+c) = (a+b)+c$; 2° $a+c \leqslant b+c$, if and only if $a \leqslant b$; 3° for every $a, b \in A$ there exists an element $c \in A$ such that $a+c = b$. The symbol 0 denotes the element satisfying $a+0 = a$. An element a is *positive* if $a > 0$ (see footnote (¹¹) of [9], p. 128).

For an ω_μ-metric space $\mathfrak{X}$, we can introduce the natural topology by setting (3) $\overline{X} = E[p;\ \varrho(p, X) = 0]$, where X is an arbitrary subset of $\mathfrak{X}$ and $\varrho(p, X) = 0$ means that for every positive $\varepsilon \in A$ there exists a $p \in X$ such that $\varrho(p, p') < \varepsilon$. And, then, the sets $E[p;\ \varrho(p, p_0) < \varepsilon]$, where $p_0 \in \mathfrak{X}$ is arbitrarily given and ε is an arbitrary positive element of A, form a basis of the open sets of $\mathfrak{X}$. It can be proved that such spaces are ω_μ-additive. For this purpose it is only necessary to prove that the intersection of every α-sequence ($\alpha < \omega_\mu$) of open sets $\{G_\xi\}$ is open. Let p_0 be an arbitrary point of $\prod_\xi G_\xi$; then for each G_ξ there exists a positive element $\varepsilon_{\eta_\xi} \in A$ such that $\eta_\xi < \omega_\mu$, and if $\varrho(p, p_0) < \varepsilon_{\eta_\xi}$ then $p \in G_\xi$. Let ξ_0 be an ordinal which is greater than every η_ξ and $\xi_0 < \omega_\mu$; then for $\varrho(p, p_0) < \varepsilon_{\xi_0}$ we have $p \in \prod_\xi G_\xi$, whence p_0 is an interior point of $\prod_\xi G_\xi$; this proves that $\prod_\xi G_\xi$ is an open set.

The ω_μ-metric spaces were considered by Hausdorff [3], Cohen and Goffman [2], Sikorski [9], and others. As Sikorski had pointed out in [9], many topological theorems about separable metric spaces can be generalized to the present case, but some singularities concerning compactness and completeness may occur.

In the above, if A is the set of all real numbers and b) is replaced by

b′) $\varrho(p, q) = p(q, p)$,

then ϱ is called a *pseudo-metric* on $\mathfrak{X}$. Let us call an *almost-metric space* each set $\mathfrak{X}$ with a family $P = \{\varrho_\xi\}$ of pseudo-metrics and satisfying

d) If for every $\varrho_\xi \in P$ $\varrho_\xi(p, q) = 0$, then $p = q$.

Moreover, we can assume that, for P, the following statement holds:

e) For every $\varrho_{\xi_1}, \varrho_{\xi_2} \in P$ there exist $\varrho_\xi \in P$ such that $\varrho_\xi(x, y) \geqslant \max\{\varrho_{\xi_1}(x, y);\ \varrho_{\xi_2}(x, y)\}$.

If the power of P is equal to m, $\mathfrak{X}$ is called an *m-almost metric space*. One can introduce the topology for $\mathfrak{X}$ by setting

$$\overline{X} = \prod_{\varrho_\xi \in P} E[p;\ \varrho_\xi(p, X) = 0],$$

where $X \subset \mathfrak{X}$, i.e. the family of sets $E[p;\ \varrho_\xi(p, p_0) < d]$, where $p_0 \in \mathfrak{X}$, $d > 0$, $\varrho_\xi \in P$, is a basis for this topology.

The m-almost-metric spaces were introduced and investigated by Mrówka [5-7]. In fact, such spaces are equivalent in the sense of uniform and topological structure to the Hausdorff uniform spaces (for the terminology of Hausdorff uniform spaces, see [4], p. 180) with the basis of power m, i.e. a uniformity has a basis of the power m if and only if it is generated by a family of pseudo-metrics of power m.

(3) The symbol $E[p; \varphi(p)]$ denotes the set of points $p \in \mathfrak{X}$ which satisfies the condition φ, i.e. the proposition $\varphi(p)$ is true.

For brevity, in the following sections, the topological space $\mathfrak{X}$ is said to be a *$(U)_m$-space* if its topology can be derived from a uniformity with a basis of power m, where m is supposed to be the smallest possible; the topological space $\mathfrak{X}$ is said to be *ω_μ-metrisable*, if it is possible to define an ω_μ-metric ϱ such that the topology induced by ϱ agrees with the original topology of $\mathfrak{X}$. By the *basis* of $\mathfrak{X}$ we always mean the open basis.

In the following two theorems, given by Mrówka, the original "$\mathfrak{X}$ is an m-almost-metrisable space" is replaced by "$\mathfrak{X}$ is a $(U)_m$-space".

THEOREM M$_1$. *A normal space $\mathfrak{X}$ is a $(U)_m$-space if and only if it has an m-basis (i.e. this basis is formed by the union of at most m locally finite systems).*

THEOREM M$_2$. *A completely regular space $\mathfrak{X}$ is a $(U)_m$-space if and only if there exist a basis $\{U\}$ and a family $\{f_U\}$ of continuous functions such that $0 \leqslant f_U(p) \leqslant 1$; $f_U(p) \equiv 1$ for $p \in U$ and the sets $E[p; f_U(p) > 0]$ can be divided into a family of locally finite (discrete) systems of power at most m.*

The present paper is divided into the following four parts. In § 2 necessary and sufficient conditions for a $(U)_m$-space to be ω_μ-additive are obtained.

In § 3, we study the relationship between $(U)_m$-spaces and ω_μ-metrisable spaces.

In § 4, some necessary and sufficient conditions for an ω_μ-additive space to be ω_μ-metrisable are obtained. The well-known Nagata-Smirnov metrisation theorem is contained in one of our theorems. Finally, some remarks on compactness and bicompactness are also made in § 5.

§ 2. The necessary and sufficient conditions for a $(U)_m$-space to be ω_μ-additive. We now prove

PROPOSITION 1. *If $\mathfrak{X}$ is an ω_μ-additive space, then, unless $\mathfrak{X}$ is discrete or $\mu = 0$ (while every topological space is ω_0-additive), its topology cannot be derived from a uniformity with the basis of power $< \aleph_\mu$.*

Proof. Let $\mathfrak{X}$ be given as above. For our purpose it is only necessary to prove that its topology cannot be derived by a family of pseudo-metrics (in the sense of § 1) of power $< \aleph_\mu$. Suppose it is not the case, i.e. its topology can be derived by a family of pseudo-metrics $P = \{\varrho_\xi\}$ of power $< \aleph_\mu$. Let p_0 be an arbitrarily given point of $\mathfrak{X}$. Then, if $\mu > 0$, by

$$E[p;\ \varrho_\xi(p, p_0) = 0] = \prod_{n=1}^{\infty} E\left[p;\ \varrho_\xi(p, p_0) < \frac{1}{n}\right],$$

we know that the set $E[p;\ \varrho_\xi(p, p_0) = 0]$ is open, and by

$$\prod_{\varrho_\xi \in P} E[p;\ \varrho_\xi(p, p_0) = 0] = \{p_0\}$$

we know that the set $\{p_0\}$ is open, and hence, if $\mu > 0$, $\mathfrak{X}$ must be discrete, which contradicts the hypothesis of our proposition.

Thus, a $(U)_m$-space is ω_μ-additive (for $\mu > 0$) only when $m \geqslant \aleph_\mu$ (4). It is natural to ask under what conditions the $(U)_m$-space $\mathfrak{X}$ would be ω_μ-additive, where $m \geqslant \aleph_\mu$.

Since every topological space (and hence every uniform space) is ω_0-additive, in the rest of this section $\mu > 0$ is assumed.

Let $\mathfrak{X}$ be a set and $P = \{\varrho_\xi\}$ a family of pseudo-metrics on $\mathfrak{X}$. Including in P the functions $d\varrho_\xi$, $\max\{\varrho_{\xi_1}, \ldots, \varrho_{\xi_n}\}$ (where d is an arbitrary positive rational number, n a natural number and $\varrho_{\xi_i}, \varrho_\xi \in P$) we get a new family P^*, which is called the *completion* of P; for P^* we have a), b′), c), d), e) and the following:

f) For every positive rational d and $\varrho_\xi \in P^*$, $d\varrho_\xi \in P^*$.

DEFINITION 1. Let $\mathfrak{X}$, P be given as above. If, for every subfamily $P' \subseteq P$, $\overline{\overline{P'}} < m$ and every point $p_0 \in \mathfrak{X}$, there exist $\varrho_\xi \in P$ and a neighbourhood $V(p_0)$ of p_0 such that $\varrho_\xi(p, q) \geqslant \varrho_\eta(p, q)$ holds for $\varrho_\eta \in P'$ and $p, q \in V(p_0)$, then we say that P is an *m-locally direct family.*

THEOREM 1. *For a $(U)_m$-space $\mathfrak{X}$ to be ω_μ-additive (where $\mu > 0$), it is necessary and sufficient that $m \geqslant \aleph_\mu$ and its topology can be derived from a uniformity which is generated by a family of pseudo-metrics $P = \{\varrho^\xi\}$ such that the completion P^* is an $\aleph_\mu$-locally direct family.*

Proof. Sufficiency. Let $\{G_\xi\}$, $\xi < a$ $(a < \omega_\mu)$ be an a-sequence of open sets, p_0 an arbitrary point of $\prod_\xi G_\xi$. Then there exist a positive number d (by e) one can assume $d = 1$) and a subfamily $\{\varrho_{\eta_\xi}\} \subseteq P^*$ such that

$$E[p;\ \varrho_{\eta_\xi}(p, p_0) < 1] \subseteq G_\xi \quad \text{for} \quad 0 \leqslant \xi < a.$$

By the $\aleph_\mu$-locally directness of P^*, there exist $\varrho_\eta \in P^*$ and a neighbourhood $V(p_0)$ such that $\varrho_\eta \geqslant \varrho_{\eta_\xi}$ $(0 \leqslant \xi < a)$ holds in $V(p_0)$. Then

$$V(p_0)\cdot E[p;\ \varrho_\eta(p, p_0) < 1] \subseteq V(p_0)\cdot \prod_{0 \leqslant \xi < a} E[p;\ \varrho_{\eta_\xi}(p, p_0) < 1]$$

$$\subseteq V(p_0)\cdot \prod_{0 \leqslant \xi < a} G_\xi \subseteq \prod_{0 \leqslant \xi < a} G_\xi;$$

this proves that p_0 is an interior point of $\prod_\xi G_\xi$, whence $\prod_\xi G_\xi$ is an open set.

(4) Throughout the rest of the paper, topological spaces always mean non-discrete topological spaces.

Necessity. Let the uniformity of the $(U)_m$-space $\mathfrak{X}$ be generated by a family P of pseudo-metrics; P^* is the completion of P. For an arbitrarily given $P' \subseteq P^*$ and if $\overline{\overline{P'}} < \aleph_\mu$, let p_0 be an arbitrary point of $\mathfrak{X}$. Then the set

$$V(p_0) = \prod_{\varrho_{\eta_\xi} \in P'} E[p;\ \varrho_{\eta_\xi}(p, p_0) = 0] = \prod_{n=1}^{\infty} \prod_{\varrho_{\eta_\xi} \in P'} E\left[p;\ \varrho_{\eta_\xi}(p, p_0) < \frac{1}{n}\right]$$

is an open set containing p_0, i.e. $V(p_0)$ is a neighbourhood of p_0 satisfying the condition that for every pair $p, q \in V(p_0)$ and every $\varrho_\eta \in P^*$ we have $\varrho_\eta(p, q) \geqslant \varrho_{\eta_\xi}(p, q) = 0$. Therefore P^* is an $\aleph_\mu$-locally direct family.

DEFINITION 2. Let $\mathfrak{X}$, P be given as in def. 1; if for every subfamily $P' \subseteq P$ with $\overline{\overline{P'}} < m$ and every point $p_0 \in \mathfrak{X}$, there exists a neighbourhood $V(p_0)$ of p_0 such that $\varrho_{\eta_\xi}(p, q) \equiv 0$ for $\varrho_{\eta_\xi} \in P'$ and $p, q \in V(p_0)$, then we say that P is an *m-locally zero family.*

A more convenient test to see if a $(U)_m$-space $\mathfrak{X}$ is ω_μ-additive is the following

THEOREM 2. *For a $(U)_m$-space $\mathfrak{X}$ to be ω_μ-additive, it is necessary and sufficient that $m \geqslant \aleph_\mu$ and its topology can be derived from a uniformity which is generated by an $\aleph_\mu$-locally zero family of pseudo-metrics.*

Proof. Sufficiency. We observe that the completion P^* is also an $\aleph_\mu$-locally zero family; the sufficient part is a corollary of Theorem 1.

Necessity. The proof is completely the same as the proof of the necessary part of Theorem 1.

§ 3. The relationship between ω_μ-metrisable spaces and $(U)_m$-spaces. We now prove

PROPOSITION 2. *If $\mathfrak{X}$ is an ω_μ-metrisable space and $\mathfrak{F}$ is an open covering of $\mathfrak{X}$, then there exists an $\aleph_\mu$-discrete refinement $\mathfrak{F}'$ of $\mathfrak{F}$ (i.e. $\mathfrak{F}'$ is the union of $\aleph_\mu$ families of discrete open sets, $\mathfrak{F}'$ is a covering of $\mathfrak{X}$ and for every $U \in \mathfrak{F}'$ there is a $V \in \mathfrak{F}$ such that $U \subseteq V$). Moreover, for $\mu > 0$ we can require that $\mathfrak{F}'$ be formed by sets both open and closed.*

Proof. The first part is essentially the same as in the case of $\mu = 0$. Order the elements of $\mathfrak{F}$ by the relation $<$. For each $U \in \mathfrak{F}$ let (5) $U_\xi = E[p;\ \varrho(p, \mathfrak{X} - U) > \varepsilon_\xi]$; then, $\varrho(U_\xi, \mathfrak{X} - U_{\xi+1}) > \varepsilon_\xi - \varepsilon_{\xi+1}$. We put $U'_\xi = U_\xi - \Sigma\{V_{\xi+1};\ V \in \mathfrak{F}$ and $V < U\}$; since one of the relations $U < V$ and $V < U$ must hold, therefore if U, V are distinct elements of $\mathfrak{F}$, we have $\varrho(U'_\xi, V'_\xi) > \varepsilon_\xi - \varepsilon_{\xi+1}$. Choose two elements $\varepsilon'' < \varepsilon'$ of A such that $2\varepsilon' = \varepsilon' + \varepsilon' < \varepsilon_\xi - \varepsilon_{\xi+1}$ (to verify this possibility is easy), and define

(5) The meaning of A and ε_ξ has been given in § 1.

$$U_\xi^* = E[p;\ \varrho(p, U'_\xi) < \varepsilon'], \quad V_\xi^* = E[p;\ \varrho(p, V'_\xi) < \varepsilon'],$$
$$U_\xi^{**} = E[p;\ \varrho(p, U'_\xi) \leqslant \varepsilon''], \quad V_\xi^{**} = E[p;\ \varrho(p, V'_\xi) \leqslant \varepsilon''].$$

Then U_ξ^* (and V_ξ^*) is open and U_ξ^{**} (V_ξ^{**}) is closed, $U_\xi^{**} \subseteq U_\xi^*$. If $\mu > 0$, then there exists an open-closed set [9] $\widetilde{U}_\xi$ such that $U_\xi^* \subseteq \widetilde{U}_\xi \subseteq U_\xi^{**}$. In the following we prove that the family $\{\widetilde{U}_\xi^*\}$ (or $\{U_\xi\}$, if $\mu > 0$), where $\xi < \omega_\mu$ and $U \in \mathfrak{F}$, is required.

Firstly, the sets U_ξ^* (or $\widetilde{U}_\xi$, if $\mu > 0$) for fixed ξ are discrete. To prove this, let $U \neq V$, $U, V \in \mathfrak{F}$ and $p \in U_\xi^*$, $q \in V_\xi^*$ be arbitrarily given; then we have $\varrho(p, U'_\xi) < \varepsilon'$ and $\varrho(q, V'_\xi) < \varepsilon'$. From $\varrho(U'_\xi, V'_\xi) < \varepsilon_\xi - \varepsilon_{\xi+1}$ it follows that $\varrho(p, q) > (\varepsilon_\xi - \varepsilon_{\xi+1}) - 2\varepsilon' > 0$, i.e. $p \neq q$. Therefore $U_\xi^* \cdot V_\xi^* = \emptyset$. Secondly, let $p \in \mathfrak{X}$ be an arbitrary point and let U be the first member of $\mathfrak{F}$ to which p belongs. Then surely $p \in U_\xi^{**}$ for some ξ, that is $p \in U'_\xi$ (for $\mu > 0$, $p \in \widetilde{U}_\xi$). Finally, it is evident that $U_\xi^* \subseteq U$ (and $\widetilde{U}_\xi \subseteq U$ for $\mu > 0$). Hence the family $\{U_\xi^*\}$, or $\{\widetilde{U}_\xi^*\}$ if $\mu > 0$, is the required family.

THEOREM 3. *Every ω_μ-metrisable space $\mathfrak{X}$ is a $(U)_{\aleph_\mu}$-space.*

Proof. By proposition 2, Theorem 3 follows from Theorem M_1 immediately. (By theorem (viii) of [9], $\mathfrak{X}$ is a normal space).

It will be observed that Theorem 3 can be proved in a direct way.

THEOREM 4. *Every ω_μ-additive $(U)_{\aleph_\mu}$-space is ω_μ-metrisable.*

Proof. Let $\mathfrak{X}$ be an ω_μ-additive $(U)_{\aleph_\mu}$-space. Then its topology can be derived from a family $P = \{\varrho_\xi\}$ of pseudo-metrics of power $\aleph_\mu$.

If $\mu = 0$, then $P = \{\varrho_n\}$. Put

$$\varrho(p, q) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \varrho_n(p, q)\};$$

then ϱ is a metric on $\mathfrak{X}$, whence $\mathfrak{X}$ is ω_0-metrisable. We now prove the case of $\mu > 0$ as follows. Let A be the set of all ω_μ-sequences of real numbers. For every pair of elements $a, b \in A$, where

$$a = \{a_0, a_1, \ldots, a_\xi, \ldots\},$$
$$b = \{b_0, b_1, \ldots, b_\xi, \ldots\},$$

$\xi < \omega_\mu$, if there exists $\xi_0 < \omega_\mu$ such that $a_\xi = b_\xi$ for $\xi < \xi_0$ but $a_{\xi_0} < b_{\xi_0}$, then we say that a is smaller than b, $a < b$. The sum and the difference are defined by $a \pm b = \{a_0 \pm b_0, \ldots, a_\xi \pm b_\xi, \ldots\}$.

It is not difficult to verify that A is an ordered group of character ω_μ: to see this we only take $\varepsilon_\xi = \{a_0^\xi, a_1^\xi, \ldots, a_\eta^\xi, \ldots\}$, where $a_\eta^\xi = 0$ for $\eta < \xi$ and $a_\eta^\xi = 1$ for $\eta \geqslant \xi$ ($\eta < \omega_\mu$).

If $P = \{\varrho_\xi\}$, $\xi < \omega_\mu$, we put

$$\varrho(p, q) = \{\varrho_0(p, q), \ldots, \varrho_\xi(p, q), \ldots\};$$

$\mathfrak{X}$ is now an ω_μ-metric space, and we have to prove that its topology T^2 agrees with the original topology T^1. For brevity, by T^1 (or T^2)—open, we always mean a set which is open with respect to the topology T^1 (or T^2); the same applies to "T^1 (or T^2)—closed".

(I) The set $E[p;\ \varrho(p, p_0) < \varepsilon]$ is T^1—open for $\varepsilon \in A$, where $p_0 \in \mathfrak{X}$ is arbitrarily given.

In fact, if $\varepsilon = \{a_0, a_1, \ldots, a_\xi, \ldots\}$ then (I) follows from the equations

$$E[p;\ \varrho(p, p_0) < \varepsilon] = \sum_{0 \leqslant \eta < \omega_\mu} \prod_{0 \leqslant \xi < \eta} E[p;\ \varrho_\xi(p, p_0) = a_\xi] \cdot E[q;\ \varrho_\eta(q, p_0) < a_\eta],$$

and

$$E[p;\ \varrho_\xi(p, p_0) = a_\xi] = \prod_{n=1}^{\infty} E\left[p;\ \varrho_\xi(p, p_0) > a_\xi - \frac{1}{n}\right] \cdot E\left[p;\ \varrho_\xi(p, p_0) < a + \frac{1}{n}\right].$$

(II) The sets $E[p;\ \varrho(p, p_0) < a_\eta]$ are T^2-open, where $p_0 \in \mathfrak{X}$, a_η is a positive real number $\eta < \omega_\mu$ and $\varrho_\eta \in P$.

From

$$E[p;\ \varrho_\eta(p, p_0) < a_\eta] = \sum_{\{a_\xi\}_{\xi<\eta}} \prod_{0 \leqslant \xi < \eta} E[p;\ \varrho_\xi(p, p_0) = a_\xi] \cdot E[p;\ \varrho_\eta(p, p_0) < a_\eta]$$

it is evident that (II) follows from

(II′) For every $\eta < \omega_\mu$ and an arbitrary η-sequence $\{a_\xi\}$, $\xi < \eta$, the sets

$$(\Delta)_\eta = \prod_{0 \leqslant \xi < \eta} E[p;\ \varrho_\xi(p, p_0) = a_\xi] \cdot E[p;\ \varrho_\eta(p, p_0) < a_\eta]$$

and

$$(\Delta)'_\eta = \prod_{0 \leqslant \xi < \eta} E[p;\ \varrho_\xi(p, p_0) = a_\xi] \cdot E[p;\ \varrho_\eta(p, p_0) > a_\eta]$$

are both T^2-open and T^2-closed sets.

We prove it by the following two steps:

(a) The sets $E[p;\ \varrho_0(p, p_0) < a_0]$ and $E[p;\ \varrho_0(p, p_0) > a_0]$ are both T^2-open-closed sets.

In fact, let $\varepsilon^{(n)} = \left\{a_0 - \frac{1}{n}, a_1, \ldots, a_\xi, \ldots\right\}$, where $\xi < \omega_\mu$, and $a_0, \ldots, a_\xi, \ldots$ are fixed as n varies; then

$$E[p;\ \varrho_0(p, p_0) < a_0] = \sum_{n=1}^{\infty} E[p;\ \varrho(p, p_0) < \varepsilon^{(n)}],$$

which implies the T^2-openness of the set $E[p;\ \varrho_0(p,p_0)<a_0]$. (Similarly, the T^2-openness of $E[p;\ \varrho_0(p,p_0)>a_0]$ can be proved.) To prove that they are T^2-closed it suffices to take the complements, for example

$$E[p;\ \varrho_0(p,p_0)<a_0]=\mathfrak{X}-\prod_{n=1}^{\infty}E\left[p;\ \varrho_0(p,p_0)>a_0-\frac{1}{n}\right].$$

(b) By the principle of transfinite induction, assume that (II′) holds for all ordinals $\xi<a$, to prove the case of a $(a<\omega_\mu)$.

(i) If a is an isolated ordinal, let $\varepsilon^{(n)}=\{a_\xi^{(n)}\}$, where $\xi<\omega_\mu$ and $a_\xi^{(n)}=a_\xi$ for $\xi\neq a$ and $a_a^{(n)}=a_a-\dfrac{1}{n}$; then

$$E[p;\ \varrho(p,p_0)<\varepsilon^{(n)}]$$
$$=\sum_{0\leqslant\eta<\omega_\mu}\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi^{(n)}]\cdot E[p;\ \varrho_\eta(p,p_0)<a_\eta^{(n)}].$$

Subtracting from the above set the following T^2-closed set (hypothesis of (b))

$$\sum_{0\leqslant\eta<a}\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi^{(n)}]\cdot E[p;\ \varrho_\eta(p,p_0)<a_\eta^{(n)}]$$

one obtains the following T^2-open set:

$$\sum_{a\leqslant\eta<\omega_\mu}\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi^{(n)}]\cdot E[p;\ \varrho_\eta(p,p_0)<a_\eta^{(n)}];$$

its union with respect to n, $a_{a+1},\ldots,a_\xi,\ldots$ $(\xi<\omega_\mu)$, is the T^2-open set $(\Delta)_a$. In a similar way one can prove that $(\Delta)'_a$ is T^2-open.

By taking the complements we can prove that the sets $(\Delta)_a$ and $(\Delta)'_a$ are T^2-closed, e.g. from

$$\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi]\cdot E[p;\ \varrho_\eta(p,p_0)\geqslant a_\eta]$$
$$=\prod_{n=1}^{\infty}\prod_{0\leqslant\xi<\eta}E\left[p;\ \varrho_\xi(p,p_0)>a_\xi-\frac{1}{n}\right]\cdot E\left[p;\ \varrho_\xi(p,p_0)<a_\xi+\frac{1}{n}\right]\cdot$$
$$\cdot E\left[p;\ \varrho_\eta(p,p_0)>a_\eta-\frac{1}{n}\right];$$

and

$$\mathfrak{X}-(\Delta)_a=\sum_{0\leqslant\xi<a}E[p;\ \varrho_\xi(p,p_0)>a_\xi]$$
$$+\sum_{0\leqslant\xi<a}E[p;\ \varrho_\xi(p,p_0)<a_\xi]+\prod_{0\leqslant\xi<a}E[p;\ \varrho_\xi(p,p_0)=a_\xi]\cdot E[p;\ \varrho_a(p,p_0)\geqslant a_a],$$

one can prove that $(\Delta)_a$ is T^2-closed.

(ii) If a is a limit ordinal, then from the following equation

$$\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi]\cdot E[p;\ \varrho_\eta(p,p_0)\leqslant a_\eta]$$
$$=\prod_{n=1}^{\infty}\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi]\cdot E\left[p;\ \varrho_\eta(p,p_0)<a_\eta+\frac{1}{n}\right],$$

and by the hypothesis of (b), we know that, for each $\eta<a$, the set

$$\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi]\cdot E[p;\ \varrho_\eta(p,p_0)\leqslant a_\eta]$$

is a T^2-open set. By intersecting the above sets with respect to $\eta<a$ we obtain the following T^2-open set:

$$\prod_{0\leqslant\xi<a}E[p;\ \varrho_\xi(p,p_0)=a_\xi].$$

The intersection of the above set with the T^2-open set $E[p;\ \varrho(p,p_0)<\varepsilon^{(n)}]$, where $\varepsilon^{(n)}$ assumes the same meaning as in (i), is the following T^2-open set:

$$\sum_{a\leqslant\eta<\omega_\mu}\prod_{0\leqslant\xi<\eta}E[p;\ \varrho_\xi(p,p_0)=a_\xi^{(n)}]\cdot E[p;\ \varrho_\eta(p,p_0)<a_\eta^{(n)}];$$

by making a union of the above sets with respect to n, $a_{a+1},\ldots$, the T^2-open set $(\Delta)_a$ is obtained. In a similar way one can prove that $(\Delta)'_a$ is T^2-open.

The proof that $(\Delta)_a$ and $(\Delta)'_a$ are T^2-closed sets is completely the same as in case (i), whence it is omitted here.

From Theorems 3 and 4 we have

THEOREM 5. *ω_μ-metrisable spaces and ω_μ-additive $(U)_{\aleph_\mu}$ spaces are identical, in particular ω_0-metrisable spaces and ordinary metrisable spaces are identical.*

§ 4. ω_μ-metrisation theorems (6). We prove

THEOREM 6. *For a regular ω_μ-additive space to be ω_μ-metrisable, it is necessary and sufficient that there exist an $\aleph_\mu$-basis.*

Let us recall that the family $\mathfrak{F}$ of open sets is called an *$\aleph_\mu$-basis* of the topological space if $\mathfrak{F}$ is a basis and $\mathfrak{F}$ can be written as $\mathfrak{F}=\sum_{0\leqslant a<\omega_\mu}\mathfrak{F}_a$, where $\mathfrak{F}_a$ are locally finite systems of open sets.

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(6) Let us observe that in our metrisation theorems the notion of ordered algebraic field (see [9], p. 129) W_μ is not used.

Proof of Theorem 6. As the necessary part has been contained in the proof of proposition 2, we need to prove the sufficient part only.

From Theorems 5 and M_1, we need only to prove that $\mathfrak{X}$ is a normal space (this is an improvement of theorem (vii) of [9]).

In fact, let F_1 and F_2 be disjointed closed sets; since $\mathfrak{X}$ is regular, for every pair of points $p \in F_1$, $q \in F_2$ there exist neighbourhoods $U_p \in \mathfrak{F}_{\xi(p)}$ and $U_q \in \mathfrak{F}_{\xi(q)}$ such that $\overline{U}_p \cdot F_2 = 0$ and $\overline{U}_q \cdot F_1 = 0$. Let $U_\eta^{(1)} = \sum_{\xi(p)=\eta} U_p$ and $U_\eta^{(2)} = \sum_{\xi(q)=\eta} U_q$ ($p \in F_1$ and $q \in F_2$); then $\overline{U}_\eta^{(1)} = \sum_{\xi(p)=\eta} \overline{U}_p$ and $\overline{U}_\eta^{(2)} = \sum_{\xi(q)=\eta} \overline{U}_q$ since $\mathfrak{F}_\eta$ is a locally finite family.

Put

$$U_\xi^* = U_\xi^{(1)} - \sum_{\eta<\xi} \overline{U}_\eta^{(2)}, \qquad U_\xi^{**} = U_\xi^{(2)} - \sum_{\eta<\xi} \overline{U}_\eta^{(1)},$$

$$U^* = \sum_{0\leqslant\xi<\omega_\mu} U_\xi^*, \qquad U^{**} = \sum_{0\leqslant\xi<\omega_\mu} U_\xi^{**}.$$

The sets U^* and U^{**} are disjointed open sets containing F_1 and F_2 respectively. Thus $\mathfrak{X}$ is normal. Therefore, theorem 6 is proved.

COROLLARY 1 (R. Sikorski [9]). *If $\mathfrak{X}$ is an ω_μ-additive normal space with a basis of power $\aleph_\mu$, then $\mathfrak{X}$ is ω_μ-metrisable.*

COROLLARY 2 (Nagata-Smirnov). *For a regular space to be metrisable, it is necessary and sufficient that there exist an $\aleph_0$-basis.*

THEOREM 7. *For $\mu > 0$, for an ω_μ-additive space to be ω_μ-metrisable it is necessary and sufficient that there exist an $\aleph_\mu$-basis consisting of sets both open and closed.*

Proof. Necessity. It is contained in the proof of proposition 2.

Sufficiency (7). Let $\mathfrak{F}$ be an $\aleph_\mu$-basis of $\mathfrak{X}$ and let $\mathfrak{F} = \sum_{0\leqslant\xi<\omega_\mu} \mathfrak{F}_\xi$ where $\mathfrak{F}_\xi$ are locally finite (discrete) systems consisting of open-closed sets (Proposition 2). For $U \in \mathfrak{F}_\xi$ define

$$f_U(p) = \begin{cases} 1 & \text{for} \quad p \in U, \\ 0 & \text{for} \quad p \notin U. \end{cases}$$

The family $P = \{\max(\varrho_{\xi_1}, \ldots, \varrho_{\xi_n})\}$ of functions,

$$\varrho_{\xi_i}(p, q) = \sum_{U \in \mathfrak{F}_{\xi_i}} |f_U(p) - f_U(q)|,$$

makes $\mathfrak{X}$ as $\aleph_\mu$-almost metric space its topology is the same as the original. In fact, the ϱ_ξ are continuous functions by the local finiteness of $\mathfrak{F}_\xi$. Conversely, for an arbitrarily given open set G and $p_0 \in G$, one

(7) The proof given here is not based on Theorem M_1.

can find $U \in \mathfrak{F}_\xi$ (for some ξ) such that $p_0 \in U \subseteq G$, whence $\varrho_\xi(p_0, \mathfrak{X} - U) \geqslant 1$ and therefore $E[p; \varrho_\xi(p, p_0) < 1] \subseteq U \subseteq G$. Thus, $\mathfrak{X}$ is an ω_μ-additive $(U)_{\aleph_\mu}$-space, and theorem 7 follows from Th. 4 (or Th. 5) immediately.

From theorem 7 we can derive some results which are closely related to Theorem M_2.

COROLLARY 1. *For $\mu > 0$, for an ω_μ-additive space $\mathfrak{X}$ to be ω_μ-metrisable it is necessary and sufficient that there exist a collection of families of continuous functions $P = \{P_\xi\}$ and $P_\xi = \{f_\eta^\xi\}$, where $\xi < \omega_\mu$, such that the families of sets $E[p; f_\eta^\xi(p) > 0]$ for fixed ξ are locally finite (discrete) systems, and the family of sets $E[p; f_\eta^\xi(p) > 1]$ (where $\xi < \omega_\mu$ and $f_\eta^\xi \in P_\xi$) is a basis of $\mathfrak{X}$.*

Proof. Necessity. It suffices to put in theorem 7

$$f_U(p) = \begin{cases} 2 & \text{for} \quad p \in U, \\ 0 & \text{for} \quad p \notin U, \end{cases} \qquad \text{for every } U \in \mathfrak{F}_\xi, \quad \xi < \omega_\mu.$$

Sufficiency. The families of sets $E[p; f_\eta^\xi(p) > 1]$, for fixed ξ, are locally finite systems, consisting of sets both open and closed:

$$E[p; f_\eta^\xi(p) > 1] = \sum_{n=1}^{\infty} E\Big[[p; f_\eta^\xi(p) \geqslant 1 + \frac{1}{n}\Big].$$

COROLLARY 2. *For an ω_μ-additive space to be ω_μ-metrisable, it is necessary and sufficient that there exist a family of functions $\{f_U\}$ which are continuous and $0 \leqslant f_U(p) \leqslant 1$ and that the family of sets $E[p; f_U(p) > 0]$ form an $\aleph_\mu$-basis of $\mathfrak{X}$.*

Proof. Sufficiency. Completely the same as the proof of the sufficient part of theorem 7.

Necessity. The case $\mu > 0$ is contained in theorem 7. Let $\mu = 0$, and let $\mathfrak{F}$ be an $\aleph_0$-basis of $\mathfrak{X}$, $\mathfrak{F} = \sum_{n=1}^{\infty} \mathfrak{F}_n$, where $\mathfrak{F}_n$ are locally finite (discrete) systems. For $U \in \mathfrak{F}$ we put

$$f_U(p) = \varrho(p; \mathfrak{X} - U),$$

where ϱ is the metric function of $\mathfrak{X}$. Then $\{f_U\}$ fulfils the requirement of Cor. 2.

§ 5. Compactness and bicompactness. The terminology of compactness and bicompactness has been given by Sikorski [9]. We say that the topological space $\mathfrak{X}$ has the *$\aleph_\mu$-Lindelöf property*, if from every covering of $\mathfrak{X}$ one can select a subcovering of power $\leqslant \aleph_\mu$.

PROPOSITION 3. *If $\mathfrak{X}$ is a regular ω_μ-additive space which has the $\aleph_\mu$-Lindelöf property, then $\mathfrak{X}$ is normal.*

Proof. It is completely the same as in the case of $\mu = 0$, which is classical and well known ([4], p. 113), whence omitted.

The above proposition had been given by Parovicenko in [8].

THEOREM 8. *If $\mathfrak{X}$ is an ω_μ-metric space and is compact (in the sense of* [9]*), then $\mathfrak{X}$ has a basis of power $\leqslant \aleph_\mu$, whence is bicompact (in the sense of* [9]*).*

Proof. By Th. 3, $\mathfrak{X}$ is a $(U)_{\aleph_\mu}$-space. Since $\mathfrak{X}$ is compact, every subset X of power $\geqslant \aleph_\mu$ has in $\mathfrak{X}$ a contact point of order $\geqslant 2$ (p_0 being a *contact point* of X of order $\geqslant 2$ means that for every neighbourhood $V(p_0)$ of p_0 the set $X \cdot V(p_0)$ contains at least two points of X, [10]), then from Theorem of [10], $\mathfrak{X}$ has a basis of power $\leqslant \aleph_\mu$. Then Th. 8 follows from Lemma 2 of [10] immediately.

Recalling Cor. 1 of Th. 6, we have the following

THEOREM 9. *For a Hausdorff ω_μ-additive compact (in the sense of* [9]*) space to be ω_μ-metrisable, it is necessary and sufficient that it have a basis of power $\leqslant \aleph_\mu$.*

Proof. Sufficiency. Follows from Th. 6 immediately.

Necessity. Follows from Th. 8 immediately.

The case $\mu = 0$ of this theorem is the well-know second metrisation theorem of P. Urysohn.

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On lattice-ordered groups

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Introduction. We shall be concerned with a lattice-ordered group G, written additively though not necessarily abelian, with the set P of its positive (i.e. $x \geqslant 0$) elements, and with homomorphisms, epimorphisms, etc. from G to other such groups (mainly totally ordered ones and their products) which are *always* understood to be non-trivial, and lattice-ordered group homomorphisms, i.e. meet and join as well as sum preserving. If $K \subset G$ is an l-ideal in G then G/K denotes the quotient group as lattice ordered group, i.e. with the partial ordering defined by the image of P under the natural mapping $G \to G/K$, and we recall that for lattice-ordered groups and their homomorphisms the First Isomorphism Theorem holds, i.e. if f: $G \to G'$ is an epimorphism and $f = g \circ h$ its factorization into the natural mapping h: $G \to G/\mathrm{Ker}(f)$ and the induced mapping g: $G/\mathrm{Ker}(f) \to G'$ then h is an epimorphism and g an isomorphism (1). Our main object is to study the epimorphisms from G to totally ordered groups T, to obtain characterizing conditions for the existence of "sufficiently many" of these and hence of embeddings of G into products of such T, and to consider particular types of such embeddings. Some of our results can be regarded as an extension of those of Ribenboim [6] who restricted himself to the abelian case. The possibility of this extension is suggested by Lorenzen's theorem on regular lattice ordered groups [5] for which a proof is given in the present setting. The methods used here differ from the approach in [5] or in [6], the latter since we are able to dispense with Jaffard's notion of filet [4] in the proof of Proposition 3.

Particular subsets of P which will be of interest in the following are:

(i) the *filters* in P: the non-void subsets $F \subseteq P$ with $x \wedge y \in F$ for any $x, y \in F$ and $x \in F$ for any $x \geqslant y$ where $y \in F$;

(ii) the *prime filters* (2) in P: the proper filters Q in P for which $x + y \in Q$, x and y in P, implies $x \in Q$ or $y \in Q$;

(1) Terminology as in [2] unless stated otherwise.

(2) We use the term "prime" with respect to the group operation here rather than the lattice operation of forming the join. However, a prime filter in this sense is also prime with respect to join since $x + y \geqslant x \vee y$.