Remarks on \(\omega_\alpha\)-additive spaces

by

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§ 1. Preliminary notions. According to Sikorski [9], the set \(X\) is called an \(\omega_\alpha\)-additive \(^{(1)}\) space if there is defined (for every subset \(X\)) a closure operation \(X \rightarrow \overline{X}\) satisfying the following axioms:

I. \(\sum_{\alpha \in c_\alpha} X_\alpha = \sum_{\alpha \in c_\alpha} \overline{X}_\alpha\), for every \(\alpha\)-sequence of sets \(\{X_\alpha\}; \alpha < c_\alpha\);

II. \(\overline{X} = X\) for every finite subset \(X\);

III. \(\overline{X} = X\).

If \(\mu = 0\), the axiomatic system I-III coincides with the closure axiomatic system of Kuratowski, but for \(\mu > 0\) it is stronger than that system. Similar spaces were also considered by Parovicenko [8], Cohen, Goffman [1], [2], and others. A regular \(\omega_\mu\)-additive space, for \(\mu > 0\), must be \(0\)-dimensional.

Let \(A\) be an ordered group \(^{(2)}\), and if there exists a decreasing positive \(\omega_\alpha\)-sequence \(\{e_\xi\}; \xi < c_\alpha\) and \(e_\xi \in A\), satisfying the condition that for every positive element \(e \in A\) there exists an ordinal \(\xi_\alpha < c_\alpha\) such that \(e_\xi < e\) for every \(\xi > \xi_\alpha\) \((\xi < c_\alpha)\), then we say that \(A\) is of character \(c_\alpha\).

Suppose \(X\) is a set and with every given pair of points \(p, q \in X\), there is associated an element \(d(p, q) \in A\), where \(A\) is an ordered group of character \(c_\alpha\), such that

a) \(d(p, p) = 0\);

b) \(d(p, q) = d(q, p) > 0\) for \(p \neq q\);

c) \(d(p, q) \leq d(p, r) + d(r, q)\).

Then \(d\) is called an \(A\)-metric on \(X\), and \(X\) is called an \(A\)-metric space.

\(^{(1)}\) \(\omega_\alpha\) denotes a regular initial ordinal number.

\(^{(2)}\) I.e., an ordered set in which with every \(a, b \in A\) there is associated an element \(c \in A\) called the sum of \(a\) and \(b\) \(c = a + b\) and such that: 1) \(a + (b + c) = (a + b) + c\);

2) \(a + 0 = a\), if and only if \(a \leq 0\); 3) for every \(a, b \in A\) there exists an element \(c \in A\) such that \(a + c = b\). The symbol \(0\) denotes the element satisfying \(a + 0 = a\). An element \(a\) is positive if \(a > 0\) (see footnote \(^{(2)}\) of [9], p. 129).

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For an \( m \)-additive metric space \( X \), we can introduce the natural topology by setting \( \text{B}^0(p, X) = 0 \), where \( X \) is an arbitrary subset of \( X \) and \( \text{B}(p, X) = 0 \) means that for every positive \( r \in A \) there exists a \( p \in X \) such that \( B(p, r) < e \). And, then, the sets \( E[p; \text{B}(p, r)] \) form a basis of the open sets of \( X \). It can be proved that such spaces are \( m \)-additive spaces.

For this purpose it is only necessary to prove that the intersection of every \( \alpha \)-sequence \( (\alpha \leq \alpha_0) \) of open sets \( (G_\alpha) \) is open. Let \( p_0 \) be an arbitrary point of \( \prod G_\alpha \); then for each \( G_\alpha \) there exists a positive element \( \epsilon_\alpha \in A \) such that \( \epsilon_\alpha \leq p_0 \), and if \( q(p, p_0) < \epsilon_\alpha \) then \( p \notin G_\alpha \). Let \( x \) be an ordinal which is greater than every \( \epsilon_\alpha \) and \( x < \epsilon_\alpha \); then for \( q(p, p_0) < \epsilon_\alpha \), \( p \in \prod G_\alpha \); whence \( p_0 \) is an interior point of \( \prod G_\alpha \); this proves that \( \prod G_\alpha \) is an open set.

The \( m \)-additive metric spaces were considered by Hausdorff [3], Cohen and Goffman [2], Sikorski [9], and others. As Sikorski had pointed out in [9], many topological theorems about separable metric spaces can be generalized to the present case, but some singularities concerning compactness and completeness may occur.

In the above, if \( A \) is the set of all real numbers and \( b \) is replaced by \( b \) then \( q(p, q) = p(q, p) \), then \( q \) is called a pseudo-metric on \( X \). Let \( X \) be an almost-metric space, and \( X \) is called a quasi-metric space. For every \( \alpha \) there exists a \( \beta \) such that \( \alpha < \beta \) and \( \beta < \alpha \).

If \( \text{B} p \) is equal to \( 0 \), \( X \) is called an \( m \)-almost metric space.

One can introduce the topology for \( X \) by setting

\[
\text{B}(p, X) = \bigcap_{q \in \mathbb{R}^+} E[p; q(p, X) = 0],
\]

where \( X \subseteq X \), i.e. the family of sets \( E[p; q(p, X) = 0] \), where \( p \in X \), \( d > 0 \), \( q \in A \), is a basis for this topology.

The \( m \)-almost-metric spaces were introduced and investigated by Mrówka [5-7]. In fact, such spaces are equivalent in the sense of uniformity and topological structure to the Hausdorff uniform spaces (for the terminology of Hausdorff uniform spaces, see [4], p. 180) with the basis of power \( m \), i.e. a uniformity has a basis of the power \( m \) if and only if it is generated by a family of pseudo-metrics of power \( m \).

For brevity, in the following sections, the topological space \( X \) is said to be a \( (U_{m}\alpha) \)-space if its topology can be derived from a uniformity with a basis of power \( m \), where \( m \) is supposed to be the smallest possible; the topological space \( X \) is said to be \( (U_{m}\alpha) \)-metrisable, if it is possible to define an \( m \)-metric \( g \) such that the topology induced by \( g \) agrees with the original topology of \( X \). By the basis of \( X \) we always mean the open basis.

In the following two theorems, given by Mrówka, the original \( X \) is an \( m \)-almost-metrisable space" is replaced by "\( X \) is a \( (U_{m}\alpha) \)-space."
we know that the set \( E[p; 0; p; p] = 0 \) is open, and by

\[
\prod_{q \in P} E[p; q(p; p; p) = 0] = (p),
\]

we know that the set \( (p) \) is open, and hence, if \( \mu > 0 \), \( X \) must be discrete, which contradicts the hypothesis of our proposition.

Thus, a \((U)_m\)-space is \( \omega_\nu \)-additive (for \( \mu > 0 \)) only when \( m \geq \kappa_1 \).

It is natural to ask under what conditions the \((U)_m\)-space \( X \) would be \( \omega_\nu \)-additive, where \( \nu \geq \kappa_1 \).

Since every topological space (and hence every uniform space) is \( \omega_\nu \)-additive, in the rest of this section \( \mu > 0 \) is assumed.

Let \( X \) be a set and \( P = \{a\} \) a family of pseudo-metrices on \( X \).

In \( P \) the functions \( d_{a1}, \max\{d_{a1}, \ldots, d_{a\nu}\} \) (where \( a \) is an arbitrary positive rational number, \( \nu \) a natural number and \( a(1), \ldots, a(\nu) \) we get a new family \( P^* \), which is called the completion of \( P \); for \( P^* \) we have a, b, c, d, e) and the following:

f) For every positive rational \( d \) and \( a \in P^* \), \( d \in P^* \).

**Definition 1.** Let \( X, P \) be given as above. If, for every subfamily \( P \subseteq P, \overline{P} < \nu \) and every point \( p \in X \), there exist \( q \in P \) and a neighbourhood \( V(p) \) of \( p \) such that \( q(p, q) \geq \delta(p, q) \) holds for \( q \in P \) and \( q(1, q) \in \overline{P} \), then we say that \( P \) is a \( m \)-locally direct family.

**Theorem 1.** For a \((U)_m\)-space \( X \) to be \( \omega_\nu \)-additive (where \( \mu > 0 \)), it is necessary and sufficient that \( m \geq \kappa_1 \) and its topology can be derived from a uniformity which is generated by a family of pseudo-metrices \( P = \{d\} \) such that the completion \( P^* \) is an \( \kappa_1 \)-locally direct family.

**Proof.** Sufficiency. Let \( (G_\xi) \), \( \xi < \alpha \) (\( \alpha < \kappa_1 \)) be an \( \alpha \)-sequence of open sets, \( p_0 \) an arbitrary point of \( \prod G_\xi \). Then there exist a positive number \( d \) (by e) one can assume \( d = 1 \) and a subfamily \( \{\alpha_\xi\} \subseteq P^* \) such that \( E[p; \alpha_\xi(p, p) < 1] \subseteq G_\xi \) for \( 0 < \xi < \alpha \).

By the \( \kappa_1 \)-local directness of \( P^* \), there exist \( q \in P^* \) and a neighbourhood \( V(p_0) \) such that \( q > \alpha_\xi \) (\( 0 < \xi < \alpha \)) holds in \( V(p_0) \). Then

\[
V(p_0; \overline{E}[p; \alpha_\xi(p, p) < 1]) \subseteq V(p_0; \overline{E}[p; q(p, q) < 1]),
\]

this proves that \( p_0 \) is an interior point of \( \prod G_\xi \), whence \( \prod G_\xi \) is an open set.

(\(^*\)) Throughout the rest of the paper, topological spaces always mean non-discrete topological spaces.

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**Necessity.** Let the uniformity of the \((U)_m\)-space \( X \) be generated by a family \( P \) of pseudo-metrices; \( P^* \) is the completion of \( P \). For an arbitrarily given \( P \subseteq P^* \) and if \( \overline{P} < \kappa_1 \), let \( p \) be an arbitrary point of \( X \). Then the set

\[
V(p_0) = \prod_{\alpha_\xi(p, q) < 1} [p; q(p, q) < 1] = \prod_{\alpha_\xi} \prod_{\overline{P}} E[p; q(p, q) < 1]
\]

is an open set containing \( p_0 \), i.e. \( V(p_0) \) is a neighbourhood of \( p_0 \).

DEFINITION 2. Let \( X, P \) be given as in def. 1; if for every subfamily \( P \subseteq P \) with \( \overline{P} < \nu \) and every point \( p_0 \in X \), there exists a neighbourhood \( V(p_0) \) of \( p_0 \) such that \( q(p, q) = 0 \) for \( q \in P^* \) and \( p \), \( q \in V(p_0) \), then we say that \( P \) is an \( \omega \)-locally zero family.

A more convenient test to see if a \((U)_m\)-space \( X \) is \( \omega_\nu \)-additive is the following

**Theorem 2.** For a \((U)_m\)-space \( X \) to be \( \omega_\nu \)-additive, it is necessary and sufficient that \( m \geq \kappa_1 \) and its topology can be derived from a uniformity which is generated by an \( \kappa_1 \)-locally zero family of pseudo-metricals.

**Proof.** Sufficiency. We observe that the completion \( P^* \) is also an \( \kappa_1 \)-locally zero family; the sufficient part is a corollary of Theorem 1.

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**§ 3. The relationship between \( \omega_\nu \)-metrisable spaces and \((U)_m\)-spaces.** We now prove

**Proposition 2.** If \( X \) is an \( \omega_\nu \)-metrisable space and \( \overline{Y} \) is an open covering of \( X \), then there exists an \( \kappa_1 \)-discrete refinement \( \overline{Y} \) of \( \overline{Y} \), i.e. \( \overline{Y} \) is the union of \( \kappa_1 \)-families of discrete open sets, \( \overline{Y} \) is a covering of \( X \) and for every \( U \in \overline{Y} \) there is a \( V \in \overline{Y} \) such that \( U \subseteq V \). Moreover, for \( \mu > 0 \) we can require that \( \overline{Y} \) be formed by sets both open and closed.

**Proof.** The first part is essentially the same as in the case of \( \mu = 0 \). Order the elements of \( \overline{Y} \) by the relation \( < \). For each \( U \in \overline{Y} \) let \( U_1 = U_{1+1} \cap \overline{Y} \). Then, \( U_1, U_2, \ldots, U_{\nu+1} \). We put \( U_{\nu+1} = \bigcup_{\nu+1} V_f \cap \overline{Y} \). Since one of the relations \( V < U \) and \( V \in \overline{Y} \) must hold, therefore if \( U, V \) are distinct elements of \( \overline{Y} \), we have \( q(U, V) > \kappa_1 \). Choose two elements \( s, t, s' \) of \( A \) such that \( 2\kappa_1 > s, t, s', t' < \kappa_1 \) (to verify this possibility is easy), and define

(\(^*\)) The meaning of \( A \) and \( \epsilon \) has been given in § 1.
\[ U_1^* = \mathbb{E}_1(p; \varphi(p, V)^c) \setminus \mathcal{E}_1(p; \varphi(p, V)^c), \]
\[ U_2^* = \mathbb{E}_2(p; \varphi(p, V)^c). \]

Then \( U_1^* \) (and \( V_1^* \)) is open and \( U_2^* \) (and \( V_2^* \)) is closed, \( U_2^* \subset U_1^* \). If \( \mu > 0 \), then there exists an open-closed set \( \{ \mathcal{G}_3 \} \) such that \( U_2^* \subset \mathcal{G}_3 \subset U_2^* \).

In the following we prove that the family \( \{ \mathcal{U}_1^* \} \) (or \( \{ \mathcal{U}_2^* \} \)) for \( \mu > 0 \), where \( \xi < \omega_0 \) and \( U \in \mathcal{G}_3 \), is required.

Firstly, the sets \( U_1^* \) (or \( U_2^* \), if \( \mu > 0 \)) for fixed \( \xi \) are discrete. To prove this, let \( U \neq \emptyset \), \( U \in \mathcal{G}_3 \) and \( p \in U_1^* \). \( g \in V_1^* \) be arbitrary given; then we have \( \varphi(p, U_1) < \xi \) and \( \varphi(p, V_1) < \xi \). From \( \varphi(U_1, V_1) < \xi - \xi_2 + \xi_3 \), it follows that \( \varphi(p, g) > (\xi_2 - \xi_3) - 2\xi > 0 \), i.e. \( p \neq g \). Therefore \( U_1^* \cap V_1^* = \emptyset \).

Secondly, let \( p \in X \) be an arbitrary point and let \( U \) be the first member of \( \mathcal{G}_3 \) to which \( p \) belongs. Then surely \( p \in U_2^* \) for some \( \xi_2 \), that is \( p \in U_3 \) (for \( \mu > 0 \) and \( U_1 \)). Finally, it is evident that \( U_2^* \subset U \) (and \( U_1 \subset U \) for \( \mu > 0 \)), hence the family \( \{ \mathcal{U}_1^* \} \) (or \( \{ \mathcal{U}_2^* \} \)) for \( \mu > 0 \), is the required family.

**Theorem 3.** Every \( w_0 \)-metrizable space \( X \) is a \((U_2^*)_w\)-space.

**Proof.** By proposition 2, Theorem 3 follows from Theorem M, immediately. (By theorem (viii) of [9], \( X \) is a normal space.

It will be observed that Theorem 3 can be proved in a direct way.

**Theorem 4.** Every \( w_0 \)-additive \((U_2^*)_w\)-space is \( w_0 \)-metrizable.

**Proof.** Let \( X \) be an \( w_0 \)-additive \((U_2^*)_w\)-space. Then its topology can be derived from a family \( \mathcal{P} = \{ q \} \) of pseudo-metrics of power \( n \).

If \( \mu = 0 \), then \( P = \{ q \} \). Put
\[
\phi(p, q) = \sum_{n=1}^{\infty} \frac{1}{n^2} \min \{ 1, \phi(p, q_n) \};
\]
then \( \phi \) is a metric on \( X \), whence \( X \) is \( w_0 \)-metrizable. We now prove the case of \( \mu > 0 \) as follows. Let \( A \) be the set of all \( w_0 \)-sequences of real numbers. For every pair of elements \( a, b \in A \), where
\[
a = (a_0, a_1, a_2, ...),
\]
\[
b = (b_0, b_1, b_2, ...),
\]
\( \xi < \omega_0 \), if there exists \( \xi_2 < \omega_0 \) such that \( a_\xi < b_\xi \) but \( a_\xi < b_\xi \), then we say that \( a \) is smaller than \( b \), \( a < b \). The sum and the difference are defined by \( a \pm b = (a_0 \pm b_0, a_1 \pm b_1, ...) \).

It is not difficult to verify that \( A \) is an ordered group of character \( w_0 \); to see this we only take \( a_\xi = (a_0, a_1, a_2, ...), \) where \( a_\xi = 1 \) for \( \eta < \xi \) and \( a_\xi = 0 \) for \( \eta \geq \xi \) (\( \eta < \omega_0 \)).

If \( P = \{ q \} \), \( \xi < \omega_0 \), we put
\[
\phi(p, q) = (\phi(p, q_0), \phi(p, q_1), ...).
\]

\( X \) is now an \( w_0 \)-metric space, and we have to prove that its topology \( T^a \) agrees with the original topology \( T \). For brevity, by \( T \) (or \( T \)-open), we always mean a set which is open with respect to the topology \( T \) (or \( T \)); the same applies to \( T^a \) (or \( T^a \)-closed).

(1) The set \( E[p; \varphi(p, p_0) < \xi] \) is \( T^a \)-open for \( \varepsilon \in \mathbb{A} \), where \( p_0 \) is \( \mathbb{A} \) is arbitrarily given.

In fact, if \( \varepsilon = (\varepsilon_0, \varepsilon_1, ... \varepsilon, ...) \) then (1) follows from the equations
\[
E[p; \varphi(p, p_0) < \xi] = \sum_{a_0, a_1, ..., a_n, ...} E[p; \varphi(p, p_0) = a_0] \cdot E[p; \varphi(p, p_0) = a_0] \cdot E[p; \varphi(p, p_0) = a_0]
\]
and
\[
E[p; \varphi(p, p_0) = a_k] = \prod_{n=1}^{\infty} E[p; \varphi(p, p_0) = a_k - \frac{1}{n}] \cdot E[p; \varphi(p, p_0) = a_k + \frac{1}{n}]
\]

(I) The sets \( E[p; \varphi(p, p_0) < \xi] \) are \( T^a \)-open, where \( p_0 \in X \), \( a_0 \) is a positive real number \( \eta < \omega_0 \) and \( \phi(\varepsilon) \).

From
\[
E[p; \varphi(p, p_0) < a_0] = \sum_{a_0, a_1, ..., a_n, ...} E[p; \varphi(p, p_0) = a_0] \cdot E[p; \varphi(p, p_0) = a_0]
\]
it is evident that (I) follows from

(II) For every \( \eta < \omega_0 \) and an arbitrary \( \eta \)-sequence \( (a_k), \xi < \eta \), the sets
\[
(a) = \prod_{n=1}^{\infty} E[p; \varphi(p, p_0) = a_0] \cdot E[p; \varphi(p, p_0) < a_0]
\]
and
\[
(b) = \prod_{n=1}^{\infty} E[p; \varphi(p, p_0) = a_0] \cdot E[p; \varphi(p, p_0) > a_0]
\]
are both \( T_0 \)-open and \( T^a \)-closed sets.

We prove it by the following two steps:

(a) The sets \( E[p; \varphi(p, p_0) < a_0] \) and \( E[p; \varphi(p, p_0) > a_0] \) are both \( T_0 \)-open-closed sets.

In fact, let \( \varphi(\varepsilon) = (\varepsilon_0, \varepsilon_1, ... \varepsilon, ...) \), where \( \xi < \omega_0 \), and \( a_0, ... a_2, ... \) are fixed as \( \xi \) varies; then
\[
E[p; \varphi(p, p_0) < a_0] = \sum_{a_0, a_1, ..., a_n, ...} E[p; \varphi(p, p_0) < a_0]
\]
which implies the $T^\alpha$-openness of the set $E[p; \varrho(p, p_a) < a_a]$. (Similarly, the $T^\alpha$-openness of $E[p; \varrho(p, p_a) > a_a]$ can be proved.) To prove that they are $T^\alpha$-closed it suffices to take the complements, for example

$$E[p; \varrho(p, p_a) < a_a] = \mathcal{X} - \prod_{n=1}^{\infty} E[p; \varrho(p, p_a) > a_a - \frac{1}{n}].$$

(b) By the principle of transfinite induction, assume that (II') holds for all ordinals $\xi < \alpha$, to prove the case of $\alpha = \alpha_\xi$.

(i) If $\alpha$ is a limit ordinal, let $\alpha(\xi) = (\alpha^{(\xi)})$, where $\xi < \alpha_\xi$ and $\alpha^{(\xi)} = \alpha_\xi$ for $\xi \neq \alpha$ and $\alpha^{(\alpha)} = a_a - \frac{1}{n}$; then

$$E[p; \varrho(p, p_a) < \alpha^{(\xi)}] = \sum_{\xi < \alpha_\xi} \prod E[p; \varrho(p, p_a) = \alpha^{(\xi)}] E[p; \varrho(p, p_a) < \alpha^{(\xi)}].$$

Subtracting from the above set the following $T^\alpha$-closed set (hypothesis of (b))

$$\sum_{\alpha < \alpha_\xi} \prod E[p; \varrho(p, p_a) = \alpha^{(\xi)}] E[p; \varrho(p, p_a) < \alpha^{(\xi)}]$$

one obtains the following $T^\alpha$-open set:

$$\sum_{\alpha < \alpha_\xi} \prod E[p; \varrho(p, p_a) = \alpha^{(\xi)}] E[p; \varrho(p, p_a) < \alpha^{(\xi)}];$$

its union with respect to $n_1, a_{n_1}, ..., a_{n_i} ... (i < \alpha_\xi)$, is the $T^\alpha$-open set $(\Delta)$. In a similar way one can prove that $(\Delta)^\alpha$ is $T^\alpha$-open.

By taking the complements we can prove that the sets $(\Delta)$ and $(\Delta)^\alpha$ are $T^\alpha$-closed, e.g.

$$\prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = \alpha] E[p; \varrho(p, p_a) > \alpha_a]$$

$$= \prod_{n=1}^{\infty} \prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) > a_a - \frac{1}{n}] E[p; \varrho(p, p_a) < a_a + \frac{1}{n}];$$

and

$$\mathcal{X} - (\Delta) = \sum_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) > a_a] + \sum_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) < a_a] + \prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a] E[p; \varrho(p, p_a) > a_a],$$

one can prove that $(\Delta)_\alpha$ is $T^\alpha$-closed.

(ii) If $\alpha$ is a limit ordinal, then from the following equation

$$\prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a] E[p; \varrho(p, p_a) < a_a]$$

$$= \prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a] E[p; \varrho(p, p_a) < a_a + \frac{1}{n}]$$

and by the hypothesis of (b), we know that, for each $\eta < \alpha$, the set

$$\prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a] E[p; \varrho(p, p_a) < a_a]$$

is a $T^\alpha$-open set. By intersecting the above sets with respect to $\eta < \alpha$ we obtain the following $T^\alpha$-open set:

$$\prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a].$$

The intersection of the above set with the $T^\alpha$-open set $E[p; \varrho(p, p_a) < a_a^{(\xi)}]$, where $a_a^{(\xi)}$ assumes the same meaning as in (i), is the following $T^\alpha$-open set:

$$\sum_{\alpha < \alpha_\xi} \prod_{\alpha < \alpha_\xi} E[p; \varrho(p, p_a) = a_a^{(\xi)}] E[p; \varrho(p, p_a) < a_a^{(\xi)}];$$

by making a union of the above sets with respect to $n_1, a_{n_1}, ..., a_{n_i} ... (i < \alpha_\xi)$, the $T^\alpha$-open set $(\Delta)_{\alpha}$ is obtained. In a similar way one can prove that $(\Delta)^\alpha_{\alpha}$ is $T^\alpha$-open.

The proof that $(\Delta)$ and $(\Delta)^\alpha$ are $T^\alpha$-closed sets is completely the same as in case (i), whence it is omitted here.

From Theorems 3 and 4 we have

**Theorem 5.** $\omega_\alpha$-metrisable spaces and $\omega_\alpha$-additive $(U)_{\omega_\alpha}$ spaces are identical, in particular $\omega_\alpha$-metrisable spaces and ordinary metrisable spaces are identical.

**§ 4. $\omega_\alpha$-metrisation theorems (iv).** We prove

**Theorem 6.** For a regular $\omega_\alpha$-additive space to be $\omega_\alpha$-metrisable, it is necessary and sufficient that there exist an $\kappa_\alpha$-basis.

Let us recall that the family $\mathcal{B}$ of open sets is called an $\kappa_\alpha$-basis of the topological space if $\mathcal{B}$ is a basis and $\mathcal{B}$ can be written as $\mathcal{B} = \sum_{\alpha < \alpha_\xi} \mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ are locally finite systems of open sets.

(iv) Let us observe that in our metrisation theorems the notion of ordered algebraic base (see (93), p. 129) $\mathcal{B}$ is not used.
Proof of Theorem 6. As the necessary part has been contained in the proof of proposition 2, we need to prove the sufficient part only.

From Theorems 5 and 4, we need only to prove that X is a normal space (this is an improvement of theorem (vii) of [9]).

In fact, let $F_1$ and $F_2$ be disjointed closed sets; since $X$ is regular, for every pair of points $p \in F_1$, $q \in F_2$, there exist neighbourhoods $U_p \subseteq U_q$ such that $U_p \cap F_2 = \emptyset$ and $U_q \cap F_1 = \emptyset$. Let $U_p^{(i)} = \sum_{a \in p} U_a$ and $U_q^{(j)} = \sum_{a \in q} U_a$, then $U_p^{(i)} = \sum_{a \in p} U_a$ and $U_q^{(j)} = \sum_{a \in q} U_a$, since $\beta X$ is a locally finite family.

Put $U_* = U_*^{(i)} - \sum_{a \in p} D_a^{(i)}$, $U** = U**^{(i)} - \sum_{a \in q} D_a^{(i)}$.

The sets $U_*$ and $U**$ are disjointed open sets containing $F_1$ and $F_2$, respectively. Thus $X$ is normal. Therefore, theorem 6 is proved.

Corollary 1 (R. Sikorski [9]). If $X$ is an $\omega_\alpha$-additive normal space with a basis of power $\kappa$, then $X$ is $\omega_\alpha$-metrizable.

Corollary 2 (Nagata-Smircnov). For a regular space to be metrizable, it is necessary and sufficient that there exist an $\kappa$-basis.

Theorem 7. For $\mu > 0$, for an $\omega_\alpha$-additive space to be $\omega_\alpha$-metrizable, it is necessary and sufficient that there exist an $\kappa$-basis consisting of sets both open and closed.

Proof. Necessity. It is contained in the proof of proposition 2.

Sufficiency (3). Let $\mathscr{B}$ be an $\kappa$-basis of $X$ and let $\mathscr{G} = \sum_{a \in [\mathscr{B}]} G_a$ where $\mathscr{G}$ are locally finite (discrete) systems consisting of open-closed sets (Proposition 2). For $U \in \mathscr{G}$ define

$$f_U(p) = \begin{cases} 1 & \text{for } p \in U, \\ 0 & \text{for } p \notin U. \end{cases}$$

The family $P = \{ \max(\varnothing, \ldots, \varnothing) \}$ of functions,

$$\varnothing(\varnothing, \varnothing) = \sum_{U \in \mathscr{G}} f_U(p) - f_U(q),$$

makes $X$ as $\kappa$-almost metric space its topology is the same as the original. In fact, the $\varnothing$ are continuous functions by the local finiteness of $\mathscr{G}$. Conversely, for an arbitrarily given open set $G$ and $p \in G$, one can find $U \in \mathscr{G}$ (for some $\varnothing$) such that $p \in U \subseteq G$, whence $\varnothing(p, \varnothing) \subseteq G$. Thus $X$ is an $\omega_\alpha$-additive ($U_\kappa$)-space, and theorem 7 follows from Th. 4 (or Th. 5) immediately.

From theorem 7 we can derive some results which are closely related to Theorem $M_\alpha$.

Corollary 1. For $\mu > 0$, for an $\omega_\alpha$-additive space $X$ to be $\omega_\alpha$-metrizable it is necessary and sufficient that there exist a collection of families of continuous functions $F = (F_i)$ and $P = (P)$, where $\xi < \omega_\alpha$, such that the families of sets $E_x \subseteq \omega_\alpha^x$, for fixed $\xi$ are locally finite (discrete) sets, and the family of sets $E_x \subseteq \omega_\alpha^x$, where $\xi < \omega_\alpha$ and $\xi < \omega_\alpha$, is a basis of $X$.

Proof. Necessity. It suffices to put in theorem 7

$$f_U(p) = \begin{cases} 2 & \text{for } p \in U, \\ 0 & \text{for } p \notin U, \text{ for every } U \in \mathscr{G}, \xi < \omega_\alpha. \end{cases}$$

Sufficiency. The families of sets $E_x \subseteq \omega_\alpha^x$, for fixed $\xi$, are locally finite systems, consisting of sets both open and closed:

$$E_x < \omega_\alpha^x \subseteq 1 + 1^1.$$
Proof. It is completely the same as in the case of \( \mu = 0 \), which is classical and well known ([4], p. 113), whence omitted.

The above proposition had been given by Parovicenko in [8].

**Theorem 8.** If \( X \) is an \( \omega_\alpha \)-metric space and is compact (in the sense of [9]), then \( X \) has a basis of power \( \leq \kappa_\alpha \), whence is bicomplete (in the sense of [9]).

Proof. By Th. 3, \( X \) is a \( (U_n)_{n \in \omega} \)-space. Since \( X \) is compact, every subset \( \mathcal{X} \) of power \( \geq \kappa_\alpha \) has a \( X \)-contact point of order \( > 2 \) (\( p_x \) being a contact point of \( X \) of order \( > 2 \) means that for every neighbourhood \( V(p_x) \) of \( p_x \) the set \( X \cap V(p_x) \) contains at least two points of \( X \) (cf. [10]), then from Theorem of [10], \( X \) has a basis of power \( \leq \kappa_\alpha \). Then Th. 8 follows from Lemma 2 of [10] immediately.

Recalling Cor. 1 of Th. 6, we have the following

**Theorem 9.** For a Hausdorff \( \omega_\alpha \)-additive compact (in the sense of [9]) space to be \( \omega_\alpha \)-metrisable, it is necessary and sufficient that it have a basis of power \( \leq \kappa_\alpha \).

Proof. Sufficiency. Follows from Th. 8 immediately.

Necessity. Follows from Th. 8 immediately.

The case \( \mu = 0 \) of this theorem is the well-know second metrisation theorem of P. Urysohn.

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**References**


**On lattice-ordered groups**

by

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**Introduction.** We shall be concerned with a lattice-ordered group \( G \), written additively though not necessarily abelian, with the set \( P \) of its positive (i.e. \( x \geq 0 \)) elements, and with homomorphisms, epimorphisms, etc. from \( G \) to other such groups (mainly totally ordered ones and their products) which are always understood to be non-trivial, and lattice-ordered group homomorphisms, i.e. meet and join as well as sum preserving. If \( K \subseteq G \) is an \( l \)-ideal in \( G \) then \( G/K \) denotes the quotient group as lattice ordered group, i.e. with the partial ordering defined by the image of \( P \) under the natural mapping \( G \to G/K \), and we recall that for lattice-ordered groups and their homomorphisms the First Isomorphism Theorem holds, i.e. if \( f : G \to G' \) is an epimorphism and \( f = g \circ h \) its factorization into the natural mapping \( h : G \to G/\text{Ker}(f) \) and the induced mapping \( g : G/\text{Ker}(f) \to G' \) then \( h \) is an epimorphism and \( g \) an isomorphism. Our main object is to study the epimorphisms from \( G \) to totally ordered groups \( T \), to obtain characterizing conditions for the existence of "sufficiently many" of these and hence of embeddings of \( G \) into products of such \( T \), and to consider particular types of such embeddings. Some of our results can be regarded as an extension of those of Ribenboim [6] who restricted himself to the abelian case. The possibility of this extension is suggested by Lorenzen's theorem on regular lattice ordered groups [5] for which a proof is given in the present setting. The methods used here differ from the approach in [5] or in [6], the latter since we are able to dispense with Jaffard's notion of filter [4] in the proof of Proposition 3.

**Particular subsets of \( P \) which will be of interest in the following are:**

(i) the filters in \( P \); the non-vord subsets \( F \subseteq P \) with \( \neq y \leq F \) for any \( x \leq y \leq F \) and \( x \leq F \) for any \( x \geq y \) where \( y \leq F \);

(ii) the prime filters \( (\dagger) \) in \( P \); the proper filters \( Q \) in \( P \) for which \( x \not\leq y \leq Q \), \( x \) and \( y \) in \( P \), implies \( x \not\leq Q \) or \( y \not\leq Q \);

\( (\dagger) \) Terminology as in [2] unless stated otherwise.

\( (\dagger) \) We use the term "prime" with respect to the group operation here rather than the lattice operation of forming the join. However, a prime filter in this sense is also prime with respect to join since \( x \vee y \neq x \vee y \).