*pour chaque $h > 0$ est un polynôme de degré $n$ au plus.* Notre théorème 2 est dans un certain sens plus fort. Il permet de remplacer dans la formule (35) le signe d'égalité par celui d'une inégalité faible ($\leqslant$ ou $\geqslant$) pour tous les $h$ sauf un. (Dans ce travail on a demandé l'égalité (35) pour $h = 1$, mais cette condition n'est pas essentielle. Dans les théorèmes 1 et 2 on peut remplacer l'accroissement $h = 1$ par un autre accroissement arbitraire $h > 0$.)

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# On maximally resolvable spaces

by

**J. G. Ceder** (Santa Barbara, Calif.)

In [2] E. Hewitt posed the problem of determining the largest number of disjoint, dense subsets possible in a topological space. As one result in this direction, W. Sierpiński [8] has proved that a metric space $X$ each non-void open subset of which constains $\geqslant \boldsymbol{m} \geqslant \aleph_0$ points, is the union of $\boldsymbol{m}$ disjoint sets each of which contains at least $\boldsymbol{m}$ points of each non-void open set in $X$. It is the purpose of this note to generalize Sierpiński's result in two ways, one way of which will enable us to extend some of Hewitt's results in [2] so as to determine the largest possible number of disjoint, dense subsets in certain spaces, including locally compact Hausdorff spaces and first countable spaces.

In the sequel, we will consider ordinals and cardinals as defined, for example, in J. L. Kelley ([4], appendix), so that each ordinal is equal to the set of its predecessors and a cardinal is an ordinal which is not equipollent with any smaller ordinal (1). The cardinal number of a set $A$ will be denoted by $|A|$. The symbols $\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{n}$ will always denote specific cardinals and the Greek letters $\alpha, \beta, \gamma$, etc. will denote general ordinals. A subset $A$ of topological space is said to be $\boldsymbol{m}$*-dense* if $|A \cap U| = \boldsymbol{m}$ for each non-void open subset $U$ of $X$.

Our first generalization of Sierpiński's result is

THEOREM 1. *Let $X$ be any topological space with an infinite base $\mathfrak{B}$ such that $|\mathfrak{B}| \leqslant \boldsymbol{n} \leqslant \boldsymbol{m}$. Then, if $A$ is an $\boldsymbol{m}$-dense subset of $X$, $A$ is the union of $\boldsymbol{m}$ disjoint, $\boldsymbol{n}$-dense subsets of $X$.*

Proof. We will first take the case when $|\mathfrak{B}| = \boldsymbol{n}$ and induct on the cardinals $\boldsymbol{m} \geqslant \boldsymbol{n}$.

For $\boldsymbol{n} = \boldsymbol{m} = |\mathfrak{B}|$, let us well-order $\mathfrak{B}$ so that $\mathfrak{B} = \{B_\alpha\}_{\alpha<\boldsymbol{n}}$. Since $|B_\alpha \cap A| = \boldsymbol{n}$ for each $\alpha < \boldsymbol{n}$, we can by a result of K. Kuratowski ([5], Lemma 1) find a disjoint family $\{H_\alpha\}_{\alpha<\boldsymbol{n}}$ such that $H_\alpha \subseteq B_\alpha \cap A$ and $|H_\alpha| = \boldsymbol{n}$ for each $\alpha < \boldsymbol{n}$. Since $|\boldsymbol{n}\cdot\boldsymbol{n}| = \boldsymbol{n}$, we can put each $H_\alpha = \bigcup_{\alpha<\boldsymbol{n}} H_{\alpha,\beta}$ where $|H_{\alpha,\beta}| = \boldsymbol{n}$ and the sets $H_{\alpha,\beta}$ are disjoint. For $0 < \beta < \boldsymbol{n}$

(1) For facts about ordinal and cardinal arithmetic we employ in the sequel the reader is referred to Sierpiński [9].

put $A_\beta = \bigcup_{a<n} H_{a,\beta}$ and put $A_0 = (A - \bigcup_{a<n} H_a) \cup (\bigcup_{a<n} H_{a,0})$. Then it is easily checked that $\{A_a\}_{a<n}$ gives the desired decomposition of $A$.

Now suppose that $\boldsymbol{m}$ is a cardinal $> \boldsymbol{n}$. Since $|A| = \boldsymbol{m}$, we can well-order $A$ by the cardinal $\boldsymbol{m}$. Also, well-order $\mathcal{B}$ by $\boldsymbol{n}$ so that $\mathcal{B} = \{B_a: 0 < a < \boldsymbol{n}\}$. Let $\Gamma$ be the intersection of all subsets $C$ of $\boldsymbol{m}$ satisfying the three properties: (i) $0 \in C$; (ii) if $a \in C$, then $a + \boldsymbol{n} \in C$; (iii) the union of any subset of $C$ is $\boldsymbol{m}$ or is in $C$. Then clearly each $a < \boldsymbol{m}$ can be uniquely written as $\gamma + \xi$ where $\gamma \in \Gamma$ and $\xi < \boldsymbol{n}$. Since $|\Gamma| = \boldsymbol{m}$, we can also find a one-to-one increasing function $f$ from $\boldsymbol{m}$ onto $\Gamma$.

By the "first $\boldsymbol{n}$-points" of a subset $C$ of the well-ordered set $A$ (where $|C| \geqslant \boldsymbol{n}$) we mean the smallest initial segment (relative to the well-ordering of $C$ that is induced by $A$) of $C$ having cardinality $\boldsymbol{n}$. Now begin by choosing $C_0$ to consist of the first $\boldsymbol{n}$-points of $A$. Having chosen $C_a$ for all $a < \beta < \boldsymbol{m}$, consider $\beta$. If $\beta \in \Gamma$ we let $C_\beta$ consist of the first $\boldsymbol{n}$-points of $A - \bigcup_{a<\beta} C_a$. If $\beta \notin \Gamma$, then $\beta = \gamma + \xi$ where $\gamma \in \Gamma$ and $0 < \xi < \boldsymbol{n}$, and we let $C_\beta$ consist of the first $\boldsymbol{n}$-points of the set $D_\beta = (A \cap B_\xi) - \bigcup_{a<\beta} C_a$. Finally put $A_a = \bigcup \{C_\beta: f(a) \leqslant \beta < f(a+1) = f(a) + \boldsymbol{n}\}$ for each $a < \boldsymbol{m}$.

Then $\{A_a\}_{a<\boldsymbol{m}}$ yields the desired decomposition of $A$. Since $|A \cap B_\xi| = \boldsymbol{m}$ and $|\bigcup_{a<\beta} C_a| \leqslant |\boldsymbol{n}| \cdot |\beta| < \boldsymbol{m}$, we have $|D_\beta| = \boldsymbol{m}$ for each $\beta \notin \Gamma$; hence the choice of $C_\beta$ is possible. The disjointness and the $\boldsymbol{n}$-density of the sets $A_a$ are immediate. The fact that $\bigcup_{a<\boldsymbol{m}} A_a$ exhausts $A$ follows from the choice of $C_\beta$ for $\beta \in \Gamma$.

For the case when $|\mathcal{B}| < \boldsymbol{n}$, we decompose $\boldsymbol{m}$ so that $\boldsymbol{m} = \bigcup_{a<\boldsymbol{m}} M_\beta$ where $|M_\beta| = \boldsymbol{n}$ and the $M_\beta$'s are disjoint. From above we know that $A = \bigcup_{a<\boldsymbol{m}} A_a$ where $A_a$ is $|\mathcal{B}|$-dense. Now define $A'_a = \bigcup \{A_\xi: \xi \in M_a\}$. Then $A$ is the union of the $\boldsymbol{m}$ disjoint, $\boldsymbol{n}$-dense sets $A'_a$, which completes the proof of the theorem.

Of particular interest is the decomposition of an $\boldsymbol{m}$-dense subset of the plane. Here, the decomposants can be made to satisfy some interesting geometrical properties. For instance,

THEOREM 2. *Any $\boldsymbol{m}$-dense subset of the plane is the union of $\boldsymbol{m}$ disjoint, mutually homeomorphic, countably dense sets each of which has the properties that no three of its points are collinear and that each vertical or horizontal line intersects it at most once.*

Proof. We follow the pattern of the proof of theorem 1. In the case $\boldsymbol{m} = \aleph_0 = \omega$ we have from theorem 1 that $A = \bigcup_{k=1}^{\infty} C_k$, where the $C_k$'s are disjoint and $\omega$-dense. Now well-order $A$ by the ordinal $\omega^2$.



Let $\{B_k: 0 < k < \omega\}$ be a countable basis for the plane. By induction upon $\omega^2$, pick $x_0$ to be the first point in $A$. Having chosen $x_a$ for all $a < \beta < \omega^2$, consider $\beta$, which can be uniquely expressed as $\omega \cdot n + k$ where $n, k < \omega$. If $k = 0$, let $x_\beta$ be the first point in $A - \{x_a: a < \beta\}$. If $k \neq 0$, choose $x_\beta$ to be the first point in the set $(B_k \cap C_n) - \bigcup_{i=0}^{k-1} K(x_{\omega\cdot n+i}) - \bigcup \{L(x_{\omega\cdot n+i}, x_{\omega\cdot n+j}): 0 < i \leqslant j < k)\}$ where $K(z) = \{v \in E^2: z$ and $v$ have a common coordinate$\}$ and $L(z, v)$ is the line passing through $z$ and $v$, where $z \neq v$. Then put for each $n < \omega$ $A_n = \{x_\beta: \omega \cdot n \leqslant \beta < \omega \cdot (n+1)\}$. Then it can be shown without difficulty that $\{A_n\}_{n<\omega}$ yields the desired decomposition of $A$.

In the case $\boldsymbol{m} > \aleph_0$, we simply repeat the corresponding argument in the proof of theorem 1 with the modifications that $\boldsymbol{n} = \aleph_0$; $C_\beta$ is chosen to consist of the first point in the set

$$D_\beta - \bigcup_{i=0}^{k-1} K(x_{f(\xi)+i}) - \bigcup \{L(x_{f(\xi)+i}, x_{f(\xi)+j}): 0 \leqslant i < j < k\}$$

whenever $\beta = f(\xi) + k$, $k \neq 0$; $C_\beta$ is chosen to consist of the first point in $A - \bigcup_{a<\beta} C_a$ whenever $\beta \in \Gamma$. The rest of the proof is as before. Finally, the decomposants are mutually homeomorphic since any two countably dense subsets of the plane are homeomorphic (Sierpiński [7]).

For a given topological space $X$, let us put $\Delta(X) = \min\{|U|: U$ is a non-void open subset of $X\}$; $\chi(X) = \min\{|\mathcal{B}|: \mathcal{B}$ is a base for $X\}$; and $\chi_X(\beta) = \min\{|\mathfrak{U}|: \mathfrak{U}$ is a local base at $p \in X\}$. We will say that a space $X$ is *$\boldsymbol{m}$-resolvable*, where $\boldsymbol{m}$ is a cardinal $\geqslant 2$, if $X$ is the union of $\boldsymbol{m}$ disjoint, dense subsets each of which intersects each non-void open subset of $X$ in at least $\boldsymbol{m}$ points. (Hewitt's "resolvable" is our 2-resolvable without the last restriction). It is clear that $X$ can not be $\boldsymbol{m}$-resolvable for any $\boldsymbol{m} > \Delta(X)$. So we say that $X$ is *maximally resolvable* if $X$ either has isolated points or is $\Delta(X)$-resolvable. Now we proceed to obtain some sufficient conditions, all of which are generalizations of Hewitt's results on "resolvable" spaces, for a space to be maximally resolvable. We begin by again generalizing Sierpiński's result, this time to

THEOREM 3. *If $\aleph_0 \leqslant \chi(X) \leqslant \Delta(X)$, then $X$ is maximally resolvable.*

Proof. The case when $\Delta(X) = \chi(X)$ is identical to that in theorem 1. For the case when $\chi(X) < \Delta(X)$, we construct the sets $\{A_a\}_{a<\boldsymbol{m}}$, $\boldsymbol{m} = \Delta(X)$, as done in the proof of theorem 1. Here, however $X$ may not be equal to $\bigcup_{a<\boldsymbol{m}} A_a$, in which case we define $X_a = A_a$ for $a \neq 0$ and $X_0 = A_0 \cup (X - \bigcup_{a<\boldsymbol{m}} A_a)$. Then it is easily checked that $\{X_a\}_{a<\boldsymbol{m}}$ yields the desired resolution of $X$.

Only in the case $X$ is Hausdorff can we dispense with the requirement that $\aleph_0 \leqslant \chi(X)$ in theorem 3 (as well as in Theorem 9), for then, if $\chi(X)$ were finite, $X$ would have the discrete topology and hence, be maximally resolvable. To show that $\aleph_0 \leqslant \chi(X)$ is necessary in a non-Hausdorff space, take a two point set with the indiscrete topology. Next we show that a "locally $m$-resolvable" space is $m$-resolvable.

THEOREM 4. *If each non-void open subset of $X$ contains an $\boldsymbol{m}$-resolvable subspace, then $X$ itself is $\boldsymbol{m}$-resolvable.*

Proof. Let $A_0$ be a subset of $X$ which is $\boldsymbol{m}$-resolvable. Let $\{C_0^\gamma\}_{\gamma<\boldsymbol{m}}$ be its corresponding resolution. Now suppose we have chosen for each $\alpha<\beta$ an $\boldsymbol{m}$-resolvable set $A_\alpha$ with resolution $\{C_\alpha^\gamma\}_{\gamma<\boldsymbol{m}}$. Consider $G_\beta = X - (\bigcup_{\alpha<\beta} A_\alpha)^-$. If $G_\beta \neq \emptyset$, we select by hypothesis an $A_\beta \subseteq G_\beta$ which is $\boldsymbol{m}$-resolvable. If $G_\beta = \emptyset$, put $A_\beta = \emptyset$. Let $\beta_0$ be the least ordinal $\beta$ such that $G_\beta = \emptyset$. Then put $C^\gamma = \bigcup_{\alpha<\beta_0} C_\alpha^\gamma$ for each $\gamma<\boldsymbol{m}$. Then it can be shown without difficulty that $\{C^0 \cup (X - \bigcup_{\gamma<\boldsymbol{m}} C^\gamma)\} \cup \{C^\gamma\colon 0<\gamma<\boldsymbol{m}\}$ gives the desired $\boldsymbol{m}$-resolution of $X$.

THEOREM 5. *If each point $x \in X$ has an open neighborhood $U(x)$ such that $\aleph_0 \leqslant \chi(U(x)) \leqslant \Delta(U(x))$, then $X$ is maximally resolvable.*

Proof. By theorem 3 each $U(x)$ is $\Delta(U(x))$-resolvable, hence $\Delta(X)$ resolvable. Since open subsets of $\boldsymbol{m}$-resolvable spaces are obviously $m$-resolvable, the hypothesis of theorem 4 is satisfied for $\boldsymbol{m} = \Delta(X)$. Thus, $X$ is maximally resolvable.

THEOREM 6 (Hewitt ([2], Th. 46)). *Let $X$ be a $T_0$-space devoid of isolated points and having the property that each non-void open subset of $X$ contains a non-void open subset $H$ such that for each point $p \in H$, $\chi_X(p) \leqslant |H|$. Then each non-void open subset of $X$ contains a non-void open subset $G$ so that $\chi(G) \leqslant \Delta(G)$.*

The next four theorems provide us with a large class of maximally resolvable spaces.

THEOREM 7. *Any locally compact Hausdorff space is maximally resolvable.*

Proof. It is well known that a locally compact Hausdorff space devoid of isolated points satisfies the hypothesis of theorem 6 (cf. [1], p. 67). Then apply theorems 3 and 4 where $\boldsymbol{m} = \Delta(X)$ to complete proof.

THEOREM 8. *Any $T_0$-space in which each point has a local basis linearly ordered by inclusion is maximally resolvable.*

Proof. It is easily seen that the hypothesis of theorem 6 is fulfilled. Then apply theorems 3 and 4.

In particular then, 1st countable $T_0$-spaces (including metric spaces), and linearly ordered sets equipped with the order-topology are maximally resolvable. We are unable to prove that an arbitrary product of maximally resolvable spaces is maximally resolvable, however we do have



THEOREM 9. *For each $\alpha \in A$ let $X_\alpha$ be a maximally resolvable space with $\aleph_0 \leqslant \chi(X_\alpha)$. Then, the product space $\prod_{\alpha\in A} X_\alpha$ is maximally resolvable, provided there does not exist an infinite subset $M \subseteq A$ for which $|M| < \sup_{\alpha\in A} |X_\alpha|$ and $\Delta(X_\alpha) < |X_\alpha| < \chi(X_\alpha)$ for all $\alpha \in M$.*

Proof. First we prove that a finite product of maximally resolvable spaces is maximally resolvable. By induction, we need only consider two maximally resolvable spaces $X$ and $Y$ with resolutions $\{X_\alpha\colon \alpha < \Delta(X)\}$ and $\{Y_\alpha\colon \alpha<\Delta(Y)\}$, respectively. Then, since $\Delta(X\times Y) = |\Delta(X)\cdot\Delta(Y)|$, $\{X_\alpha \times Y_\beta\colon \alpha<\Delta(X),\ \beta<\Delta(Y)\}$ will become a $\Delta(X\times Y)$ resolution for $X\times Y$.

Let us now put $X = \prod_{\alpha\in A} X_\alpha$, $\boldsymbol{k}_\alpha = |X_\alpha|$, $\boldsymbol{n}_\alpha = \chi(X_\alpha)$ and $\boldsymbol{m}_\alpha = \Delta(X_\alpha)$ for each $\alpha \in A$. Now we decompose the index set $A$ into four disjoint sets: $A_1 = \{\alpha\in A\colon \boldsymbol{k}_\alpha = \boldsymbol{m}_\alpha\}$; $A_2 = \{\alpha\in A\colon \boldsymbol{n}_\alpha \leqslant \boldsymbol{m}_\alpha\} - A_1$; $A_3 = \{\alpha \in A\colon \boldsymbol{m}_\alpha < \boldsymbol{k}_\alpha < \boldsymbol{n}_\alpha\}$; and $A_4 = \{\alpha\in A\colon \boldsymbol{m}_\alpha < \boldsymbol{n}_\alpha \leqslant \boldsymbol{k}_\alpha\}$. Since $X$ is homeomorphic to $\prod_{i=1}^{4} X_i$ where $X_i = \prod_{\alpha\in A_i} X_\alpha$ $(i = 1, 2, 3, 4)$, we need only show that each $X_i$ is maximally resolvable.

(1) For $X_1$, put $\boldsymbol{m} = |A_1|$ and let $\{C_\beta^\alpha\colon \beta<\boldsymbol{m}_\alpha\}$ be a $\boldsymbol{m}_\alpha$-resolution for $X_\alpha$. Then the family $\{\prod C_{\beta_\alpha}^\alpha\colon \beta_\alpha < \boldsymbol{m}_\alpha \text{ and } \alpha<\boldsymbol{m}\}$ is disjointed and has cardinality $\prod_{\alpha<\boldsymbol{m}} \boldsymbol{m}_\alpha$. Moreover, each non-void open set intersects each member of this family in $\geqslant \prod_{\alpha<\boldsymbol{m}} \boldsymbol{m}_\alpha$ points. Clearly there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $A_1$ for which $\Delta(X_1) = \prod_{i=1}^{n} \boldsymbol{m}_{\alpha_i} \times \prod_{\alpha\neq\alpha_i} \boldsymbol{k}_\alpha = \prod_{\alpha<\boldsymbol{m}} \boldsymbol{m}_\alpha$. Hence, $X_1$ is maximally resolvable.

(2) For $X_2$, we have a finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of $A_2$ for which $\Delta(X_2) = \prod_{i=1}^{n} \boldsymbol{m}_{\alpha_i} \times \prod_{\alpha\neq\alpha_i} \boldsymbol{k}_\alpha$. Let $\mathcal{A}_i$ consist of all finite subsets of $A_i$; then $|\mathcal{A}_2| = |A_2|$ unless $A_2$ is finite in which case $X_2$ is already maximally resolvable. Then obviously $\chi(X_2) = \sum\{\prod_{\alpha\in F} \boldsymbol{n}_\alpha\colon F \in \mathcal{A}_2\}$. Then it is easily established that $\chi(X_2) \leqslant \sum_{\alpha\in A_2} \boldsymbol{n}_\alpha \leqslant \sum_{\alpha\in A_2} \boldsymbol{m}_\alpha \leqslant \Delta(X_2)$. Now we apply Theorem 3 to conclude that $X_2$ is maximally resolvable.

(3) For $X_3$, put $A_5 = \{\alpha\in A_3\colon \sup_{\alpha\in A_3} \boldsymbol{k}_\alpha \leqslant |A_3|\}$ and $A_6 = A_3 - A_5$. By assumption, $A_6$ is finite so that $X_6 = \prod_{\alpha\in A_6} X_\alpha$ is maximally resolvable. Assuming $|A_5| = \boldsymbol{m}$ is infinite, we have, since $\boldsymbol{n}_\alpha \leqslant 2^{\boldsymbol{k}_\alpha}$, that

$$\chi(X_5) \leqslant \sum\Big\{\prod_{\alpha\in F} 2^{\boldsymbol{k}_\alpha}\colon F\in\mathcal{A}_5\Big\} \leqslant |\mathcal{A}_5|\cdot 2^\gamma = \boldsymbol{m}\cdot 2^\gamma \quad \text{where} \quad \gamma = \sup_{\alpha\in A_3} \boldsymbol{k}_\alpha.$$

Also, since there exists a set $\{a_1, \ldots, a_n\}$ for which $\prod_{i=1} \boldsymbol{m}_{a_i} \times \prod_{a \neq a_i} \boldsymbol{k}_a = \Delta(X_5)$, we can prove that $\gamma^{\boldsymbol{m}} = \prod_{a \neq a_i} \boldsymbol{k}_a \leqslant \Delta(X_5)$. But the assumption that $\sup_{a \in A_3} \boldsymbol{k}_a \leqslant \boldsymbol{m}$ implies that $\gamma \leqslant \boldsymbol{m}$ and $\chi(X_5) \leqslant \boldsymbol{m} \cdot 2^{\gamma} \leqslant \gamma^{\boldsymbol{m}} \leqslant \Delta(X_5)$. Hence, by theorem 3, $X_5$ is maximally resolvable, as is $X_3$.

(4) For $X_4$, we have as in (3) that $\gamma^{\boldsymbol{m}} \leqslant \Delta(X_4)$ where we can assume without loss of generality that $|A_4| = \boldsymbol{m}$ is infinite. Then $\chi(X_4) \leqslant \sum \{\prod_{a \in F} \boldsymbol{n}_a \colon F \in \mathcal{A}_4\} \leqslant \sum \{\prod_{a \in F} \boldsymbol{k}_a \colon F \in \mathcal{A}_4\} \leqslant |\mathcal{A}_4| \cdot \gamma = \boldsymbol{m} \cdot \gamma$. But since $\boldsymbol{m}\gamma \leqslant \gamma^{\boldsymbol{m}}$ for all $\boldsymbol{m}$, we have $\chi(X_4) \leqslant \Delta(X_4)$ and thus, $X_4$ is maximally resolvable.

There do exist maximally resolvable spaces $X$ for which $\aleph_0 \leqslant \Delta(X) < |X| < \chi(X)$. For example, let $Y$ be the linearly ordered space obtained by inserting a copy of the rationals between each two consecutive ordinals $< 2^{\aleph_0}$. Let $X$ be the subspace $Y \cup \{P\}$ of $\beta(Y)$ (the Stone-Čech compactification of $Y$) where $p \in \beta(Y) - Y$. Then $X$ will be maximally resolvable with $\aleph_0 = \Delta(X) < 2^{\aleph_0} = \boldsymbol{c} = |X| < 2^{\boldsymbol{c}} = \chi(X)$. Now as an application of Theorem 9 we obtain

THEOREM 10. *A product space $\prod_{a \in A} X_a$ is maximally resolvable if each $X_a$ is Hausdorff and $\chi_{X_a}(p) \leqslant |X_a|$ for each $a \in A$ and $p \in X_a$. (In particular, if each $X_a$ is Hausdorff and is either locally compact or has a local linearly-ordered base at each point.)*

Proof. Let $B = \{a \in A \colon \aleph_0 \leqslant |X_a|\}$ and $C = A - B$. For $a \in B$ we have for $X = \prod_{a \in B} X_a$ that $\aleph_0 \leqslant \chi(X) \leqslant \sum_{p \in X} \chi_X(p) \leqslant |X|\,|X| = |X|$ and upon application of theorem 9 we have that $X$ is maximally resolvable. If $C$ is finite, then $\prod_{a \in C} X_a = Y$ has an isolated point and we are finished. If $C$ is infinite then $\Delta(Y) = \prod_{a \in C} |X_a|$ and $\chi(Y) = |C|$ unless $Y$ has an isolated point. Therefore, unless $Y$ has an isolated point $\aleph_0 \leqslant \chi(Y) = |C| \leqslant \prod_{a \in C} |X_a| = \Delta(Y)$. Hence, $Y$ is maximally resolvable, which finishes the proof. The parenthetical statement is a consequence of theorems 7 and 8.

In particular, a product of real intervals is maximally resolvable. Although open subsets of maximally resolvable spaces are obviously maximally resolvable, arbitrary subsets may not be. For example, Katětov [3] has shown that there exist normal, zero-dimensional spaces of arbitrary infinite cardinality which are not 2-resolvable. In particular then, there exists a countable, normal space (hence, regular Lindelöf) which is not 2-resolvable. Finally it should be mentioned that Padmavally [6] has constructed a connected Hausdorff space which is not 2-resolvable.



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UNIVERSITY OF CALIFORNIA
at SANTA BARBARA

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