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Reçu par la Rédaction le 27. 2. 1963

Sequentially pointwise continuous linear functionals

by

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1. Introduction. If L is a linear space of real-valued functions on a non-empty set X and φ is a linear functional on L which is continuous with respect to pointwise convergence on X of nets in L, then it is an immediate consequence of the theory of duality in linear spaces that there are points $x_1, \ldots, x_n$ in X and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\varphi(f) = \sum_{i=1}^{n} \lambda_i f(x_i) \tag{P}$$

for all f in L. If, however, we ask under what conditions every linear functional on L which is continuous with respect to pointwise convergence on X of *sequences* in L (*sequentially* or *σp-continuous*) is of this form then the problem is much more complicated. For example, Mrówka [5] constructs a positive σp-continuous linear functional on a linear space of continuous real-valued functions on the unit interval, $[0,1]$, of the reals which is far from being of this form. On the other hand, he shows [3] that if L is the linear space $C(X)$ of all continuous real-valued functions on a completely regular Hausdorff space X, then a necessary and sufficient condition is that X be *real-compact* (i.e. complete with respect to the weak uniformity induced by $C(X)$). In [4] and [5] he gives necessary and sufficient conditions for certain algebras of real-valued functions and linear lattices of bounded real-valued functions.

The present paper is chiefly concerned with the case where L is a linear lattice. For such an L we prove, (a) that the non-trivial σp-continuous linear lattice [1] functionals are, up to a positive multiple, in (1-1) correspondence with the proper prime σ-filters of zero sets on X (these are precisely the σ-filters of zero sets which are maximal proper filters of zero sets) (Theorem 1); and (b) that for any σp-continuous linear functional, φ, on L there is a unique class, $\mathcal{S}_\varphi$, of non-trivial σp-continuous

[1] Definitions are given below.

linear lattice functionals on L such that for every $f \in L$, $\psi(f) = 0$ for all but a finite number of ψ in $\mathcal{S}_\varphi$ and $\varphi(f) = \sum\{\psi(f)\colon \psi \in \mathcal{S}_\varphi\}$ (Theorem 2). These two results together give necessary and sufficient conditions when, for example, all functions in L are bounded or when, given $\{f_i\}_{i=1}^n \subset L$, there is always an $f \in L$ which is zero only when every f_i is zero; but it is left undecided whether or not $\mathcal{S}_\varphi$ can be infinite. In Section 3 necessary and sufficient conditions are found when L belongs to a fairly wide class of algebras, including those which contain $f^2(1+f^2)^{-1}$ when they contain f and those whose functions are all bounded.

Throughout the paper notations similar to $[f \geqslant g] = \{x \in X\colon f(x) \geqslant g(x)\}$ will be understood. The zero set, $\boldsymbol{Z}(f)$, of $f \in L$ is $[f = 0]$, and if $\mathfrak{Z} \subset L$ then $\boldsymbol{Z}(\mathfrak{Z}) = \{\boldsymbol{Z}(f)\colon f \in \mathfrak{Z}\}$. $L^+ = \{f \in L\colon [f \geqslant 0] = X\}$ and a linear functional, φ, on L is called *positive* if $\varphi(f) \geqslant 0$ when $f \in L^+$. The class of all bounded functions in L is written L^* and the empty set is denoted by $\emptyset$.

2. Linear lattices. In this section L is a linear lattice, so $\boldsymbol{Z}(L)$ is a lattice under set union and intersection. The usual apparatus of ideal theory is therefore available, though we shall find it more convenient to speak in terms of filters (i.e. dual ideals).

A *filter* is a subclass, $\mathcal{F}$, of $\boldsymbol{Z}(L)$ such that if $\mathfrak{Z}$ is a finite subclass of $\mathcal{F}$ with $\bigcap \mathfrak{Z} \subset Z \in \boldsymbol{Z}(L)$ then $Z \in \mathcal{F}$; it is a *σ-filter* if "finite" can be replaced by "countable" in this definition. A filter, $\mathcal{F}$, is *prime* if $\boldsymbol{Z}(L)\setminus\mathcal{F}$ is closed under finite unions and is *proper* if $\mathcal{F} \neq \boldsymbol{Z}(L)$.

A linear functional, φ, on L is a *lattice linear functional* if $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$ for all $f, g \in L$.

THEOREM 1. *The following statements are equivalent for any $\mathcal{F} \subset \boldsymbol{Z}(L)$;*

1. *$\mathcal{F}$ is a proper prime σ-filter,*

2. *There is a non-trivial σp-continuous lattice linear functional, φ, on L such that $\mathcal{F} = \{\boldsymbol{Z}(f)\colon \varphi(f) = 0\}$,*

3. *$\mathcal{F}$ is a σ-filter and a maximal proper filter.*

Proof. We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. Take $e \in L^+$ such that $\boldsymbol{Z}(e) \notin \mathcal{F}$. For any $f \in L$, let $A_f = \{r\colon r$ is a rational with $[f \geqslant re] \in \mathcal{F}\}$. If r is a rational not in A_f then $[f \leqslant re] \in \mathcal{F}$; if r is a rational less than a member of A_f then $r \in A_f$. Thus, A_f is bounded above, otherwise $\bigcap_{n=1}^{\infty} [f \geqslant ne] \cap (X\setminus\boldsymbol{Z}(e)) \neq \emptyset$; and A_f is non-empty, otherwise $\bigcap_{n=1}^{\infty} [f \leqslant ne] \cap (X\setminus\boldsymbol{Z}(e)) \neq \emptyset$. So $a = \sup A_f$ is a real number and $[f = ae] = \bigcap\{[f \geqslant re]\colon r \in A\} \cap \bigcap\{[f \leqslant re]\colon r$ is a rational not in $A_f\}$ is in $\mathcal{F}$. Clearly a is the only real number for which this holds; we may therefore denote it by $\widetilde{\mathcal{F}}(f/e)$. If $\{f_i\}_{i=1}^{\infty}$ is any countable subset of L, then $\left(\bigcap_{i=1}^{\infty}[f_i = \widetilde{\mathcal{F}}(f_i/e)e]\right) \cap (X\setminus\boldsymbol{Z}(e)) \neq \emptyset$; from which it follows that $\widetilde{\mathcal{F}}(\cdot/e)$ is a σp-continuous lattice linear functional with $\widetilde{\mathcal{F}}(e/e) = 1$. By our construction $\widetilde{\mathcal{F}}(f/e) = 0$ if and only if $Z(f) = [f = 0e] \in \mathcal{F}$.

$2 \Rightarrow 3$. If $\varphi(f_i) = 0$, $i = 1, 2, \ldots$, and $\boldsymbol{Z}(f) \supset \bigcap_{i=1}^{\infty} \boldsymbol{Z}(f_i)$, then $|f| \wedge \wedge\, (n\sum_{i=1}^{n}|f_i|) \nearrow |f|$; whence $\varphi(|f|) = 0$, $Z(f) \in \mathcal{F}$ and $\mathcal{F}$ is a σ-filter. Since we have in fact proved that $\varphi(f) = 0$ if and only if $\boldsymbol{Z}(f) \in \mathcal{F}$, $\mathcal{F}$ is proper. Now suppose $\boldsymbol{Z}(g) \notin \mathcal{F}$ and consider the filter $\mathcal{F}_1$ generated by $\mathcal{F} \cup \{\boldsymbol{Z}(g)\}$. For any $f \in L$, $[\varphi(f)g = \varphi(g)f] \cap Z(g) \subset \boldsymbol{Z}(f)$, so $\boldsymbol{Z}(f) \in \mathcal{F}_1$. Thus $\mathcal{F}$ is a maximal proper filter.

$3 \Rightarrow 1$. The standard type of argument proves that $\mathcal{F}$ is prime.

If $\mathcal{F}$ is any proper prime σ-filter and $Z(e) \notin \mathcal{F}$, then we shall write $\widetilde{\mathcal{F}}(\cdot/e)$ for the σp-continuous lattice linear functional with $\mathcal{F}$ for the filter of zero sets of the functions with zero value and for which the value at e is 1.

The following two lemmas are needed to simplify the proof of the main result (Theorem 2); the first one is, of course, true for more general systems.

LEMMA 1. *Any filter $\mathcal{F}$ is the intersection of the minimal prime filters containing it.*

Proof. If $Z \in \boldsymbol{Z}(L)\setminus\mathcal{F}$ we show that there is a minimal prime filter $\mathcal{I}$ containing $\mathcal{F}$ but not containing Z. In fact, let $\mathcal{M}$ be a maximal filter containing $\mathcal{F}$ but not containing Z. Then $\mathcal{M}$ is prime and we can take $\mathcal{I}$ to be a minimal prime filter containing $\mathcal{F}$ and contained in $\mathcal{M}$.

LEMMA 2. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two filters neither of which is contained in the other then there are $f_1, f_2 \in L^+$ with $f_1 \wedge f_2 = 0$, $\boldsymbol{Z}(f_1) \in \mathcal{F}_2\setminus\mathcal{F}_1$ and $\boldsymbol{Z}(f_2) \in \mathcal{F}_1\setminus\mathcal{F}_2$.*

Proof. If $g_1, g_2 \in L^+$ with $\boldsymbol{Z}(g_1) \in \mathcal{F}_2\setminus\mathcal{F}_1$ and $\boldsymbol{Z}(g_2) \in \mathcal{F}_1\setminus\mathcal{F}_2$ we take $f_1 = (g_1 - g_2)^+$ and $f_2 = (g_1 - g_2)^-$. Then $\boldsymbol{Z}(f_1) \supset \boldsymbol{Z}(g_1)$, so $\boldsymbol{Z}(f_1) \in \mathcal{F}_2$, and $\boldsymbol{Z}(f_1) \cap Z(g_2) \subset \boldsymbol{Z}(g_1)$, so $\boldsymbol{Z}(f_1) \notin \mathcal{F}_1$. Similarly $\boldsymbol{Z}(f_2) \in \mathcal{F}_1\setminus\mathcal{F}_2$.

We can now prove

THEOREM 2. *If φ is a σp-continuous linear functional on L, then there is, on L, a unique lineally independent class $\mathcal{S}_\varphi$ of lattice linear functionals, or the negatives of lattice linear functionals, such that for each $f \in L$, $\psi(f) = 0$ for all but a finite number of ψ in $\mathcal{S}_\varphi$ and*

$$\varphi(f) = \sum\{\psi(f)\colon \psi \in \mathcal{S}_\varphi\}.$$

Furthermore every $\psi \in \mathcal{S}_\varphi$ is σp-continuous.

Proof. 1. It is easily seen that if $f \in L^+$, the set $\{\varphi(g)\colon |g| \leqslant f\}$ is bounded; so that ([2], pp. 35-36) we can define a positive linear functional $|\varphi|$ on L by:

$$|\varphi|(f) = \sup\{\varphi(g)\colon |g| \leqslant f\}\,, \qquad f \in L^+,$$
$$|\varphi|(f) = |\varphi|(f^+) - |\varphi|(f^-)\,, \qquad f \in L\,.$$

Clearly

$$|\varphi(f)| \leqslant |\varphi|(|f|)\,, \qquad f \in L\,.$$

Furthermore, $|\varphi|$ is σp-continuous. For if $\{f_i\}_{i=1}^\infty \subset L$ and $f_i \to 0$ then $|f_i| \to 0$. Now choose $\{g_i\}_{i=1}^\infty \subset L$ with $|g_i| \leqslant f_i$ and $|\varphi|(|f_i|) < \varphi(g_i) + 1/i$. Then $g_i \to 0$ and so $|\varphi|(|f_i|) \to 0$; whence $|\varphi|(f_i) \to 0$.

2. Following Mrówka [3], we define:

$$N_\varphi = \{f \in L^+\colon |\varphi|(f) = 0\}\,, \qquad \mathfrak{Z}_\varphi = \boldsymbol{Z}(N_\varphi)\,.$$

$\mathfrak{Z}_\varphi$ is a σ-filter and if $\boldsymbol{Z}(f) \in \mathfrak{Z}_\varphi$, then $|\varphi|(|f|) = 0$.

If $\{f_i\}_{i=1}^\infty \subset N_\varphi$ and $\boldsymbol{Z}(f) \supset \bigcap_{i=1}^{\infty} \boldsymbol{Z}(f_i)$, then $|f| \wedge n(\sum_{i=1}^{n} f_i) \nearrow |f|$; so $|\varphi|(|f|) = 0$ and $\boldsymbol{Z}(f) \in \mathfrak{Z}_\varphi$.

3. Let $\mathfrak{P}_\varphi$ be the class of minimal prime filters containing $\mathfrak{Z}_\varphi$.

For each $\mathfrak{J}_0 \in \mathfrak{P}_\varphi$ there is a $Z \in \boldsymbol{Z}(L)$ such that $\{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z \notin \mathfrak{J}\} = \{\mathfrak{J}_0\}$.

For if not, there is a $\mathfrak{J}_0 \in \mathfrak{P}_\varphi$ such that if $Z \in \boldsymbol{Z}(L) \setminus \mathfrak{J}_0$, then $\{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z \notin \mathfrak{J}\}$ is infinite. Pick $\mathfrak{J}_1 \in \mathfrak{P}_\varphi \setminus \mathfrak{J}_0$ and choose f_1, g_1 with $f_1 \wedge g_1 = 0$, $\boldsymbol{Z}(f_1) \in \mathfrak{J}_0 \setminus \mathfrak{J}_1$ and $\boldsymbol{Z}(g_1) \in \mathfrak{J}_1 \setminus \mathfrak{J}_0$. Since $\boldsymbol{Z}(e_1) \notin \mathfrak{J}_0$ there is $\mathfrak{J}_2 \in \mathfrak{P}_\varphi$ with $\boldsymbol{Z}(g_1) \notin \mathfrak{J}_2$. Choose $h_1, k_1 \in L$ with $h_1 \wedge k_1 = 0$, $\boldsymbol{Z}(h_1) \in \mathfrak{J}_0 \setminus \mathfrak{J}_2$ and $\boldsymbol{Z}(k_1) \in \mathfrak{J}_2 \setminus \mathfrak{J}_0$. Put $f_2 = g_1 \wedge h_1$, $g_2 = g_1 \wedge k_1$. Then $0 \leqslant f_1 \wedge f_2 \leqslant f_1 \wedge g_1 \wedge h_1 = 0$, $\boldsymbol{Z}(f_1) \notin \mathfrak{J}_1$, $\boldsymbol{Z}(f_2) \notin \mathfrak{J}_2$ and $\boldsymbol{Z}(g_2) \notin \mathfrak{J}_0$. Continuing in this way we construct a sequence $f_1, f_2, \ldots$ of functions in L^+ such that $f_i \wedge f_j = 0$ when $i \neq j$ and $\varphi(f_i) > 0$ for all i (since each $\mathfrak{J}$ in $\mathfrak{P}_\varphi$ contains $\mathfrak{Z}_\varphi$). But this gives the contradiction, $\varphi\big(f_i/\varphi(f_i)\big) \equiv 1$ while $f_i/\varphi(f_i) \to 0$.

4. *Each $\mathfrak{J} \in \mathfrak{P}_\varphi$ is a proper σ-filter and so a maximal proper filter.*

If $Z_i \in \mathfrak{J}_0 \in \mathfrak{P}_\varphi$, $i = 1, 2, \ldots$, and $\bigcap_{i=1}^{\infty} Z_i \subset Z \in \boldsymbol{Z}(L)$, then, picking $Z_0 \in \boldsymbol{Z}(L)$ such that $\{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z_0 \notin \mathfrak{J}\} = \{\mathfrak{J}_0\}$, we find that $Z \cup Z_0 \supset \bigcap_{i=1}^{\infty} (Z_i \cup Z)$. Now $Z_i \cup Z_0 \in \mathfrak{Z}_\varphi$ for all i; so $Z \cup Z_0 \in \mathfrak{Z}_\varphi \subset \mathfrak{J}_0$ and, $\mathfrak{J}_0$ being prime, $Z \in \mathfrak{J}_0$.

5. *For each $Z \in Z(L)$ the set $\{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z \notin \mathfrak{J}\}$ is finite.*

For if this is not so for $Z_0 \in Z(L)$, then the class $\mathcal{F} = \{Z \in \boldsymbol{Z}(L)\colon \{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z \notin \mathfrak{J}\}$ is finite$\}$ is a filter containing $\mathfrak{Z}_\varphi$ but not containing Z_0. Let $\mathcal{M}$ be a maximal filter containing $\mathfrak{Z}_\varphi$ but not containing Z_0. Then $\mathcal{M}$ is prime, so it contains an element, $\mathfrak{J}$ say, of $\mathfrak{P}_\varphi$. Now $\mathfrak{J}$ is a maximal proper filter and $\mathcal{M}$ is proper, so $\mathcal{M} = \mathfrak{J}$; but this is impossible by 3.

6. *For each $\mathfrak{J} \in \mathfrak{P}_\varphi$ pick $e_{\mathfrak{J}} \in L^+$ with $\boldsymbol{Z}(e_{\mathfrak{J}})$ in every element of $\mathfrak{P}_\varphi$ but $\mathfrak{J}$ and with $\varphi(e_{\mathfrak{J}}) = 1$. If $\mathcal{S}_\varphi = \{\widetilde{\mathfrak{J}}(\cdot/e_{\mathfrak{J}})\colon \mathfrak{J} \in \mathfrak{P}_\varphi\}$, then, for any $f \in L$, $\psi(f) = 0$ for all but a finite number of $\psi \in \mathcal{S}_\varphi$ and*

$$\varphi(f) = \Sigma\{\psi(f)\colon \psi \in \mathcal{S}_\varphi\}\,.$$

For let $\mathfrak{P}_{\varphi,f} = \{\mathfrak{J} \in \mathfrak{P}_\varphi\colon \boldsymbol{Z}(f) \notin \mathfrak{J}\}$. This is finite, and $\widetilde{\mathfrak{J}}(f/e) = 0$ when $\mathfrak{J} \notin \mathfrak{P}_{\varphi,f}$. Now

$$\boldsymbol{Z}\big(f - \Sigma\{\widetilde{\mathfrak{J}}(f/e)e\colon \mathfrak{J} \in \mathfrak{P}_{\varphi,f}\}\big) \in \mathfrak{Z}_\varphi\,,$$

since it is in every element of $\mathfrak{P}_\varphi$. Therefore

$$\varphi(f) - \Sigma\{\widetilde{\mathfrak{J}}(f/e_{\mathfrak{J}})\colon \mathfrak{J} \in \mathfrak{P}_f\} = 0\,,$$

that is,

$$\varphi(f) = \Sigma\{\psi(f)\colon \psi \in \mathcal{S}_\varphi\}\,.$$

7. Proof of uniqueness. Suppose $\mathcal{S}'_\varphi$ is a lineally independent class of lattice linear functionals or negatives of lattice linear functionals such that, for each $f \in L$, f has zero value for all but a finite subclass of $\mathcal{S}'_\varphi$ and $\varphi(f) = \Sigma\{\psi'(f)\colon \psi' \in \mathcal{S}'_\varphi\}$. If $\psi'_0 \in \mathcal{S}'_\varphi$ is not a non-zero multiple of any $\psi \in \mathcal{S}_\varphi$ we could find $e \in L$ with $\psi'_0(e) = 1$, $\psi'(e) = 0$ for $\psi' \in \mathcal{S}'_\varphi \setminus \{\psi'_0\}$ and with $\psi(e) = 0$ for all $\psi \in \mathcal{S}_\varphi$. But this gives the contradiction, $1 = \Sigma\{\psi'(e)\colon \psi' \in \mathcal{S}'_\varphi\} = \varphi(e) = \Sigma\{\psi(e)\colon \psi \in \mathcal{S}_\varphi\} = 0$. Therefore $\psi'_0 = \lambda\psi_0$ for some $\psi_0 \in \mathcal{S}_\varphi$ and some $\lambda \neq 0$. A similar sort of argument now shows that $\lambda = 1$.

It is tempting to try to show that $\mathcal{S}_\varphi$ must be finite. So far I have not succeeded in doing this, nor do I have an example showing that it need not be finite. However, the following corollaries give a number of useful conditions guaranteeing that $\mathcal{S}_\varphi$ is finite:

Corollary 1. *If for every countable $\mathfrak{Z} \subset \boldsymbol{Z}(L)$ there is a $Z \in \boldsymbol{Z}(L)$ with $Z \subset \bigcap \mathfrak{Z}$ (in particular if L contains a function which is nowhere zero), then $\mathcal{S}_\varphi$ is finite.*

Proof. If $\{\mathfrak{J}_i\}_{i=1}^\infty$ are distinct elements of $\mathfrak{P}_\varphi$, pick $Z_i \in \boldsymbol{Z}(L) \setminus \mathfrak{J}_i$, $i = 1, 2, \ldots$, and $Z \subset \bigcap_{i=1}^{\infty} Z_i$. Then Z is in no $\mathfrak{J}_i$, contradicting Part 3 of the proof.

Corollary 2. *If $Z(L)$ is closed under countable unions, then $\mathcal{S}_\varphi$ is finite.*

Proof. If $\{\mathfrak{J}_i\}_{i=1}^\infty$ are distinct elements of $\mathfrak{P}_\varphi$, then for each i pick $e_i \in L^+$ such that $\{\mathfrak{J} \in \mathfrak{P}_\varphi\colon Z(e_i) \notin \mathfrak{J}\} = \{\mathfrak{J}_i\}$ and for each $n = 1, 2, \ldots$ choose a finite sequence $\{f_{i,n}\}_{i=1}^n \subset L^+$ with $f_{i,n} \wedge f_{j,n} = 0$, $i \neq j$, $f_{i,n} \leqslant e_i$ and $[f_{i,n} = e_i] \in \mathfrak{J}_i$. Put $X_i = \bigcap_{n=i}^{\infty} [f_{i,n} = e_i]$. Pick $g_i \in L^+$ with $\boldsymbol{Z}(g_i) = Z_i = \bigcap_{n=i}^{\infty} \boldsymbol{Z}(f_{i,n})$. Now $Z_i \cap X_i \subset \boldsymbol{Z}(e_i)$ so $Z_i \notin \mathfrak{J}_i$ and $|\varphi|(g_i) \neq 0$. From this we get the contradiction: $|\varphi|\big(g_i/|\varphi|(g_i)\big) \equiv 1$ while $\big(g_i/|\varphi|(g_i)\big) \to 0$.

Let us say that $\{X, L\}$ is *σ-bounded* under decomposition $\{X_i\}_{i=1}^{\infty}$ if $\{X_i\}_{i=1}^{\infty}$ is an increasing sequence of subsets of X with $\bigcup_{i=1}^{\infty} X_i = X$ and each $f \in L$ is bounded on every X_i. (Any linear lattice of continuous real-valued functions on a σ-compact space (e.g. R^n) is σ-bounded.) For such a system let L^σ be the class of all limits of sequences of functions from L which converge uniformly on each X_i. It is easily proved that $\{X, L^\sigma\}$ is σ-bounded under decomposition, $\{X_i\}_{i=1}^{\infty}$, that $L^{\sigma\sigma} = L^\sigma$ and that any σp-continuous linear functional, φ, on L extends uniquely to a σp-continuous linear functional, φ^σ, on L^σ. The preceding remarks are, of course, applicable to any linear space.

We can now state:

COROLLARY 3. *If $\{X, L\}$ is σ-bounded under decomposition $\{X_i\}_{i=1}^{\infty}$ then $\mathcal{S}_\varphi$ is finite.*

Proof. φ extends to φ^σ on L^σ. If $\{f_i\}_{i=1}^{\infty} \subset L^\sigma$ and for any $i = 1, 2, \ldots$ and any $g \in L^\sigma$ we let $\|g\|_i = \sup\{|g(x)|: x \in X_i\}$, then $f = \sum_{i=1}^{\infty} (|f_i|(2^i\|f\|_i)^{-1}) \in L$ and $\boldsymbol{Z}(f) = \bigcap_{i=1}^{\infty} \boldsymbol{Z}(f_i)$, so $\{X, L^\sigma\}$ satisfies the conditions of Corollary 1. Therefore $\mathcal{S}_{\varphi^\sigma}$ is finite and, by uniqueness, $\mathcal{S}_\varphi$ is finite.

Let $B(L)$, the *Baire class generated by* L, be the smallest class of real-valued functions on X which contains L and is closed under pointwise convergence of sequences. $B(L)$ is a linear lattice. We have:

LEMMA 2. *Any σp-continuous lattice linear functional on L can be uniquely extended to a σp-continuous lattice linear functional, φ^B, on $B(L)$.*

Proof. Define $\mathfrak{Z}_\varphi = \{\boldsymbol{Z}(f): f \in L \text{ and } \varphi(f) = 0\}$ and take $e \in L^+$ with $\varphi(e) = 1$. For any real number λ, let $B_\lambda = \{f: f$ is a real-valued function X with $[f = \lambda e]$ containing the intersection of a countable number of sets from $\mathfrak{Z}_\varphi\}$. Put $B = \bigcup\{B_\lambda: \lambda \text{ real}\}$. Then B is a linear lattice containing L and closed under pointwise convergence of sequences. Further, if we define $\bar\varphi(f) = \lambda$ for all $f \in B_\lambda$, then this is a proper definition and φ is a σp-continuous lattice linear functional on B with $\varphi_B = \bar\varphi | B(L)$ as required.

The following corollary to Theorem 2 is now immediate:

COROLLARY 4. *If L_0 is a linear lattice with $L_0 \subset L \subset B(L_0)$ and if $\mathcal{S}_{\varphi|L_0}$ is finite, then $\mathcal{S}_\varphi$ is finite.*

To combine the two theorems just proved let us say that a prime proper (maximal proper) σ-filter $\mathcal{F}$ is *determined* by $x \in X$ if $\mathcal{F} = \{Z \in \boldsymbol{Z}(L): x \in Z\}$. Let the system $\{X, L\}$ be called *real-compact* if every prime proper σ-filter is determined by a point in X. The justification for this definition is that when L is $C(X)$ for a completely regular Hausdorff space X it is often taken as the definition of real-compactness for X. We have

THEOREM 3. *If $\{X, L\}$ is real-compact and φ is a σp-continuous linear functional on L, then there are points $\{x_i\}_{i\in I}$ in X and real numbers $\{\lambda_i\}_{i\in I}$ such that for any $f \in L$, $f(x_i) = 0$ for all but a finite number of $i \in I$ and*

$$\varphi(f) = \sum_{i\in I} \lambda_i f(x_i).$$

If $\{X, L\}$ is not real-compact, then there is a non-trivial σp-continuous lattice linear functional which is not of this form.

Corollaries analogous to those of Theorem 2 are immediately obvious.

3. Algebras.

In this section L is an algebra of real-valued functions. Results are obtained by using the techniques of the proof of the Stone Weierstrass Theorem ([1], p. 55) to reduce the problem, where possible, to one in linear lattices.

A linear functional, φ, on L is called *multiplicative* if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in L$.

A *stationary set* for $f \in L$ is a set of the form $[f = a]$ for some real number a. A *complete class* of stationary sets is one that contains one set for each $f \in L$; it has the *countable intersection property* if the intersection of any countable subclass is non-empty.

LEMMA 4. *Let L be a linear lattice besides being an algebra. Then a non-trivial functional, φ, on L is a p-continuous positive multiplicative linear functional on L if and only if $\{[f = \varphi(f)]: f \in L\}$ is a complete class of stationary sets with the countable intersection property.*

Proof. (i) *if*. The proof of this is immediate.

(ii) *only if*. Suppose $\bigcap_{i=1}^{\infty} [f_i = \varphi(f_i)] = \emptyset$. Then if $g \in L^+$ with $\varphi(g) > 0$ we have $g \wedge n \sum_{i=1}^{n} [f_i g - \varphi(f_i) g]^2 \nearrow g$. Whence $\varphi(g) = 0$, contrary to hypothesis.

COROLLARY 1. *Every σp-continuous positive multiplicative linear functional, φ, on L is uniquely extendable to a σp-continuous positive multiplicative linear functional, φ_B, on $B(L)$.*

Proof. If $\varphi = 0$ the result is trivial. If not, then the class of all real-valued functions which are constant on some countable intersection of sets of the form $[f = \varphi(f)]$ contains $B(L)$. If $f \in B(L)$ is always equal to λ on such a set define $\varphi_B(f) = \lambda$. Uniqueness is immediate.

COROLLARY 2. *Every σp-continuous positive multiplicative linear functional, φ, on L is of the form $f \to f(x)$ for some $x \in X$ if and only if every* *complete class of stationary sets with the countable intersection property has a non-empty intersection.*

LEMMA 5. *If L is a linear lattice and an algebra, then every non-trivial σp-continuous positive lattice linear functional, φ, on L is a positive multiple of a multiplicative linear functional.*

Proof. If $e \in L^+$ with $\varphi(e) = 1$, then $\varphi(e^2) > 0$, since $Z(e) = Z(e^2)$. Let $\psi(f) = \varphi(e^2)\varphi(f)$ for all $f \in L$. If $f, g \in L$, then there is an $x \in X \setminus Z(e)$ with

$$\varphi(f) = \varphi(e^2)\varphi(f) = \frac{e^2(x)}{e(x)} \cdot \frac{f(x)}{e(x)} = f(x), \quad \psi(g) = g(x) \quad \text{and} \quad \psi(fg) = (fg)(x).$$

So $\psi(fg) = \psi(f)\psi(g)$.

We now have

THEOREM 4. *If L is σ-bounded under decomposition $\{X_i\}_{i=1}^{\infty}$ and φ is a σp-continuous linear functional on L, then there are non-trivial σp-continuous positive multiplicative linear functionals $\varphi_1, \ldots, \varphi_n$ on L and real numbers $\lambda_1, \ldots, \lambda_n$ such that $\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i$.*

Proof. L^σ is a σ-bounded algebra under decomposition $\{X_i\}_{i=1}^{\infty}$ and $L^{\sigma\sigma} = L^\sigma$. For any $f \in L^\sigma$ and any X_i there is a polynomial, $P_i(f)$, in f such that $P_i(0) = 0$ and $\|P_i(f) - |f|\|_i < 1/i$, so L^σ is a linear lattice. Now φ extends to a σp-continuous linear functional φ^σ on L. The result now follows from Theorem 2 and Lemma 5.

COROLLARY. *If L is an algebra with a σ-bounded subalgebra L_0 such that $B(L_0) \supset L$ (e.g. if $B(L^*) \supset L$), then the conclusion still holds.*

Proof. This follows from the Theorem and Corollary 2 to Lemma 4.

Among the algebraic conditions on L which guarantee $B(L^*) \supset L$ we might mention:

(i) $f/(1+f) \in L^+$ when $f \in L^+$.

(ii) $f^2/(1+f^2) \in L^+$ for all $f \in L$.

From Theorem 4, its corollary and Lemma 4, Corollary 2 we have

THEOREM 5. *If L is as in Theorem 4 (or its corollary), then a necessary and sufficient condition that every σp-continuous linear functional, φ, on L be of form (P) is that every complete class of stationary sets with countable intersection property have non-empty intersection.*

Note added in proof. Results closely related to those of this paper have been obtained, under stronger hypotheses, by J. R. Isbell and E. S. Thomas Jr., Proc. Amer. Math. Soc. 14 (1963). pp. 644—647.

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Reçu par la Rédaction le 4. 3. 1963