arc $A_{pi}P_{pi}Q_{pi}B_{pi}$ intersects every link of D_p, every non-end link of D_p contains a link of γ'_{pi}. In particular, the non-end link Y of D_p which intersects E_{pZ} contains a link of γ'_{pi} and therefore a point Q of $K+k_1+k_2$. Then Q is at distance from Z less than $2/p<\varepsilon$, a contradiction. Thus, no composant of T contains two components of M.

The following theorem can be proven by an argument which is a simplification of the argument for Theorem 10:

THEOREM 11. *A closed and compact point set M is a subset of a chainable continuum if and only if every component of M is either a single point or a chainable continuum.*

Indeed, if M is such a closed and compact point set, there exists an indecomposable chainable continuum T, containing M, no composant of which contains two components of M.

References

[1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), pp. 729-742.

[2] — *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653-663.

[3] G. W. Henderson, *Proof that every compact decomposable continuum which is topologically equivalent to each of its nondegenerate subcontinua is an arc*, Ann. of Math., Ser. 2, 72 (1960), pp. 421-428.

[4] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), pp. 581-594.

[5] — *A note on the pseudo-arc*, ibidem 67 (1949), pp. 57-58.

[6] R. L. Moore and J. R. Kline, *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Ann. of Math., Ser. 2, 20 (1919), pp. 218-223.

[7] R. L. Moore, *Foundations of Point Set Theory*, Amer. Math. Soc. Colloquium Publications 13, revised edition, New York 1962.

THE UNIVERSITY OF TEXAS

Reçu par la Rédaction le 19. 2. 1963

Metric characterizations of Banach and Euclidean spaces

by

E. Z. Andalafte and **L. M. Blumenthal** (Columbia, Missouri)

Introduction. One of the most important classes of metric spaces, both intrinsically and for its great usefulness in theoretical physics, is formed by the (metrically) complete, normed, linear spaces. This class was axiomatized and studied by Banach in 1922, and in the same year (and quite independently) the class was defined and investigated both by Hahn and by Wiener.

The usual formulation of the abstract Banach space (over the reals) is in terms of three primitive concepts: (1) *addition* (with each (ordered) pair of elements x, y there is associated a unique element $x+y$—the *ordered sum* of x and y); (2) *scalar multiplication* (with each real number λ and each element x there is associated a unique element $\lambda\cdot x$—the *scalar multiple* of x by λ); and (3) *normation* (with each element x there is associated a unique real number $\|x\|$—the *norm* of x). The three primitive notions are subjected to ten postulates, which are stated in another part of this paper. A normed linear space is a Banach space provided it is *complete* (that is, if $\{x_n\}$ is an infinite sequence of elements such that

$$\lim_{i,j\to\infty}\|x_i+(-1)x_j\|\equiv\lim_{i,j\to\infty}\|x_i-x_j\|=0,$$

then an element x exists such that $\lim\limits_{i\to\infty}|x-x_i|=0$).

The concept of *distance* is introduced in a normed linear space by defining the distance xy of two elements x, y to be the norm of their difference,

$$xy=\|x-y\|,$$

and it is easily seen that in terms of this definition, every normed linear space is a *metric* space (that is, (1) $xy\geqslant 0$, (2) $xy=0$ if and only if $x=y$, (3) $xy=yx$, and (4) $xy+yz\geqslant xz$ for each three elements x, y, z of the space). The class of Banach spaces is, therefore, a (proper) subclass of the class of all metric spaces, and the problem arises of characterizing metrically this subclass among the members of the whole class. More precisely, the problem is to obtain conditions, expressed *wholly* and *explicitly* in terms of the metric, in order that an arbitrary metric space may be a Banach space, with distance expressed in terms of the norm in the usual way. Alternatively, the problem is to define a Banach space in terms of just *one* primitive notion (distance) instead of the customary three concepts stated above. This problem is solved in Part I of this paper, with the unimportant (and easily removed) restriction that there is not more than one metric line joining two distinct points of the space.

Previous contributions to this problem were made by Aronszajn and by Fréchet.

In 1935 two short articles by Aronszajn [1] appeared in the *Comptes Rendus de l'Académie des Sciences*, Paris, in which a metric characterization of normed linear spaces was stated. The characterization, which appeared without proof, made use of the concept of center of symmetry, $m(p, q)$, of a symmetry f transforming a point p into a point q, and the following interesting property: if x, y, and z are points of a metric space G, then $m(x, z)m(y, z) = \frac{1}{2}xy$, where juxtaposition of two symbols is used to denote the distance of the points represented by them. This property is a metric restatement and generalization of a condition used by Young [6] in 1911 to distinguish Euclidean geometry from hyperbolic and elliptic geometries in a space satisfying the axioms of Hilbert's groups I, II, III, and V, namely the axioms of connection, order, congruence, and continuity.

In 1958 Fréchet [5] provided metric definitions of sum, product by a scalar, and norm for elements in general metric spaces, by means of the concept of "center of gravity" of pairs of points. His work also provided conditions under which the sum, scalar product, and norm are uniquely defined, but it *did not yield a characterization of normed linear spaces among metric spaces.*

For our characterization theorem of Part I we make basic use of our complete metrization of Aronszajn's assumption (which we had, in fact, formulated before observing its connection with the form in which Aronszajn used it) which we call the Young Postulate because of its similarity to the condition used by Young:

THE YOUNG POSTULATE. *If p, q, and r are points of a metric space M, and if q' and r' are the mid-points of p and q, and of p and r, respectively, then $q'r' = \frac{1}{2}qr$.*

The first major result we obtain is that *a complete metric space with a unique metric straight line joining any pair of its distinct points, is a normed linear space* (*Banach space*) *if and only if it satisfies the Young Postulate.*

Part II of this article is based on our metrization of a norm postulate used by Ficken [4] who showed that an inner product can be defined in a complete, real, normed linear space if and only if, for elements p and q of the space, the equality of the norms of p and q,

$$\|p\| = \|q\|,$$

implies $\|\lambda\cdot p+\mu\cdot q\| = \|\mu\cdot p+\lambda\cdot q\|$ for all real numbers λ and μ. It is natural to seek a metrization of this statement and to apply it to finitely compact metric spaces with unique straight lines which satisfy the Young Postulate, since such spaces are shown in Part I to be normed linear spaces.

Ficken's postulate may be written

$$\|p+q\| = \|p-q\| \quad \text{implies} \quad \|\lambda\cdot p+\mu\cdot q\| = \|\lambda\cdot p-\mu\cdot q\|,$$

for all real numbers λ and μ, so it is natural to metrize the property in the following way:

THE FICKEN POSTULATE. *If f is a foot of a point p on a line L (p not on L) and if q and r are points of L with $fq = fr$, then $sq = sr$ for each point s of a line $L(p, f)$ joining p and f.*

This postulate is then used to provide a metric characterization of Euclidean spaces as *finitely compact metric spaces with unique straight lines, in which the Ficken and Young postulates are satisfied.* It is to be noted that *the proof of this characterization theorem is purely metric in form, making no explicit use of the properties of normed linear spaces.* We thus obtain, incidentally, a generalization of Ficken's theorem which concerned only normed linear spaces.

I. Characterization of normed linear spaces with unique metric lines

1. Preliminary remarks. Before proceeding with the characterization of Euclidean spaces and certain normed linear spaces, it is necessary to give some fundamental definitions and state some theorems needed in the sequel. It is assumed that the reader is familiar with the notion of metric space. Let us suppose, then, that M is a metric space, with the distance of points p and q of M denoted by pq.

If p and q are distinct points of M, a point r of M is *between* p and q provided $p \neq r \neq q$ and $pr+rq = pq$. This relation is denoted by the symbol prq. The metric space M is said to be *convex* provided it contains, for each pair of its distinct points, at least one between point. M is *externally convex* provided for every pair of points p and q of M, with $p \neq q$, there exists a point r of M such that the relation pqr holds. A metric space is *complete* provided every Cauchy sequence of its points has a limit in the space, and is *finitely compact* provided every bounded infinite subset of its points has an accumulation point.

Two subsets S and S' of metric spaces are *congruent* provided there exists a one-to-one, distance-preserving mapping f of S onto S'. The mapping f is referred to as a *congruence*. Defining a *metric line* as a subset of M which is congruent with the Euclidean line E_1, we have the following theorem.

THEOREM 1.1. *Each two distinct points of a complete, convex externally convex metric space are on a metric line of the space.*

A triple of points is said to be *linear* provided it is congruent with a triple of points of the Euclidean line.

DEFINITION 1.1. A metric space has the *two-triple property* provided for each of its quadruples of pairwise distinct points, linearity of any two triples implies linearity of the remaining two triples.

Now let M denote a metric space of at least two points which is (1) complete, (2) convex, (3) externally convex, and (4) has the two-triple property. For such spaces M the following theorem has been established:

THEOREM 1.2. *Each two distinct points of M are elements of a unique metric line.*

Hereafter the metric line of distinct points p and q of M is referred to as the *line* of p and q, denoted $L(p,q)$. A (metric) *segment* joining p and q is any subset of M containing p and q, which is congruent with a Euclidean line segment of length pq. The unique segment joining distinct points p and q is denoted by $S(p,q)$.

For proofs of Theorems 1.1 and 1.2, and a detailed study of the concepts introduced above, the reader is referred to Blumenthal [2].

We conclude this section by introducing the concept *foot of a point on a line*, and proving two remarks concerning it. Let p be a point of M and let L denote a line of M.

DEFINITION 1.2. A point f_p of a line L is a *foot of the point p on L* provided $pf_p = \text{g.l.b.}\{px|\ x \text{ in } L\}$.

It can be shown (Blumenthal [2]) that if p is a point of M, and L is a line of M, there exists at least one foot f_p of p on L.

Remark 1. *If p is a point of M not lying on L, and if k is a real number, $k \geqslant pf_p$, then in each half-line of L determined by and containing f_p, there is a point s, such that $ps = k$.*

Proof. By the triangle inequality, $px > |f_px - f_pp|$ for each point x of L. But f_pp is a constant, and for x in either of the two half-lines of L determined by and containing f_p, it follows from the congruence of L with E_1 that f_px assumes every non-negative value. If $N > k$, then each such half-line contains a point t such that

$$pt \geqslant |f_pt - f_pp| > N.$$

But px is a continuous function of x on the segment $S(f_p, t)$, and consequently takes on every value between any two of its values. Thus, since $pf_p \leqslant k < N < pt$, there is a point s in the segment $S(f_p, t)$ of L, with $ps = k$.

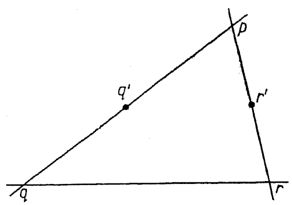
Remark 2. *If f_p is any foot of p on L and if s is a point of L with $f_ps > 2\cdot pf_p$, then $ps > pf_p$, i.e., s is not a foot of p on L.*

Proof. If $f_ps > 2\cdot pf_p$, then $f_pp + ps \geqslant f_ps > 2\cdot pf_p$, and hence $ps > pf_p$, completing the proof.

2. Some consequences of the Young postulate.

As was indicated in the Introduction, the Young Postulate plays an important role in the characterization of complete normed linear spaces with unique metric lines. Consequently, we begin this study with an investigation of some of the consequences of the supposition that a metric space M with properties 1, 2, 3, and 4 satisfies the Young Postulate. The Young Postulate leads first to many important intersection theorems for lines and triangles, then to a definition of a plane, which is uniquely determined by any non-collinear triple of its points, and finally to a proof of the Pasch axiom and the Euclidean parallel postulate.

One of the first questions arising from consideration of the Young Postulate is that of its extension to a theorem stating that if q' and r' are points of segments $S(p,q)$ and $S(p,r)$ respectively, with $pq'/pq = \lambda = pr'/pr$, then $q'r' = \lambda\cdot qr$. This section is devoted to a proof of this extension theorem.



Notation. For Theorems 2.1 to 2.4, let p, q, and r denote pairwise distinct, non-linear points of M, and let q' and r' denote the mid-points of p and q, and of p and r, respectively, denoted by $q' = m(p,q)$ and $r' = m(p,r)$, respectively. The triangle determined by distinct points p, q, and r of M is denoted by $T(p,q,r)$. The *sides* of $T(p,q,r)$ are the lines $L(p,q)$, $L(p,r)$, and $L(q,r)$.

THEOREM 2.1. *If t is a point of $L(q,r)$ there is a point t' of $L(q',r')$ such that t' is the mid-point of p and t.*

Proof. Letting $t' = m(p,t)$, it follows from consideration of $T(p,q,t)$ that $q't' = \frac{1}{2}qt$, and from $T(p,r,t)$ that $r't' = \frac{1}{2}rt$. But $q'r' = \frac{1}{2}qr$, so it follows immediately that any relation of betweenness satisfied by points q, r, and t is also satisfied by points q', r', and t', and hence t' is on $L(q',r')$.

THEOREM 2.2. *If t' is a point of $L(q',r')$ there is a point t of $L(q,r)$ such that t' is the mid-point of p and t.*

Proof. The proof is similar to the above, where t is chosen so that $pt't$ holds and $pt = 2\cdot pt'$.

THEOREM 2.3. *If f is a foot of p on $L(q,r)$ and if f' is the intersection of $L(q',r')$ and $L(p,f)$, then f' is a foot of p on $L(q',r')$.*

Proof. If f' were not a foot, there would exist a point g' of $L(q', r')$ with $pg' < pf'$. But then there exists a point g on $L(q, r)$ with $pg'g$ holding and $pg = 2\cdot pg'$. Hence

$$pg = 2\cdot pg' < 2\cdot pf' = pf,$$

contrary to the fact that f is a foot of p on $L(q, r)$.

Similarly the following result may be obtained:

THEOREM 2.4. *If f' is a foot of p on $L(q', r')$ and if f is the intersection of $L(q, r)$ and $L(p, f')$, then f is a foot of p on $L(q, r)$.*

LEMMA 2.5. *If p, q, and r are non-linear points of M and q' and r' are points of $S(p, q)$ and $S(p, r)$ respectively with $pq' = \frac{3}{4}pq$ and $pr' = \frac{3}{4}pr$, then $q'r' = \frac{3}{4}qr$.*

Proof. Let q^*, r^*, and m be the mid-points of $S(p, q)$, $S(p, r)$, and $S(q, r)$ respectively, and let $s = m(q^*, m)$, and $t = m(r^*, m)$. Now

$$q's = \tfrac{1}{2}qm = \tfrac{1}{4}qr, \quad st = \tfrac{1}{2}q^*r^* = \tfrac{1}{4}qr, \quad tr' = \tfrac{1}{2}rm = \tfrac{1}{4}qr.$$

To show linearity of the points q', s, t, and r', let $n = m(q, m)$. Then there exists a point n' of $S(q', s)$ such that $n' = m(q^*, n)$. Applying Theorem 2.1 to $T(r, q, p)$ it follows that t lies in $S(q^*, r)$. Thus considering $T(q^*, n, r)$ it is seen that $S(q^*, m')$ intersects $L(t, n')$ at s, so s is on $L(t, n') = L(s, n') = L(q', s)$, and hence q, s, and t are linear. A similar argument shows that r' is on $L(q', s)$ and so s and t are on $L(q', r')$, and it follows that

$$q'r' = q's + st + tr' = \tfrac{3}{4}qr.$$

THEOREM 2.5. *If q' and r' are points of $S(p, q)$ and $S(p, r)$, respectively, with $pq' = \lambda\cdot pq$, and $pr' = \lambda\cdot pr$ $(0 \leqslant \lambda \leqslant 1)$, then $q'r' = \lambda\cdot qr$.*

Proof. The proof proceeds by showing inductively the validity of the theorem for dyadically rational numbers between 0 and 1 and then extending the result by the continuity of the metric.

Let $n > 0$, choose an integer k so that $1 \leqslant k \leqslant 2^n-1$, and select points a^*, a, and a' of $S(p, q)$ and b^*, b, and b' of $S(p, r)$ such that

$$pa^* = \frac{k-1}{2^n}pq, \quad pa = \frac{k}{2^n}pq, \quad pa' = \frac{k+1}{2^n}pq,$$

$$pb^* = \frac{k-1}{2^n}pr, \quad pb = \frac{k}{2^n}pr, \quad pb' = \frac{k+1}{2^n}pr.$$

Let $S(a^*, b^*)$ be divided into $k-1$ equal segments by points $m_1^*, m_2^*, \ldots, m_{k-2}^*$; let $S(a, b)$ be similarly divided into k equal segments by $m_1, m_2, \ldots, m_{k-1}$, and $S(a', b')$ be divided into $k+1$ equal segments by $m_1', m_2', \ldots, m_k'$.

Make the inductive assumption that

$$a^*b^* = \frac{k-1}{2^n}qr, \quad ab = \frac{k}{2^n}qr, \quad a'b' = \frac{k+1}{2^n}qr,$$

and the betweenness relations $m_{i-1}^*m_im_{i+1}'$ and $m_i^*m_im_i'$ hold $(i = 1, \ldots, k-1)$, with the convention that $m_0^* = a^*$, $m_0 = a$, $m_0' = a'$, and $m_{k-1}^* = b^*$, $m_k = b$, $m_{k+1}' = b'$.

The lemma anchors the inductive argument, so to complete the proof it is necessary to show that if a'' and b'' are the mid-points of a and a', and of b and b', respectively, then

$$a''b'' = \frac{2k+1}{2^{n+1}}qr,$$

and that betweenness relations similar to those of the m_i^*, m_i, and m_i' above are valid.

To complete the proof define points a_i $(i = 1, \ldots, 2k)$ such that

$$a_{2i} = m(m_i, m_i'), \quad a_{2i-1} = m(m_{i-1}, m_i') \quad (i = 1, 2, \ldots, k).$$

Now

$$a''b'' \leqslant a''a_1 + a_1a_2 + \ldots + a_{2k-1}a_{2k} + a_{2k}b'' = \frac{2k+1}{2^{n+1}}qr,$$

since

$$a''a_1 = a_{2k}b'' = a_ia_{i+1} = \frac{1}{2^{n+1}}qr \quad (i = 1, 2, \ldots, 2k-1).$$

Hence it suffices to show that a_i is a point of $S(a'', b'')$ for $i = 1, 2, \ldots, 2k$.

If n is a point of $S(a', m_1')$, then there exists a point n'' of $S(a_1, a'')$ such that $an''n$ holds. By the inductive assumption, the relation $a^*m_1m_2'$ holds, so aa_2m_2' holds, by Theorem 2.1. Applying Theorem 2.1 to $T(a, n, m_2')$, since $a_1 = m(a, m_1')$, then a_1 is a point of $S(n'', a_2)$ so a_1 lies on $L(n'', a_2) = L(a_1, a_2)$. But n'' lies on $L(a'', a_1)$ and hence $L(a'', a_1) = L(a_1, a_2)$. Similarly it can finally be shown that for each $i = 1, 2, \ldots, 2k$, a_i is in $S(a'', b'')$ and it follows that

$$a''b'' = \frac{2k+1}{2^{n+1}}qr.$$

The required betweenness, corresponding to the relations $m_{i-1}^*m_im_{i+k}'$ and $m^*m_im_i'$, follows from Lemma 2.5. Thus by induction on n, the theorem is valid for all dyadically rational numbers λ, with $0 \leqslant \lambda \leqslant 1$. But by continuity of the metric, the result then follows for each real λ $(0 \leqslant \lambda \leqslant 1)$ and the proof is complete.

COROLLARY. *Theorem 2.5 is valid (mutatis mutandis) for any real number $\lambda \geqslant 0$.*

The result of Theorem 2.5 permits an important extension of Theorems 2.1 through 2.4. The proof depends on the following easily proved lemma:

LEMMA 2.6. *If p is the mid-point of q and q', and of r and r', then $qr = q'r'$.*

Proof. If $p' = m(q', r)$ then $qr = 2\cdot pp' = q'r'$, by application of the Young Postulate to $T(q', q, r)$ and $T(r, q', r')$, respectively.

From this and theorem 2.5, the final theorem is easily obtained:

THEOREM 2.6. *If p, q, and r are non-linear points of M, q' and r' are points of $L(p,q)$ and $L(p,r)$ respectively, with q and q', and r and r', both on the same side of p or both on opposite sides of p, and if $pq/pq' = pr/pr'$, then for each point t of $L(q,r)$ there is a point t' common to $L(q', r')$ and $L(p,t)$ with $pt/pt' = pq/pq'$. Furthermore, if f is a foot of p on $L(q,r)$ then the intersection f' of $L(q', r')$ and $L(p,f)$ is a foot of p on $L(q', r')$.*

3. Intersection theorems for triangles. In order to arrive at a fruitful definition of the plane, it is necessary to understand something of the behavior of a line which intersects some pair of sides of a triangle. Certainly it can not be expected that every line intersecting a pair of sides of a triangle will intersect the third side, for the behavior of the lines $L(q,r)$ and $L(q',r')$ of Theorem 2.6 provides an example to the contrary. However, the Young Postulate does permit a proof that such lines form the only exception to the desired rule; that is, with the exception of the lines $L(q',r')$ of the triangle $T(p,q,r)$ of Theorem 2.6, any line intersecting two sides of a non-degenerate triangle in distinct points will intersect the third side of the triangle.

This section is devoted to the proof of this important result.

THEOREM 3.1. *If p, q, and r are non-linear points of M and the relations $pq'q$ and $pr'r$ hold for points q' and r' of M, with $pq'/pq \neq pr'/pr$, then there exists a point s of $L(q,r)$ such that $q'r's$ or $sq'r'$ holds.*

Proof. Let $pq'/pq = \lambda$ $(0<\lambda<1)$, $pr'/pr = \mu$ $(0<\mu<1)$, and relabel, if necessary, so that $\lambda > \mu$. Now since $pr'/\lambda < pr'/\mu = pr$, a point t of $S(p,r)$ can be found with $pt = pr'/\lambda$. Then $pr'/pt = pq'/pq = \lambda$, so it follows that $q'r'/qt = \lambda$. Now

$$rt = rp - pt = rp - \frac{pr'}{\lambda} = rp - \frac{\mu}{\lambda}\cdot rp = \frac{\lambda-\mu}{\lambda}\cdot rp$$

so

(1) $$\frac{rt}{rp} = \frac{\lambda-\mu}{\lambda}.$$

Since $0<(\lambda-\mu)/\lambda<1$, a point u exists such that rqu holds and

$$ru = \frac{\lambda}{\lambda-\mu}\cdot rq.$$

Thus $rq/ru = (\lambda-\mu)/\lambda$, so by (1) and Theorem 2.5, it follows that

$$tq = \frac{\lambda-\mu}{\lambda}\cdot pu.$$

But since $q'r' = \lambda\cdot qt$, it follows that

(2) $$q'r' = (\lambda-\mu)\cdot pu.$$

Now $qq' = pq - pq' = (1-\lambda)\cdot pq$, so $qq'/pq = 1-\lambda$. Since $0<1-\lambda<1$, there is a point s of $S(q,u)$ such that $qs = (1-\lambda)\cdot qu$, so $qs/qu = 1-\lambda = qq'/qp$, and Theorem 2.5 yields

(3) $$q's = (1-\lambda)\cdot pu.$$

To complete the proof, it is observed that

$$rs = ru-(qu-qs) = ru-qu+(1-\lambda)\cdot qu = ru-\lambda\cdot qu.$$

But

$$qu = ru - qr = \left(1-\frac{\lambda-\mu}{\lambda}\right)\cdot ru = \frac{\mu}{\lambda}\cdot ru,$$

so $rs = (1-\mu)\cdot ru$. Now $rr' = pr - pr' = (1-\mu)\cdot pr$, and again Theorem 2.5 yields

(4) $$r's = (1-\mu)\cdot pu.$$

But (2), (3), and (4) imply

$$r's = (1-\mu)\cdot pu = sq' + p'r',$$

so the relation $r'q's$ holds, and s is the desired intersection of $L(q,r)$ and $L(q',r')$.

THEOREM 3.2. *If p, q, and r are non-collinear points of M and if q' and r' are points of $S(p,q)$ and $S(p,r)$ respectively, with $pq'/pq = pr'/pr$, then $L(q,r)$ and $L(q',r')$ have no common point.*

Proof. If there were a point t common to $L(q,r)$ and $L(q',r')$ then by Theorem 2.6 there would be a point t' common to $L(q',r')$ and $L(p,t)$ with $pt'/pt = pq'/pq \neq 1$. But thus $t\neq t'$, which is impossible since the lines $L(q',r')$ and $L(p,t)$ meet in at most one point. This contradiction proves the theorem.

An important theorem of Euclidean plane geometry is the Theorem of Menelaus, which states that if a transversal cuts the sides of a triangle, the product of the signed ratios in which the sides are divided by the transversal is -1. At this point it is possible to give a proof of a special case of this theorem, which illustrates a method for proving the theorem in general. It should be kept in mind, of course, that in this discussion distance is an unsigned quantity, and hence the product of the ratios is unity.

THEOREM 3.3. *If p, q, and r are non-linear points of M, and q', r', and s are points of $S(p,q)$, $S(p,r)$, and $L(q,r)$ respectively, with q', r', and s linear, then*

$$\frac{pr'}{rr'}\cdot\frac{rs}{sq}\cdot\frac{qq'}{pq'}=1.$$

Proof. By Theorem 3.2, $pq'/pq \neq pr'/pr$, so the labelling may be chosen so that $\lambda = pq'/pq > pr'/pr = \mu$. As in an earlier theorem, choose a point t in $S(p,r)$ with $pt = pr'/\lambda$, and let u be a point such that rqu holds, and

$$ru = \frac{\lambda}{\lambda-\mu}\cdot rq.$$

Now $rs = (1-\mu)\cdot ru$, for, assuming the contrary, the point s' of $S(r,u)$ with $rs' = (1-\mu)\cdot ru$ would be a common point of $L(q',r')$ and $L(q,r)$, contrary to the fact that two lines meet in at most one point.

Investigating the factors of the product of ratios being considered, it is seen that $pr' = \mu\cdot pr$, and that

$$rr' = pr - pr' = (1-\mu)\cdot pr,$$

so

$$\frac{pr'}{rr'} = \frac{\mu}{1-\mu}.$$

Also

$$sq = (1-\lambda)\cdot qu = (1-\lambda)\cdot(ru-qr)$$
$$= \left(1-\frac{\lambda-\mu}{\lambda}\right)\cdot(1-\lambda)\cdot ru = \frac{\mu(1-\lambda)}{\lambda}\cdot ru,$$

so

$$\frac{rs}{sq} = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}.$$

Finally $qq' = pq - pq' = (1-\lambda)\cdot pq$, and $pq' = \lambda\cdot pq$, so

$$\frac{qq'}{pq'} = \frac{1-\lambda}{\lambda}.$$

Hence

$$\frac{pr'}{rr'}\cdot\frac{rs}{sq}\cdot\frac{qq'}{pq'} = \frac{\mu}{1-\mu}\cdot\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\cdot\frac{1-\lambda}{\lambda} = 1.$$

THEOREM 3.4. *If p, q, and r are non-linear points of M, let s denote a point of $L(q,r)$ not lying in $S(q,r)$, and q' a point in $S(p,q)$. Then $S(p,r)$ and $L(q',s)$ contain a common point r'.*

Proof. Suppose the relation sqr holds, and write $sq/sr = \nu < 1$ and $pq'/pq = \lambda < 1$. Consider now the number μ satisfying

$$\frac{\mu(1-\lambda)}{\lambda(1-\mu)} = \nu.$$

Since $\nu < 1$, $\mu(1-\lambda) < \lambda(1-\mu)$ or $\mu < \lambda < 1$. Now $\lambda > 0$ and $\mu(1-\lambda)/\lambda(1-\mu) = \nu > 0$, and $(1-\lambda)/\lambda > 0$, so $\mu/(1-\mu) > 0$. But $\mu < 1$, so $\mu > 0$, for otherwise $\mu > 1$, which is impossible. Hence $0 < \mu < 1$, so there is a point r' of $S(p,r)$ with $pr'/pr = \mu$.

Now r' is the desired point, for since $\lambda > \mu$, by Theorem 3.1 there exists a point s' common to $L(q,r)$ and $L(q',r')$ such that rqs' holds and $s'q/s'r = \mu(1-\lambda)/\lambda(1-\mu) = \nu$. But then $s' = s$, so r' is the desired point.

Similarly if r' and s are given with r' in $S(p,r)$ and s satisfying the relation sqr, and if $sq/sr = \nu$, $pr'/pr = \mu$, then as before, letting $\nu = \mu(1-\lambda)/\lambda(1-\mu)$, it is seen that $0 < \mu < \lambda$, so $\lambda > 0$, and since $(1-\lambda)/\lambda > 0$, it follows that $0 < \lambda < 1$. Thus there is a point q' of $S(p,q)$ with $pq'/pq = \lambda$, which is the desired point. This completes the proof.

THEOREM 3.5. *If p, q, and r are non-linear points of M and pqq' and prr' hold for points q' and r' of M, with $pq'/pq \neq pr'/pr$, then there exists a point s common to $L(q,r)$ and $L(q',r')$.*

Proof. Considering $T(p,q',r')$ then q is in $S(p,q')$ and r is in $S(p,r')$ with $pq/pq' \neq pr/pr'$, so by Theorem 3.1, there is a point s common to $L(q,r)$ and $L(q',r')$ and the proof is complete.

THEOREM 3.6. *If p, q, and r are non-linear points of M and sqr and prr' hold for points r' and s of M, then there exists a point q' common to $L(p,q)$ and $L(r',s)$.*

Proof. In $T(s,r,r')$, q lies in $S(r,s)$ and prr' holds, so by Theorem 3.4, there is a point q' common to $S(r',s)$ and $L(p,q)$, completing the proof.

THEOREM 3.7. *If p, q, and r are non-linear points of M and qpq' and rpr' hold for points q' and r' of M, with $pq'/pq \neq pr'/pr$, then there exists a point s common to $L(q,r)$ and $L(q',r')$.*

The proof of this theorem is omitted, as it is almost identical with that of Theorem 3.1, when use is made of Lemma 2.6.

4. Geometry of the plane.

As was mentioned earlier, the intersection theorems of the previous section lead to a fruitful definition of a plane in the space M. In the beginning this plane is regarded as being determined by a point and a line not containing it, in a manner similar to that used in [2], p. 125. The plane is defined in the following way:

DEFINITION 4.1. If p is a point of M and L is a line of M not containing p, the *plane* $\pi(p,L)$ is defined as the topological closure of the set of all points of M linear with p and a point of L.

One of the first questions airsing from this definition is that of linearity. It is desirable that the line of any two distinct points of a plane lie in the plane. Conceivably this need not be the case, as is illustrated by hyperbolic geometry, in which a non-linear point set is obtained by the above definition. However, Theorem 4.1 shows that in the space under consideration here, the plane is linear. In order to prove this theorem, the following lemma must be established:

LEMMA 4.1. *If p is a point of M and a sequence $\{q_i\}$ of points of M converges to a point q of M distinct from p, then each point t of $L(p,q)$ is the limit of a sequence $\{t_i\}$, where t_i is a point of $L(p,q_i)$ $(i=1,2,\ldots)$.*

Proof. Let t be a point of $L(p,q)$. Choose a point t_i on $L(p,q_i)$ such that p, q_i, t_i satisfy the same betweenness relation as p, q, t, and such that $pt_i/pq_i = pt/pq$. Then it follows immediately that $\lim pq_i = pq$, $\lim pt_i = pt$, and consideration of the linearity of p, q_i, t_i yields $\lim q_i t_i = qt$.

Since the sequence $\{q_i\}$ converges, it is a Cauchy sequence. It follows from Theorem 2.5 that

$$t_i t_j = \frac{pt}{pq}\cdot q_i q_j\,,$$

and so $\{t_i\}$ is also a Cauchy sequence, and hence has a limit, say t^*. By continuity of the metric, $\lim pt_i = pt^*$, $\lim q_i t_i = qt^*$, and hence p, q, and t^* are linear. But $pt = pt^*$, $qt = qt^*$, and thus $t^* = t$, completing the proof.

THEOREM 4.1. *The plane $\pi(p,L)$ is linear.*

Proof. In order to prove the theorem, it must be shown that for each pair of points x and y of $\pi(p,L)$, the line $L(x,y)$ lies in $\pi(p,L)$. It suffices to consider the case in which x and y are themselves linear with p and points x' and y' of L, respectively, for in the contrary case, sequences $\{x_i\}$ and $\{y_i\}$ of such points may be found with limits x and y respectively. Consider the lines $L(x_i,y_j)$, $j=1,2,\ldots$, for a fixed i. By Lemma 4.1 each point t of $L(x_i,y)$ is the limit of a sequence $\{t_j\}$ where t_j is a point of $L(x_i,y_j)$, $j=1,2,\ldots$ But if each of the lines $L(x_i,y_j)$, $j=1,2,\ldots$, is in $\pi(p,L)$, it follows that $L(x_i,y)$ is in $\pi(p,L)$ since the plane is a closed point set. Then since each point s of $L(x,y)$ is the limit of a sequence $\{s_i\}$ where s_i is in $L(x_i,y)$, $i=1,2,\ldots$, it follows similarly that $L(x,y)$ is in $\pi(p,L)$.

Hence it can be supposed that points x' and y' of L exist with both triples p,x,x', and p,y,y' linear. The following cases then arise.

Case I. *The relations pxx' and pyy' hold.* If $px/px' \neq py/py'$, then a point s of L exists with x, y, and s linear (Theorem 3.1). By relabelling, if necessary, it may be supposed that the relation yxs holds. Let q be an arbitrary point of $L(x,y)$. Now if either of the relations yqs or ysq holds, there exists a point q' of L with p, q, and q' linear, so q lies in $\pi(p,L)$ (Theorem 3.4 or 3.6). Similarly if syq holds and $yp/yy' \neq yq/ys$, the existence of the point q' of L again follows. Finally if syq holds and $yp/yy' = yq/ys$, a sequence $\{q_i\}$ of points of $S(q,y)$ can be found, with $\{q_i\}$ converging to q. Now since $yq_i < yq$, $i=1,2,\ldots$, it follows that $yq_i/ys \neq yp/yy'$, so for each $i=1,2,\ldots$, there is a point q_i' of L with p, q_i, and q_i' linear. But then since each q_i is a point of $\pi(p,L)$ $(i=1,2,\ldots)$, it follows that q lies in $\pi(p,L)$. This completes the argument of Case I in case $px/px' \neq py/py'$.

If $px/px' = py/py'$, then by Theorem 2.6 there corresponds to each point q of $L(x,y)$ a point q' of L with p, q, and q' linear, and so q lies in $\pi(p,L)$. This completes the proof in Case I.

Case II. *The relations xpx', ypy' hold.*

Case III. *The relations $px'x$, $py'y$ hold.*

Case IV. *The relations pxx', $py'y$ hold.*

The treatment of each of these cases is similar to that of Case I. The final case to be considered is the following.

Case V. *The relation xpx' holds, together with one of the relations pyy' or $py'y$.* In this case there is a point s common to $L(x,y)$ and L, by Theorem 3.4 or Theorem 3.6. Let q be an arbitrary point of $L(x,y)$. Considering $T(y,y',s)$, if neither of the combinations pyy' and qys, or $py'y$ and qsy holds, it follows from the preceding section that there is a point q' of L with p, q, and q' linear. If one of the above combinations does hold, and $py/yy' \neq qy/ys$, then the existence of the desired point q' of L follows from Theorem 3.7, or Theorem 3.5. Finally, if in either of these cases, $py/yy' = qy/ys$, an application of the procedure of Case I yields the result that q lies in $\pi(p,L)$.

This completes the proof that the plane $\pi(p,L)$ is linear.

In the preceding material the plane has been defined by means of a point and a line not passing through the point. It is convenient, however, to be able to regard the plane as determined by three non-collinear points. Consider now three non-collinear points p, q, and r of M. These points determine three planes, the planes $\pi\big(p,L(q,r)\big)$, $\pi\big(q,L(p,r)\big)$, and $\pi\big(r,L(p,q)\big)$. The purpose of the following theorem is to show that these planes coincide and hence that a plane is uniquely determined by three non-collinear points.

THEOREM 4.2. *Three non-collinear points p, q, and r of M uniquely determine a plane.*

Proof. It suffices to show that $\pi\big(p,L(q,r)\big)$ coincides with $\pi\big(q,L(p,r)\big)$. If x is a point of $\pi\big(p,L(q,r)\big)$ then either there is a point x' of $L(q,r)$ with p, x, and x' linear, or there is a sequence $\{x_i\}$ converging to x, such that for each $i=1,2,\ldots$, a point x_i' of $L(q,r)$ exists with p, x_i, and x_i' linear.

Case I. *The relation xpx' holds.* Considering triangle $T(x',q,x)$, it is seen that if $x'pr$ or $qx'r$ holds, then there is a point x'' of $L(p,r)$ with q, x, and x'' linear, so x lies in $P\big(q,L(p,r)\big)$. If qrx' holds and $x'r/x'q$

$\neq x'p/x'x$, then also there is a point x'' of $L(p, r)$ with x, q, and x'' linear. Finally if qrx' holds with $x'r/x'q = x'p/x'x$, a sequence $\{x_i\}$ of points of $S(p, x)$ may be chosen with $\{x_i\}$ converging to x and $x'r/x'q \neq x'p/x'x_i$. Consequently for each $i = 1, 2, \ldots$, there is a point x_i'' of $L(p, r)$ with q, x_i, and x_i'' linear, and so x lies in $\pi(q, L(p, r))$. This completes the proof in Case I.

Case II. *The relation pxx' holds.* If $x'qr$ holds and $x'x/x'p \neq x'q/x'r$, then there is a point x'' of $L(p, r)$ with q, x, and x'' linear. If $x'x/x'p = x'q/x'r$, then a sequence $\{x_i\}$ of $S(p, x)$ may be found which converges to x, and as before each x_i lies in $\pi(q, L(p, r))$, so x is also in $\pi(q, L(p, r))$. Finally, if $qx'r$ or qrx' holds, considering $T(x', q, x)$ there is always a point x'' of $S(p, r)$ with q, x, and x'' linear, so x is a point of $\pi(q, L(p, r))$.

Case III. *The relation $px'x$ holds.* This case is treated exactly as Cases I and II.

Case IV. *There is no point x' of $L(q, r)$ with p, x, and x' linear.* Then a sequence $\{x_i\}$ may be found converging to x, such that for each $i = 1, 2, \ldots$, a point x_i' of $L(q, r)$ may be found with p, x_i, and x_i' linear. But by the above argument each of the points x_i lies in $\pi(q, L(p, r))$, $i = 1, 2, \ldots$, and since the plane is a closed set, x is a point of $\pi(q, L(p, r))$.

Thus it has been shown that $\pi(p, L(q, r))$ is a subset of $\pi(q, L(p, r))$. But by relabelling, a repetition of the above argument shows that $\pi(q, L(p, r))$ is contained in $\pi(p, L(q, r))$, so the planes are identical. Similarly, another relabelling may be used to show that $\pi(q, L(p, r)) = \pi(r, L(p, q))$ and the proof is complete.

Notation. The unique plane determined by non-collinear points p, q, and r of M is denoted by $\pi(p, q, r)$.

In the preceding work it has been noted that a distinction must be made between those points of the plane $\pi(p, L)$ which are linear with p and a point of L, and those which must be expressed as limits of sequences of such points. This distinction becomes so important in the next few theorems that a notation is devised for each of the two resulting sets. The definition is as follows:

Notation. $\pi_I(p, L)$ denotes the set of points x of M which are linear with p and a point of L. $\pi_B(p, L)$ denotes the set of points of $\pi(p, L)$ which are not elements of $\pi_I(p, L)$.

The object of the following sequence of theorems is to show that a plane is determined by any three of its non-collinear points. In the course of the argument, the important result of uniqueness of the parallel to a given line through a given point is obtained.

THEOREM 4.3. *If p^* is a point of $\pi_I(p, L)$ not lying on L, then the planes $\pi(p^*, L)$ and $\pi(p, L)$ are identical.*

Proof. If x is a point of $\pi(p, L)$, we show that x is a point of $\pi(p^*, L)$. Now since p^* lies in $\pi_I(p, L)$ there is a point p' of L with p, p^*, and p' linear. The following cases then arise.

Case I. *There is a point x' on L with p, x, and x' linear.* If the relation pxx' holds, application of previous theorems to $T(p, x', p')$ gives a point x^* of L with p^*, x, and x^* linear, except when p^* is in $S(p, p')$ and $pp^*/pp' = px/px'$. In this case, however, a sequence $\{x_i\}$ of points of $S(p, x)$ can be found which converges to x, and such that for each $i = 1, 2, \ldots$, there is a point x_i^* of L with p^*, x_i, and x_i^* linear. Thus in any event, x is in $\pi(p^*, L)$.

A similar treatment is used in case the relation xpx' or $px'x$ holds.

Case II. *There is no point of L linear with p and x.* Then there is a sequence $\{x_i\}$ of points of $\pi_I(p, L)$ which converges to x. But by Case I, each point x_i lies in $\pi(p^*, L)$, $i = 1, 2, \ldots$, so since $\pi(p^*, L)$ is a closed set, x is a point of $\pi(p^*, L)$.

From Cases I and II it follows that $\pi(p, L)$ is contained in $\pi(p^*, L)$. But the same argument may be used, interchanging p and p^*, to show that $\pi(p^*, L)$ is contained in $\pi(p, L)$, and hence the identity of the two planes is established, proving the theorem.

In order to prove the uniqueness of the parallel, it is necessary to prove the following lemma:

LEMMA 4.4. *If p is a point of M and a sequence $\{q_i\}$ of points of M converges to a point q of M distinct from p, and if a line L of M distinct from $L(p, q)$ intersects $L(p, q)$ in a point r, and $L(p, q_i)$ in a point r_i $(i = 1, 2, \ldots)$ then the sequence $\{r_i\}$ converges to r.*

Proof. Choose a point t_i on each line $L(p, q_i)$ so that p, q_i, t_i satisfy the same betweenness relation as p, q, r and so that $pt_i/pq_i = pr/pq$. Since $\{q_i\}$ converges to q and since, by Theorem 2.5,

$$rt_i = \frac{pr}{pq} \cdot qq_i \quad (i = 1, 2, \ldots),$$

it follows that the sequence $\{t_i\}$ converges to r.

Now if ε is any positive number, points r^* and r' of L can be found so that r^*rr' and $0 < rr' = rr^* < \varepsilon$, and since r is not on $L(p, r')$ or $L(p, r^*)$, there is a positive number δ such that the distance from r to the nearest point of $L(p, r')$ or $L(p, r^*)$ is greater than δ. Now since $\{t_i\}$ converges to r, there is a positive integer N such that if $i > N$ then $rt_i < \delta$. By hypothesis there is a point r_i common to L and $L(p, t_i)$, $i = 1, 2, \ldots$ The following cases then arise.

Case I. *The relation r_it_ip or t_ir_ip holds.* If r_i lies in $S(r', r^*)$ it follows that $rr_i < \varepsilon$. Otherwise, considering triangle $T(r, r_i, t_i)$, it follows from Theorem 3.4 that there is a point t^* or a point t' at which $L(r, t_i)$ intersects

$L(p, r^*)$ or $L(p, r')$, respectively, with rt^*t_i or $rt't_i$ holding. This contradicts the fact that $rt_i < \delta$, and both rt^* and rt' are greater than δ. Thus r_i must lie in $S(r', r^*)$ and $rr_i < \varepsilon$.

Case II. *The relation $t_i p r_i$ holds.* Then supposing r_i is not in $S(r', r^*)$ it may be assumed, by relabelling if necessary, that $r_i r r^*$ holds. Considering $T(r, r_i, t_i)$, since p is in $S(t_i, r_i)$ and $r_i r r^*$ holds, it follows that $L(p, r^*)$ intersects $L(r, t_i)$ in a point t^* such that rt^*t_i holds. This is impossible, since $rt_i < \delta < rt^*$.

Thus in all cases, $rr_i < \varepsilon$ for all $i > N$, and hence $\{r_i\}$ converges to r, completing the proof.

THEOREM 4.4. *If q is a point of $\pi_I(p, L)$ not lying on L, then there is a unique line L^* of $\pi(p, L)$ which passes through q and does not intersect L.*

Proof. Let t be an arbitrary point of L, and select a point r such that tqr holds. Then if t^* is any point of L distinct from t, a point q^* of $S(r, t^*)$ can be found with $rq^*/rt^* = rq/rt$, since $0 < rq/rt < 1$. Let L^* be the line of q and q^*. By Theorem 3.2, the line L^* does not intersect L.

Thus it has been shown that there is at least one line of $\pi(p, L)$ passing through q and not intersecting L. It must be shown that this line is independent of the manner of construction. First it is remarked that if two intersecting lines were cut in different ratios by their point of intersection and the lines L and L^*, then by Theorem 3.2 the lines L and L^* would intersect, contrary to fact. Hence for any other choice of t on L and any arbitrary point r with tqr holding, and any t^* of L ($t^* \neq t$) it follows by the intersection theorems that there is again a point q^* common to L^* and $S(r, t^*)$, and by the above remarks, $rq^*/rt^* = rq/rt$, so L^* is the line which would have been reached by this new construction as well. Thus L^* is independent of the choice of the construction points.

Suppose L'' is any line of $\pi(p, L)$ through q not intersecting L. Since q is in $\pi_I(p, L)$, there is a point q' of L with p, q, and q' linear. If q is the only such point of L'' then $L''-(q)$ is contained in $\pi_B(p, L)$ and there is a sequence $\{r_i\}$ of points of $L''-(q)$ which converges to q. Letting sequences $\{r_{ij}\}$ approach the point r_i, with $\{r_{ij}\}$ in $\pi_I(p, L)$, $i = 1, 2, \ldots$, there exist sequences $\{r'_{ij}\}$ of points of L with p, r_{ij}, and r'_{ij} linear, $i, j = 1, 2, \ldots$ Now the sequences $\{r'_{ij}\}$ are unbounded, for otherwise some point r_i would be in $\pi_I(p, L)$. Thus for each $i = 1, 2, \ldots$, a subsequence of $\{r_{ij}\}$ can be found which has as its corresponding $\{r'_{ij}\}$ a sequence of points receding monotonically from q' in one direction. Then taking a subsequence of the sequence $\{r_i\}$, sequences $\{r_{ij}\}$ are obtained with the corresponding sequences $\{r'_{ij}\}$ monotonically receding in the *same* direction *for all* $i = 1, 2, \ldots$

For any $K > 0$ there is, then, for each integer $i = 1, 2, \ldots$, a positive number N_i such that if $n > N_i$, then distance $r'_{in}q' > K$. The sequences

may be renumbered so that $r'_{ij}q' > K$ $(i, j = 1, 2, \ldots)$. A final selection of subsequences ensures that the sequence $\{r_{ii}\}$ converges to q.

Thus a sequence $\{r_{ii}\}$ has been found which converges to q, with the property that $r'_{ii}q' > K$ $(i = 1, 2, \ldots)$. But this is contrary to the fact that $\{r'_{ii}\}$ converges to q', by Lemma 4.4. Thus the assumption that q is the only point of L'' which lies in $\pi_I(p, L)$ leads to a contradiction.

Hence there is a point r of L'' distinct from q such that it is linear with p and a point r' of L. But then, since L and L'' have no common point, it follows from Theorem 3.4 that the triples p, q, q', and p, r, r' satisfy the same betweenness relation, and then by Theorem 3.1 or 3.7 it follows that $pr/pr' = pq/pq'$, and L'' is the line obtained by the original construction. Thus $L'' = L^*$ and the proof is complete.

DEFINITION 4.2. The *parallel* to a line L through a point q of $\pi_I(p, L)$ is the unique line of $\pi(p, L)$ passing through q and not intersecting L.

At this point Theorem 4.3 can be extended to include points of $\pi_B(p, L)$.

THEOREM 4.5. *If p^* is a point of $\pi(p, L)$ not lying on L, then the planes $\pi(p, L)$ and $\pi(p^*, L)$ are identical.*

Proof. If p^* lies in $\pi_I(p, L)$, Theorem 4.3 gives the desired result. Suppose p^* is a point of $\pi_B(p, L)$, and let q be an arbitrary point of $\pi_I(p, L)$ not on L. Then by Theorem 4.3, the planes $\pi(p, L)$ and $\pi(q, L)$ coincide. But since $L(p^*, p)$ is the unique line of $\pi(q, L)$ passing through p^* which does not intersect L, it follows that the line $L(q, p^*)$ must intersect L, so q lies in $\pi_I(p^*, L)$, and again by Theorem 4.3, $\pi(q, L) = \pi(p^*, L)$. Hence the planes $\pi(p, L)$ and $\pi(p^*, L)$ are identical and the proof is complete.

It is now easy to complete the proof that a plane is determined uniquely by any three of its non-collinear points.

THEOREM 4.6. *A plane is uniquely determined by any non-collinear triple of its points, i.e., if p^*, q^*, and r^* are non-collinear points of $\pi(p, q, r)$ then the planes $\pi(p^*, q^*, r^*)$ and $\pi(p, q, r)$ are identical.*

Proof. Certainly one of the points p^*, q^*, r^* is not on $L(q, r)$. Supposing this is p^*, it follows that

$$\pi\big(p, L(q, r)\big) = \pi\big(p^*, L(q, r)\big) = \pi(p^*, q, r).$$

Similarly, one of the points q^* and r^* is not on $L(p^*, r)$, say q^*. Then

$$\pi(p^*, q, r) = \pi\big(q, L(p^*, r)\big) = \pi\big(q^*, L(p^*, r)\big) = \pi\big(r, L(p^*, q^*)\big) = \pi(p^*, q^*, r^*),$$

since r^* is not on $L(p^*, q^*)$. Thus the planes are identical and the proof is complete.

The section concludes with a proof that the Pasch Axiom holds in the plane. This result follows easily from the preceding intersection theorems when use is made of the fact that a plane may be considered as determined by any three of its non-collinear points.

THEOREM 4.7. *If p, q, and r are non-collinear points and a line L of $\pi(p, q, r)$, distinct from $L(p, q)$, contains a point between p and q, then L contains a point between p and r, or a point between q and r, or the point r.*

Proof. Suppose the line L contains no point between q and r, and that r is not a point of L. The plane may be considered as determined by the point r' common to L and $S(p, q)$ and the line $L(q, r)$, i.e., $\pi(p, q, r) = \pi\big(r', L(q, r)\big)$.

If L intersects $L(q, r)$, there is a point p' of L such that the relation rqp' or qrp' holds, and in either case there is a point q' on L between p and r. On the other hand, if L does not intersect $L(q, r)$, let q' be chosen as the point of $S(p, r)$ with $pq'/pr = pr'/pq$. Then $L(r', q')$ is parallel to $L(q, r)$, and hence must coincide with L, since parallels are unique. Hence in either case L has a point common to the interior of segment $S(p, r)$ and the proof is complete.

5. Theory of parallels. The previous section has provided a proof of the existence and uniqueness of a parallel to a given line passing through a given point. It is fitting at this point to investigate some properties of parallels. The first important property to be shown is the transitivity of parallelism. It will be noticed that the proof of transitivity is valid for spaces of arbitrary dimension.

THEOREM 5.1. *If L_1, L_2, and L_3 are pairwise distinct lines of M with L_1 parallel to L_2 and L_2 parallel to L_3, then L_1 is parallel to L_3.*

Proof. Let q and r be any two points of L_1 and choose q' on L_2. Then if p is an arbitrary point of $L(q, q')$ with the relation $q'qp$ holding, there is a point r' of L_2 with prr' holding, and

$$\frac{pq}{pq'} = \frac{pr}{pr'}.$$

Choose some point q'' of L_3 and let p' be a point of $L(q', q'')$ with $q'q''p'$ holding and $p'q''/p'q' < pq/pq'$. Then there exists a point r'' of L_3 such that $p'r''r'$ holds and $p'r''/p'r' = p'q''/p'q'$.

Since $pq/pq' > p'q''/p'q'$, it follows that $qq'/pq' < q'q''/q'p'$, so $L(q, q'')$ meets $L(p, p')$ in a point s such that $pp's$ holds and

$$\frac{ps}{sp'} = \frac{1}{\dfrac{p'q''}{q''q'} \cdot \dfrac{qq'}{qp}}$$

by Theorems 3.1 and 3.3. Similarly $L(r, r'')$ and $L(p, p')$ meet in a point s' with $pp's'$ holding and

$$\frac{ps'}{s'p'} = \frac{1}{\dfrac{p'r''}{r''r'} \cdot \dfrac{rr'}{rp}}.$$

But

$$\frac{r''r'}{p'r''} \cdot \frac{rr'}{rp} = \frac{p'q''}{q''q'} \cdot \frac{qq'}{qp},$$

so $ps'/s'p' = ps/sp'$, and thus $s = s'$, and the lines $L(q, q'')$ and $L(r, r'')$ intersect.

It remains to be shown that $sq''/sq = sr''/sr$. Application of Theorem 3.3 to triangles $T(s, q, p)$ and $T(s, r, p)$ yields

$$\frac{sq''}{q''q} = \frac{1}{\dfrac{qq'}{pq'} \cdot \dfrac{pp'}{sp'}} = \frac{1}{\dfrac{rr'}{pr'} \cdot \dfrac{pp'}{sp'}} = \frac{sr''}{rr''},$$

from which it follows that $sq''/sq = sr''/sr$. Hence L_1 is parallel to L_3, completing the proof.

THEOREM 5.2. *If L_1, L_2, and L_3 are three pairwise distinct mutually parallel lines of M, and if two intersecting lines L and L' cut L_1, L_2, and L_3 at q_1, q_2, q_3, and r_1, r_2, r_3, respectively, then*

$$\frac{q_1q_2}{r_1r_2} = \frac{q_2q_3}{r_2r_3}.$$

Proof. If p is the intersection of L and L', then

$$\frac{pq_1}{q_1r_1} = \frac{pq_2}{q_2r_2} = \frac{pq_3}{q_3r_3}.$$

Now

$$\frac{q_2r_2}{q_1r_1} = \frac{pq_2}{pq_1} = 1 + \frac{q_1q_2}{pq_1},$$

and also

$$\frac{q_3r_3}{q_2r_2} = 1 + \frac{q_2q_3}{pq_2}.$$

But

$$\frac{q_2r_2}{q_1r_1} = 1 + \frac{r_1r_2}{pr_1} \quad \text{and} \quad \frac{q_3r_3}{q_2r_2} = 1 + \frac{r_2r_3}{pr_2}$$

so

$$\frac{q_1q_2}{r_1r_2} = \frac{pq_1}{pr_1} \quad \text{and} \quad \frac{q_2q_3}{r_2r_3} = \frac{pq_2}{pr_2}.$$

But

$$\frac{pq_1}{pq_2} = \frac{q_1r_1}{q_2r_2} = \frac{pr_1}{pr_2},$$

so $pq_1/pr_1 = pq_2/pr_2$, and it follows that $q_1q_2/r_1r_2 = q_2q_3/r_2r_3$, completing the proof.

THEOREM 5.3. *If L_1, L_2, and L_3 are three pairwise distinct mutually parallel lines of M, and if two parallel lines L and L' intersect L_1, L_2, and L_3 at q_1, q_2, q_3, and r_1, r_2, r_3, respectively, then*

$$\frac{q_1q_2}{r_1r_2} = \frac{q_2q_3}{r_2r_3}.$$

Proof. Choose a point p of L and a point s_3 of L_3, and denote by L^* the line $L(p, s_3)$. Now from the Pasch Axiom (Theorem 4.7) it follows that L^* intersects L_2, L_1, and L' at points s_2, s_1, and p', respectively. But then by the preceding theorem

$$\frac{q_1q_2}{q_2q_3} = \frac{s_1s_2}{s_2s_3} = \frac{r_1r_2}{r_2r_3},$$

and the result follows immediately.

Defining a parallelogram as a quadrilateral whose opposite sides are segments of parallel lines, two important theorems can be proved.

THEOREM 5.4. *The diagonals of a parallelogram bisect each other.*

Proof. Let the pairs of opposite sides of the parallelogram be segments of the lines L_1 and L_2, and of lines L and L', with L_1 and L_2 intersecting L and L' at p_1, p_2, and at q_1, q_2, respectively. Suppose $p' = m(p_1, p_2)$, $q' = m(q_1, q_2)$, and $t = m(p_2, q_1)$. Then the line $L(q', t)$ is parallel to L_1 and L_2, and by the Pasch Axiom (Theorem 4.7) it intersects L. In fact the point of intersection is the point p', for

$$\frac{p_1p'}{p'p_2} = 1 = \frac{q_1q'}{q'q_2}.$$

By a previous theorem, there is a point s common to $S(p_1, q_2)$ and $L(q', t)$ with

$$\frac{p_1s}{sq_2} = \frac{p_1p'}{p'p_2} = 1.$$

Supposing $s \neq t$, then, if $r_1 = m(p_1, q_1)$ and $r_2 = m(p_2, q_2)$, $L(r_1, s)$ is parallel to L' and so r_2 is on $L(r_1, s)$. But since

$$\frac{p_2t}{tq_1} = \frac{p_1r_1}{r_1q_1},$$

it follows that t lies in $L(r_1, s)$. But this is impossible, for s and t are on $L(q', t)$ which is parallel to L_2. The contradiction reached here completes the proof that $s = t$, so the mid-points of the diagonals $S(p_1, q_2)$ and $S(p_2, q_1)$ coincide, and the theorem is proved.

THEOREM 5.5. *The opposite sides of a parallelogram are equal in length.*

Proof. If the parallelogram is labelled as in the preceding theorem, then since $t = m(p_1, q_2) = m(p_2, q_1)$, it follows from Lemma 2.6 that $p_2q_2 = 2\cdot p't = p_1q_1$. Similarly $p_1p_2 = q_1q_2$, and the proof is complete.

6. Introduction of segment addition. The first major objective of this study is to show that a metric space M with at least two points, which is complete, convex, externally convex, and satisfies the two-triple property and the Young Postulate, is a normed linear space. In order to achieve this goal it is necessary to define addition, scalar multiplication, and norm of an element in the metric space M. First, we review the definition of a normed linear space.

A set S of elements $p, q, r, \ldots$, is said to form a *linear space* over the field of real numbers provided a binary operation, "$+$" is defined in S (with respect to which S is closed), and for each real number λ and each element p of S there is defined a unique element $\lambda\cdot p$ of S, satisfying the following conditions, where p, q, and r are elements of S and λ and μ are real numbers:

(a) $p+q = q+p$,
(b) $(p+q)+r = p+(q+r)$,
(c) $p+x = q$ has a solution x in S,
(d) $\lambda\cdot(\mu\cdot p) = (\lambda\mu)\cdot p$,
(e) $\lambda\cdot(p+q) = \lambda\cdot p+\lambda\cdot q$,
(f) $(\lambda+\mu)\cdot p = \lambda\cdot p+\mu\cdot p$,
(g) $1\cdot p = p$.

It is easily shown that in the linear space S there is a unique element 0, called the *zero element* of S, satisfying $p+0 = p$ for every element p of S.

A linear space S over the field of real numbers is said to be *normed* provided with each element p of S is associated a non-negative real number $\|p\|$ such that

(i) $\|p\| > 0$ if and only if $p \neq 0$,
(ii) $\|\lambda\cdot p\| = |\lambda|\cdot\|p\|$,
(iii) $\|p+q\| \leqslant \|p\|+\|q\|$.

A complete normed linear space is known as a *Banach space*.

In order to define addition and scalar multiplication in the space M, it is convenient to define the reflection of a point p in a point q.

DEFINITION 6.1. If p and q are distinct points of M, a point p' of M is a *reflection* of the point p in the point q provided pqp' holds and $pq = qp'$. If $p = q$, the reflection of p in q is the point p.

It follows from the uniqueness of the line determined by two points, and the congruence of that line with the Euclidean line E_1, that the reflection p' of p in q is uniquely determined. The operations of addition and scalar multiplication are then defined in the following way. Let o be an arbitrary but fixed point of M.

DEFINITION 6.2. If p and q are points of M, the point $p+q$ is defined as the reflection of o in the mid-point of p and q.

Remark 1. *If p and q are points of M, $p+q$ is a unique point of M.*

Proof. If $p \neq q$, it follows from the uniqueness of segments that the mid-point m of p and q is uniquely determined. But then since the reflection of o in m is unique, it follows that $p+q$ is uniquely determined. If $p = q$, their mid-point coincides with both of them, and again it follows from the uniqueness of the reflection that $p+q$ is uniquely determined.

DEFINITION 6.3. If $\lambda > 0$, and p is a point of M ($p \neq o$), the point $\lambda \cdot p$ is defined as that point p' of $L(o, p)$ with opp', $op'p$, or $p' = p$ holding, and $op' = \lambda \cdot op$. If $\lambda = 0$, $\lambda \cdot p$ is defined as the point o, i.e., $0 \cdot p = o$. For any real λ, $(-\lambda)\cdot p$ is defined as the reflection of $\lambda \cdot p$ in the point o. For every real λ, $\lambda \cdot o = o$.

Remark 2. *If p is a point of M and λ is any real number, $\lambda \cdot p$ is a unique point of M.*

Proof. If $\lambda > 0$, and $p \neq o$, it follows from the uniqueness of the line $L(o, p)$ and the congruence of this line with E_1 that $\lambda \cdot p$ is uniquely determined. If $\lambda = 0$ or $p = o$, the uniqueness follows immediately from Definition 6.3, and if $\lambda < 0$, the result follows from the uniqueness of $(-\lambda)\cdot p$ and the uniqueness of its reflection in o.

Remark 3. *Addition and scalar multiplication satisfy conditions* (a), (c), (d), (e), (f), *and* (g).

Proof. The proofs of most of the properties follow immediately from the definitions. It is remarked that in property (c), the point x satisfying $p + x = q$ is the reflection of p in the mid-point of o and q. Property (e) follows easily by use of Theorems 2.6 and 3.1 or 3.7.

It remains to be shown that addition is associative. This is accomplished in the following theorem, whose proof makes strong use of the properties of parallelograms established in the last section.

THEOREM 6.1. *Addition is associative.*

Proof. Let p, q, and r be points of M not collinear with o. By the definition of addition, in order to show that $(p+q)+r = p+(q+r)$, it suffices to show that the mid-points $m(p, q+r)$ and $m(p+q, r)$ coincide.

Let $m = m(p, q)$ and $m' = m(q, r)$. Then $pr = 2\cdot mm'$, and the distance $(p+q)(q+r) = 2\cdot mm'$, so $(p+q)(q+r) = pr$. Also $L(p, p+q)$ and $L(r, q+r)$ are both parallel to $L(o, q)$ and hence are parallel to each other. Similarly $L(p, r)$ and $L(p+q, q+r)$ are parallel, since both are parallel to $L(m, m')$. Hence, p, $p+q$, $q+r$, and r are the vertices of a parallelogram, and by Theorem 5.4, the diagonals $S(p, q+r)$ and $S(r, p+q)$ bisect each other. Thus $m(p, q+r) = m(r, p+q)$ and the proof is complete.

DEFINITION 6.4. If p is a point of M, the norm $\|p\|$ is defined as the distance op.

Remark 4. *The norm $\|p\|$ satisfies conditions* (i), (ii), *and* (iii).

The proof of Remark 4 also follows readily from the above definitions and the elementary properties of metric spaces. The results of this section combine to give the following characterization theorem:

THEOREM 6.2. *A metric space with at least two points which is complete, convex, externally convex, and satisfies the two-triple property, is a normed linear space (Banach space) if and only if it satisfies the Young Postulate.*

II. Characterization of Euclidean space

7. Some consequences of the Ficken postulate. The second part of this work is devoted to showing that a metric space M with at least two points, which is (1) finitely compact, (2) convex, (3) externally convex, and (4) satisfies the two-triple property, is Euclidean if and only if it satisfies the Ficken and Young Postulates. In later sections strong use is made of the previous results of the Young Postulate, but for the moment the Ficken Postulate assumes a place of prominence. For this reason, it is desirable to repeat the statement of this postulate.

THE FICKEN POSTULATE. *If f_p is a foot of a point p on a line L (p not on L), and if q and r are points of L with $qf_p = f_pr$, then for each point s of $L(p, f_p)$, $sq = sr$.*

The following theorems investigate some important consequences of the Ficken Postulate in the space M satisfying 1, 2, 3, and 4 above.

THEOREM 7.1. *The foot of a point on a line is unique.*

Proof. By the remarks of Section 1 there exists at least one foot, f_p, of p on L. Clearly $f_p = p$ if and only if p lies in L. If p is not on L, suppose f'_p is a foot of p on L distinct from f_p. Then $f_pf'_p > 0$, and positive real numbers a and b exist such that $pf_p = a$, and $f_pf'_p = b$. Now there exists an integer n such that $n\cdot b > 2a$. Let f'' be the point of L with $f'_pf'' = b$ and $f_pf'_pf''$ holding. Then by the Ficken Postulate $f''p = f_pp$, and f'' is thus a foot of p on L. Denote it by f''_p.

Make the inductive assumption that $f_p, f'_p, f''_p, \ldots, f^{(k)}_p$ are defined as above. Denote by $f^{(k+1)}$ the point of L with $f^{(k)}_pf^{(k+1)} = b$, and $f^{(k-1)}_pf^{(k)}_pf^{(k+1)}$ holding. Then $f^{(k+1)}$ is a foot of p on L as above, and may be denoted by $f^{(k+1)}_p$. Thus there is a foot $f^{(n)}_p$ of p on L, where $n\cdot b > 2a$. Then since L is congruent with the Euclidean line, $f_pf^{(n)}_p = n\cdot b$. But since $n\cdot b > 2a$, $f_pf^{(n)}_p > 2a = f_pp + f^{(n)}_pp$, contrary to the triangle inequality. Thus the foot of a point on a line is unique, and the proof is complete.

THEOREM 7.2. *If f_p is the foot of a point p on a line L (p not on L), then f_p is the foot on L of each point of $L(p, f_p)$.*

Proof. Denote $L^* = L(p, f_p)$, and suppose that $f_s \neq f_p$ for some point s of L^*. Choose a point t of L such that $f_s f_p t$ holds, and $f_s f_p = f_p t$. Then by the Ficken Postulate $st = sf_s$, so t is also a foot of s on L, contrary to Theorem 7.1.

THEOREM 7.3. *If f_p is the foot of a point p on a line L (p not on L), then f_p is the foot on $L(p, f_p)$ of each point of L.*

Proof. If q is a point of L with foot f_q on $L(p, f_p)$, suppose $f_q \neq f_p$. Choose a point q' of L such that qf_pq' holds, and $qf_p = f_pq'$. Then $qf_q = f_qq'$ by the Ficken Postulate. Now $f_{q'} = f_q$ for if the contrary were supposed, $q'f_q > q'f_{q'}$. But $q'f_{q'} = qf_{q'}$, so $qf_{q'} = q'f_{q'} < q'f_q = qf_q$, contrary to the fact that f_q is the foot of q on $L(p, f_p)$.

Since $f_{q'} = f_q$, it follows that $q'f_p > q'f_q$ and $qf_p > qf_q$, so that $qq' = qf_p + f_pq' > qf_q + f_qq'$, contrary to the triangle inequality. Thus $f_q = f_p$ for all points q of L.

Theorems 7.1, 7.2, and 7.3 indicate that the lines L and $L(p, f_p)$ are related in a symmetric manner by the property that their intersection is the foot on any one of the lines, of any point of the other. The fact that this relation enjoys properties similar to those of ordinary perpendicularity in E_2, leads to the following definition:

DEFINITION 7.1. A line L^* is *perpendicular* to a line L provided there exists a point p of L^* (p not in L) whose foot f_p on L is the point common to L and L^*.

Remark. *The relation of perpendicularity is symmetric, and hence if line L is perpendicular to line L^* we say that the lines are mutually perpendicular.*

THEOREM 7.4. *If f_p and f_q are the feet of distinct points p and q on a line L (p and q not on L), and if $L(p, q)$ intersects L in a point, then $f_p = f_q = f$ on L implies that f is the foot on L of each point of $L(p, q)$.*

Proof. Let $L(p, q)$ be denoted by L^* and suppose r is the common point of L and L^*. Let r' be the reflection of r in f. Then by the Ficken Postulate $rp = pr'$, and $rq = qr'$.

Case I. *The relation rpq holds.* Then $r'p + pq = rp + pq = rq = r'q$, so r' is on L^* and $r = r' = f$, since two lines intersect in at most one point.

Case II. *The relation rqp holds.* The argument is similar to that of Case I.

Case III. *The relation prq holds.* Then $pr' + r'q = pr + rq = pq$, so r' is again on L^* and $r' = r = f$.

Thus in each case $r = f$ and so by Theorem 7.2, f is the foot on L of each point of L^*.

THEOREM 7.5. *If $\lim x_i = x_0$, then $\lim f_i = f_0$, where f_i is the foot of x_i on L ($i = 0, 1, 2, \ldots$).*

Proof. Since $f_i f_0 \leqslant f_0 x_0 + x_0 x_i + x_i f_i$, and $\lim x_0 x_i = 0$, $\lim x_i f_i = x_0 f_0$, then $f_i f_0$ is bounded ($i = 1, 2, \ldots$), and consequently $\{f_i\}$ is a bounded subset of L.

Suppose f_0 is not the limit of $\{f_i\}$. Then there exists a real number $K > 0$ and a subsequence $\{f_{in}\}$ such that $f_{in} f_0 > K$ ($n = 1, 2, \ldots$). Since $\{f_{in}\}$ is bounded, some subsequence, say $\{f'_k\}$, has a limit g. Now if $\{x'_k\}$ is the subsequence of $\{x_i\}$ corresponding to $\{f'_k\}$, $\lim x'_k = x_0$. Now $x'_k f_0 > x'_k f'_k$ ($k = 1, 2, \ldots$), so $\lim x'_k f_0 \geqslant \lim x'_k f'_k$, and since $\lim x'_k f_0 = x_0 f_0$, and $\lim x'_k f'_k = x_0 g$, it follows that $x_0 f_0 \geqslant x_0 g$. But since f_0 is the foot of x_0 on L, $x_0 f_0 \leqslant x_0 g$, so $x_0 f_0 = x_0 g$, which is impossible since the foot of a point on a line is unique, and $f_0 g \geqslant K > 0$. Hence $\lim f_i = f_0$, and the theorem is proved.

Since the foot of a point p on a line L is unique, the foot f_p of p on L is a function of p, and the preceding theorem may be restated as follows:

COROLLARY. *If x is a point of M and L is a line of M, the foot f_x of x on L is a continuous function of x.*

LEMMA 7.6. *If q and r are points of L, and p is a point of M with $pq = pr$, then the foot f_s on L of each point s of $S(p, q) + S(p, r)$ lies in $S(q, r)$.*

Proof. Suppose t is a point of $S(p, q)$ whose foot f_t does not lie in $S(q, r)$. Suppose further that $f_t qr$ holds. Now $f_r = r$, and since f_x is a continuous function of x for x in $S(p, t) + S(p, r)$ there is a point s of $S(p, t) + S(p, r)$ with $f_s = q$. Now if s lies in $S(p, q)$ then $f_t = q$, by Theorem 7.2, contrary to assumption, so that the relation psr must hold. Now $sq < sr$, so

$$sq + ps < sr + ps = pr = pq,$$

by hypothesis, contradicting the triangle inequality.

If qrf_t holds, there is a point s of $S(q, t)$ such that $f_s = r$ and a similar argument holds. Similarly if t lies in $S(p, r)$ a contradiction is reached, and the proof of the lemma is complete.

THEOREM 7.6. *For each point p and line L of M, the function px is monotone increasing as x recedes along either half-line of L determined by f_p, the foot of p on L.*

Proof. Suppose px were not monotone increasing. Then there would exist points q and r of L with $f_p qr$ holding and $pq = pr$. But then by Lemma 7.6, f_p lies in $S(q, r)$, contradicting the relation $f_p qr$.

One of the more important consequences of the Ficken Postulate is the following, which is, indeed, equivalent to it:

PROPERTY (∗). *If p is a point of M not lying on a line L of M, and if q and r are points of L with $pq = pr$, then m, the mid-point of q and r, is a foot on L of each point of $L(p, m)$.*

THEOREM 7.7. *In a metric space M of at least two points with properties* (1), (2), (3), *and* (4), *satisfying the Ficken Postulate, Property* (∗) *is satisfied.*

Proof. By Theorem 7.6, M has the monotone property. Suppose that M does not have Property (∗); i.e., a point p of M and points q and r of a line L exist with $pq = pr$ and the mid-point m of q and r is not a foot of p on L.

Case I. *The relation qf_pr holds.* Let q' be the reflection of q in f_p. Now since $m \neq f_p$, $q' \neq r$, and either $f_pq'r$ or f_prq' holds, by the congruence of L with E_1. But $pr = pq = pq'$ by hypothesis and the Ficken Postulate, contrary to the monotone property.

Case II. *The relation f_pqr or f_prq holds.* Then $pr = pq$ by hypothesis, contrary to the monotone property.

Thus in both cases a contradiction is reached, so Property (∗) is satisfied, completing the proof.

8. An equivalent form for the Ficken postulate. It was suggested in the preceding section that Property (∗) is equivalent to the Ficken Postulate, and in that section it was shown that Property (∗) is a consequence of properties (1), (2), (3), (4), and the Ficken Postulate. In this section the proof of the equivalence of the two properties is completed by showing that a metric space with properties (1), (2), (3), and (4) satisfying Property (∗) also satisfies the Ficken Postulate.

Let M denote a metric space with properties (1), (2), (3), and (4) and having Property (∗).

LEMMA 8.1. *If points q and r of L are feet of p on L, then each point of the segment $S(q, r)$ is a foot of p on L.*

Proof. If q and r are feet of p on L, then $pq = pr$, so by Property (∗), their mid-point m is also a foot of p on L. Hence $pq = pm = pr$. Similarly the mid-points of q and m and of m and r are both feet of p on L. Continuing inductively a set of feet of p is obtained which is dense in $S(q, r)$ and the result follows by the continuity of the metric.

THEOREM 8.1. *A point p of M either has a unique foot on a line L of M, or the locus of all feet of p on L is a single segment of L.*

Proof. Suppose p has two distinct feet on L, say f and f'. Then by Lemma 8.1 each point of $S(f, f')$ is a foot of p on L. Let F be the set of all feet of p on L. Now F is a bounded set, for by Remark 2 of Section 1, if f is a foot of p and $fx > 2\cdot fp$, then $px > pf$, so x is not in F. Since L is congruent to E_1, and F is a bounded subset of L, F has "rightmost" and "leftmost" accumulation points, q and r respectively. Now since f and f' are distinct points of F, it follows that $q \neq r$. By continuity of the metric, q and r are both feet of p, so by Lemma 8.1, each point of $S(q, r)$ is a foot of p on L and by the definition of q and r, no other point of L is a foot of p.

THEOREM 8.2. *If each point of a segment $S(q, r)$ of L is a foot of p on L, then for each point f satisfying qfr and each point t of $L(p, f)$ distinct from p, f is the unique foot of t on L.*

Proof. Since qfr holds, there exist points of $S(q, r)$ with f as mid-point, and having equal distances from p. Then if t is on $L(p, f)$ distinct from p it follows from Property (∗) that f is a foot of t on L. Suppose t has another foot on L. If this foot is in $S(q, r)$ label it f''. If f' $(\neq f)$ is a foot of t on L, not lying in $S(q, r)$, each point of $S(f, f')$ is also a foot of t, so there exists a point f'' common to $S(q, r)$ and $S(f, f')$, distinct from f, which is a foot of t on L. Thus in any case there is a foot f'' of t in $S(q, r)$, distinct from f. Now since f'' is both a foot of t and a foot of p, $tf = tf''$ and $pf = pf''$.

Suppose the relation ptf holds. Then

$$pf'' = pf = pt + tf = pt + tf'',$$

so ptf'' also holds, contrary to the uniqueness of lines. Similar arguments are used in case tpf or pft hold, and the contradictions reached complete the proof of the theorem.

THEOREM 8.3. *If $S(q, r)$ is the set of all feet of p on L, and if t is a point of $S(p, q)$ distinct from p, then q is the unique foot of t on L.*

Proof. First, q is a foot of t on L, for if the contrary were supposed, there would be a point f of L such that $tf < tq$. But then

$$pq = pt + tq > pt + tf \geqslant pf,$$

so $pq > pf$, contrary to the fact that q is a foot of p on L.

To show the uniqueness of the foot, suppose that t has a foot $f' \neq q$. If f' is in $S(q, r)$ then f' is also a foot of p, and by the argument of the preceding theorem a contradiction is reached. If f' is not in $S(q, r)$ then

$$pf' > pq = pt + tq = pt + tf',$$

since $tq = tf'$. But this contradicts the triangle inequality, so q is the only foot of t on L, completing the proof.

THEOREM 8.4. *The foot of a point on a line is unique.*

Proof. Suppose p has more than one foot on a line L. Then the set of all feet of p on L is a segment, say $S(q, r)$. Let m be the mid-point of q and r. If t is a point satisfying tpm, then by Theorem 8.2, m is the unique foot of t, and

$$tm = tp + pm = tp + pq > tq,$$

since lines are unique. But this contradicts the fact that m is the foot of t, and completes the proof.

It is noted that in the proof of Theorem 7.5, the only result of the Ficken Postulate which is needed is that of unique feet. Since this result is given in this new context by Theorem 8.4, it follows that the theorem is valid in this context as well. It is restated in the following form:

LEMMA 8.5. *If x is a point of M and L is a line of M, the foot f_x of x on L is a continuous function of x.*

THEOREM 8.5. *If f_p is the foot of p on L, then the function px is monotone increasing as x recedes along either half-line of L determined by f_p.*

Proof. Suppose px is not monotone increasing. Then there exist points t and x_2 of L such that $f_p t x_2$ holds and $pt > px_2$. By continuity of the function px in the segment $S(f_p, t)$, there is a point x_1 of this segment with $px_1 = px_2$. But by Property (*) the mid-point m of x_1 and x_2 is a foot of p on L, contrary to the uniqueness of feet. This contradiction implies that M has the monotone property.

The proof that M satisfies the Ficken Postulate is completed by showing first that M satisfies a weak form of the Ficken Postulate.

LEMMA 8.6 (Weak Ficken Postulate). *If f_p is the foot of p on L, and if q and r are points of L with $qf_p = f_p r$, then $pq = pr$.*

Proof. Suppose $pq \neq pr$. Then one is greater, say $pq > pr$. Now by the monotone property, there exists a point r' of L with $qf_p r'$ holding and $pq = pr'$. If m is the mid-point of q and r', then m is the foot of p on L, and $f_p = m$. But then

$$r'm = mq = f_p q = rm ,$$

and since $qf_p r$ and $qf_p r'$ both hold, it follows that $r = r'$, and $pq = pr$.

THEOREM 8.6. *If f_p is the foot of a point p on a line L (p not on L), then f_p is the foot on L of each point s of $L(p, f_p)$.*

Proof. There exist points q and r of L with $qf_p = f_p r$, and $qf_p r$ holding. By Lemma 8.6, $pq = pr$, so by Property (*), f_p is the foot on L of each point of $L(p, f_p)$.

THEOREM 8.7. *In a metric space M of at least two points with properties (1), (2), (3), and (4) satisfying Property (*), the Ficken Postulate is satisfied.*

Proof. Let p be a point of M not lying on a line L, and let f_p be the foot of p on L. If s is a point of $L(p, f_p)$ then, by Theorem 8.6, $f_s = f_p$, so if $qf_p = f_p r$, Lemma 8.6 yields $sq = sr$, and the proof is complete.

9. Equidistant loci in the plane. At this point it is convenient to collect some further properties of parallels, one of which makes possible the proof of the linearity of the equidistant locus in the plane. Using this result it follows from a paper of Busemann [3] that the planes of the space M are Euclidean. This is only briefly mentioned, for another more direct method is employed later to show that the space M is Euclidean. *It is to be noted that the theorems of this section make use of the Young Postulate as well as the Ficken Postulate.*

THEOREM 9.1. *If distinct lines L_1 and L_2 of a plane π are both perpendicular to a line L, then L_1 and L_2 are parallel.*

Proof. If L_1 and L_2 are perpendicular to L at p and q respectively, then supposing L_1 and L_2 intersect at a point t, it follows that both p and q are feet of t on L, contrary to the uniqueness of feet. Hence L_1 is parallel to L_2.

THEOREM 9.2. *If L_1 and L_2 are parallel lines of a plane π, and if a line L of π is perpendicular to L_1, then L is perpendicular to L_2.*

Proof. Certainly L and L_2 have a common point, for otherwise there would be two parallels to L_2 through a single point. Let p be the common point of L and L_2. Then by Theorem 9.1, the line of π perpendicular to L at p must also be a parallel to L_1, and so must coincide with L_2. Hence L is perpendicular to L_2.

In order to make use of the results of Busemann mentioned above, it is necessary to consider the equidistant locus in a plane π of two points q and r of π.

DEFINITION 9.1. The *equidistant locus* in the plane π of two distinct points q and r of π, denoted by $H(q, r)$, is the set of points p of π such that $pq = pr$.

THEOREM 9.3. *If q and r are distinct points of a plane π of M, then $H(q, r)$ is the set of points of π whose foot on $L(q, r)$ is the mid-point m of q and r.*

Proof. If p is a point of $H(q, r)$, then $pq = pr$, by definition, and application of Property (*) gives the desired result. If p is in π and f_p on $L(q, r)$ is the mid-point of q and r, then by the Ficken Postulate $pq = pr$, and p is in $H(q, r)$.

LEMMA 9.4. *If p and q are distinct points of $H(r, r')$ for distinct points r and r' of π, and if $L(p, q)$ intersects $L(r, r')$, then $L(p, q)$ is contained in $H(r, r')$.*

Proof. Denote by L the line $L(r, r')$ and by L^* the line $L(p, q)$. If t is the common point of L and L^*, let t' be the reflection of t in the mid-point m of r and r'. Now by Theorem 9.3, m is the foot on L of p and q, so application of the Ficken Postulate gives $pt = pt'$, and $qt = qt'$.

Suppose the relation pqt holds. Then

$$pq + qt' = pq + qt = pt = pt' ,$$

so t' is on L^*, and hence must coincide with t at the point m. A similar argument is used if qpt or ptq holds. Thus in each case $t' = t = m$, so by the Ficken Postulate, m is the foot on L of every point of L^*, and consequently L^* lies in $H(r, r')$.

THEOREM 9.4. *If p and q $(p \neq q)$, are points of $H(r, r')$, for distinct points r and r' of π, then $L(p, q)$ is contained in $H(r, r')$.*

Proof. By the Lemma, it suffices to show that $L(p, q)$ intersects $L(r, r')$. Denote $L(r, r')$ by L and suppose $L(p, q)$ is parallel to L. Then by Property (*), since $pr = pr'$ and $qr = qr'$, the mid-point m of r and r' is the foot of both p and q on L, so then $L(p, m)$ and $L(q, m)$ are both perpendicular to L. But by Theorem 9.2, $L(p, m)$ and $L(q, m)$ are both perpendicular to $L(p, q)$ and hence both p and q are feet of m on $L(p, q)$, contrary to the uniqueness of feet. This contradiction implies that $L(p, q)$ and L must have a common point.

These results may now be combined with those of Busemann [3] to complete the proof that π is a Euclidean space, for in this work Busemann shows that a finitely compact metric space with unique straight lines is Euclidean or hyperbolic if and only if the equidistant locus is linear. This is now the case, for the planes π of the space M, and so each plane π of M is a Euclidean space. We shall, however, establish the Euclidean nature of M without reference to Busemann's result.

10. Reflections and the Pythagorean property.

In order to complete the proof that the space M as defined is Euclidean, it is necessary to define in the space a class of transformations known as reflections. The reflection mapping is shown to be a congruent mapping, i.e., it is one-to-one and distance-preserving. The important use of these mappings is in showing that the Pythagorean Theorem is valid in the space M. This result, in turn, leads to the fact that M is Euclidean.

The reflection mapping is defined in the following way:

DEFINITION 10.1. A point p' of M is the *reflection* of a point p in a line L of M provided the foot f_p of p on L is the mid-point of p and p'.

DEFINITION 10.2. The *reflection mapping* R_L of the plane π in a line L of π is the mapping which maps each point of π onto its reflection in L, i.e., for every point p of π, $R_L(p) = p'$, the reflection of p in L.

The mapping R_L can easily be seen to be a one-to-one mapping of π onto itself, for certainly if q is any point of π, its reflection p in L will be mapped onto q by R_L. Also if $p' = q'$, there are points p and q of π such that $f_p = m(p, p')$ and $f_q = m(q, q')$. But $f_p = f_{p'} = f_{q'} = f_q$, so $f_p = f_q$, and thus $p = q$. Thus in order to show that the mapping R_L is a congruence, it remains only to show that it preserves distance. This is accomplished in the following theorem:

THEOREM 10.1. *The mapping R_L of π onto itself is a congruence.*

Proof. It remains to be shown that if p and q are points of π, the distance $R_L(p)R_L(q) = pq$. Two cases arise.

Case I. *$L(p, q)$ intersects L.* Then there is a point r common to $L(p, q)$ and L. Since pf_pp' holds, and $pf_p = f_pp'$, where p' denotes $R_L(p)$, it follows from the Ficken Postulate that $rp = rp'$. Similarly, if $q' = R_L(q)$, it is seen that $rq = rq'$.

Let q'' be any point of π such that r, p', q'' are congruent to r, p, q, i.e., $rp' = rp$, $rq'' = rq$, and $p'q'' = pq$. Then since $f_p = m(p, p')$, it follows from Theorem 2.6 that $S(q, q'')$ and L have a common point, say m, and that m is the mid-point of q and q''. But since $qr = q''r$, m is the foot of r on $L(q, q'')$ and hence is the foot of q on L, by Theorem 7.3. Thus $m = f_q$, and $q'' = q'$. Hence r, p, q are congruent to r, p', q' and $pq = p'q'$, completing the argument in Case I.

Case II. *$L(p, q)$ and L have no common point.* Now from the definition of reflection it follows that $L(p, p')$ and $L(q, q')$ are perpendicular to L, and so, by Theorem 9.1, $L(p, p')$ is parallel to $L(q, q')$. Suppose $L(p', q')$ were not parallel to L. Then there would be a point q'' of $L(q, q')$ with $L(p', q'')$ parallel to L, so by Theorem 5.5, $f_q q'' = f_p p' = f_q q'$ and $q'' = q'$. Thus $L(p', q')$ is parallel to L, and since $L(p, p')$ is parallel to $L(q, q')$, it follows that the figure (p, q, q', p') is a parallelogram and by Theorem 5.5, it follows that $pq = p'q'$, completing the proof.

From Theorem 10.1, several important corollaries follow immediately. Thus, for example, if R_L is a reflection of a plane π in one of its lines L, and $p' = R_L(p)$, $q' = R_L(q)$, then the line of p' and q' is the image under R_L of the line $L(p, q)$. Furthermore, if r is a point of π not lying on $L(p, q)$ and if f is its foot on $L(p, q)$, then f', the image of f under R_L, is the foot of $r' = R_L(r)$ on the line $L(p', q')$. These results find an important application in the proof of Theorem 10.2.

It now becomes possible to complete the proof of the final result. This depends on a property known as the Pythagorean Property, which is defined in the following way ([2], p. 129):

DEFINITION 10.3. A space M is said to have the *Pythagorean Property* provided for each triple p, q, r of points of M such that p is a foot of q on $L(p, r)$, $(pq)^2 + (pr)^2 = (qr)^2$.

The importance of the Pythagorean Property lies in its usefulness as a means of characterizing Euclidean spaces. Blumenthal (loc. cit.) has shown that a finitely compact, convex, externally convex metric space with the Pythagorean Property is congruent with a Euclidean space of finite dimension. The following theorems thus complete the present characterization of Euclidean space, by showing that the metric space M with properties (1), (2), (3), (4), and satisfying the Young Postulate and the Ficken Postulate, satisfies the Pythagorean Property.

LEMMA 10.2. *Let p, q, and r be points of M, and p the foot of r on $L(p,q)$. If f is the foot of p on $L(q,r)$, the relation qfr holds.*

Proof. Clearly f is distinct from q and r, for otherwise $L(p,r)$ or $L(p,q)$ would be parallel to $L(q,r)$. If $pq = pr$, it follows from Property $(*)$ that qfr holds (indeed $f = m(q,r)$).

Suppose the labelling is so chosen that $pq > pr$. Then by the monotone property, rqf cannot hold. Thus one of the relations qrf or qfr must hold. But f is the foot of q on $L(p,f)$ so $qf < qp$. However, since p is the foot of q on $L(p,r)$, $qp < qr$, so $qf < qr$, which eliminates qrf. Hence qfr holds.

THEOREM 10.2. *The space M has the Pythagorean Property, i.e., if $T(p,q,r)$ is a triangle of M with p the foot of r on $L(p,q)$, then $(pq)^2 + (pr)^2 = (qr)^2$.*

Proof. Let f denote the foot of p on $L(q,r)$ which, by the preceding lemma, lies in $S(q,r)$. The proof proceeds by showing that $qr/rp = rp/rf$, and $qr/qp = qp/qf$.

Since f is the foot of r on $L(p,f)$, it follows that $rp > rf$, and consequently there is a point f' in $S(r,p)$ with $rf' = rf$. If $m = m(f,f')$, it follows from Property $(*)$ that m is the foot of r on $L(f,f')$. Then considering the reflection of triangle $T(f,p,r)$ in the line $L(m,r)$, it follows that f is reflected into f', r remains fixed, and since the reflection mapping is a congruence, p is mapped into the point p' of $S(r,q)$ such that $rp' = rp$. Using again the fact that the reflection is a congruence, it follows that f' is the foot of p' on $L(r,f')$, which coincides with $L(p,r)$. Hence the lines $L(p,q)$ and $L(f',p')$ are parallel and from the uniqueness of parallels, it follows that

$$\frac{rf'}{rp} = \frac{rp'}{rq} \quad \text{or} \quad \frac{qr}{rp} = \frac{rp}{rf}.$$

Thus $(rp)^2 = qr \cdot rf$.

In a similar manner it may be shown that $(qp)^2 = qr \cdot qf$, so it follows that $(qp)^2 + (rp)^2 = qr \cdot (qf + rf) = (qr)^2$, since the relation qfr holds. This completes the proof of the theorem

From the results obtained above, and the above mentioned theorem of Blumenthal, the final characterization theorem follows as an immediate corollary:

THEOREM 10.3. *An abstract set M forms a Euclidean space of finite dimension provided the following postulates are satisfied:*

I. Metricity. *With every pair of elements p and q of M there is associated a non-negative real number pq, such that* (1) *$pq = 0$ if and only if $p = q$,* (2) *$pq = qp$, and* (3) *if p, q, and r are elements of M, $pq + qr \geqslant pr$.*

II. Finite Compactness. *Every bounded infinite subset of M has an accumulation element.*

III. Convexity. *If p and r are distinct elements of M, there is an element q of M between p and r; that is, $pq + qr = pr$, $p \neq q \neq r$.*

IV. External Convexity. *If p and q are distinct elements of M there is an element r of M such that q is between p and r.*

V. Two-triple Property. *For any quadruple of pairwise distinct elements of M, linearity of any two triples implies linearity of the remaining two triples.*

VI. Young Postulate. *If p, q, and r are elements of M, and q' and r' are mid-elements of p and q, and of p and r, respectively, then $q'r' = \frac{1}{2}qr$.*

VII. Ficken Postulate. *If f is a foot of an element p on a line L (p not on L) and if q and r are elements of L with $fq = fr$, then $sq = sr$, for each element s of a line $L(p,f)$ joining p and f.*

In conclusion the first characterization theorem, Theorem 6.2, is restated in the following way: *Postulates* I, II, III, IV, V, VI, *define a complete normed linear space* (*Banach space*).

Bibliography

[1] N. Aronszajn, *Charactérisation métrique de l'espace de Hilbert, des espaces vectorielles, et de certaines groupes métriques*, Comptes Rendus de l'Academie des Sciences, Paris, 201 (1935), pp. 811-813, 873-875.

[2] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford 1953.

[3] H. Busemann, *On Leibnitz's Definition of Planes*, American Journal of Mathematics 63 (1941), pp. 104-111.

[4] F. A. Ficken, *Note on the Existence of Scalar Products in Normed Linear Spaces*, Annals of Mathematics, ser. 2, 45 (1944), pp. 362-366.

[5] M. Fréchet, *Définitions de la Somme et du Produit par Scalaire en Termes de Distance*, Annales Scientifiques de l'École Normale Supérieure (3), 75 (1958), pp. 223-255.

[6] W. H. Young, *On the Analytical Basis of Non-Euclidean Geometry*, American Journal of Mathematics 33 (1911), pp. 249-286.

Reçu par la Rédaction le 25. 2. 1963