Remarks on real-compact spaces

by

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The theory of real-compact spaces is in many respects analogous to the theory of compact spaces [1]. This fact is well known and has been emphasized more than one. For example, in [2] a general scheme of definitions of various compactness concepts is given, from which the compactness and the real-compactness can be obtained as particular cases. The term real-compact, which now replaces the original term Q-space, stresses both the above analogy and the fact that the real line E plays an important part in the theory of real-compact spaces. The part is analogous to that played by the interval I in the theory of compact spaces.

Many theorems concerning compact spaces have counterparts in the theory of real-compact spaces. Real-compactness is, like compactness, multiplicative and hereditary with respect to closed subspaces. The well-known characterization of compact spaces as closed subsets of products of intervals corresponds to the characterization of real-compact spaces as closed subsets of products of real lines. It is well known that for any space X one can find a compact space βX, called the Čech-Stone compactification of X, containing X as a dense subspace and such that every function f: $X \to I$ (or more generally, every mapping of X into a compact space) can be extended to βX. Similarly, for any space X one can find a real-compact space vX, called the Hewitt real-compactification of X, containing X as a dense subspace and such that any function f: $X \to E$ (or more generally, any mapping of X into a real-compact space) can be extended to vX. Finally, for many theorems about rings of all continuous functions defined on compact spaces there exist analogous theorems about rings of continuous functions defined on real-compact spaces.

The purpose of the paper is to give a counterpart of a theorem on extension of mappings with values in compact spaces and to make some simple remarks concerning the class of all real-compactifications of a space.

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(1) The definition and an outline of the theory of real-compact spaces can be found in [3]. The terminology used here is as in [4]. All spaces, if the contrary is not stated, are assumed to be completely regular (Tychonoff) and all mappings to be continuous. The (closed) unit interval $[0, 1]$ is denoted by I; E denotes the real line.

1. Extension of mappings. The following theorem is proved by Tajmanov in [5].

THEOREM 1. *Let A be a dense subspace of an arbitrary topological space X, and let $f\colon A\to Y$ be a mapping of A into the compact space Y. The mapping f has an extension from X to Y if and only if, for any pair F_1, F_2 of closed disjoint subsets of Y, we have*

$$\overline{f^{-1}(F_1)}\cap\overline{f^{-1}(F_2)}=0\,,$$

where the bar denotes the closure operation in the space X.

The same theorem, in dual formulation [2], is proved by Eilenberg and Steenrod in [1], p. 280.

The counterpart of Theorem 1 in the theory of real-compact spaces is the following theorem.

THEOREM 2. *Let A be a dense subspace of an arbitrary topological space X and let $f\colon A\to Y$ be a mapping of A into the real-compact space Y. The mapping f has an extension from X to Y if and only if, for any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of Y such that $\bigcap_{i=1}^{\infty}F_i=0$, we have*

$$\bigcap_{i=1}^{\infty}\overline{f^{-1}(F_i)}=0\,,$$

where the bar denotes the closure operation in the space X [3].

Before proving Theorem 2 we shall prove two lemmas.

LEMMA 1. *Let A be a dense subspace of an arbitrary topological space X, let $\{X_s\}_{s\in S}$ be a family of topological spaces, and let Y be a closed subspace of the product $\underset{s\in S}{P}X_s$. Any mapping $f\colon A\to Y$ has an extension from X to Y if and only if the function $f_s=p_sf\colon A\to X_s$, where $p_s\colon \underset{s\in S}{P}X_s\to X_s$ is the projection on the s-axis, has an extension from X to X_s for every s in S.*

Proof. The "only if" part is obvious. We shall prove the "if" part. Let $f_s^*\colon X\to X_s$, for every $s\in S$, be an extension of f_s. The function $f^*\colon X\to\underset{s\in S}{P}X_s$, where $f^*(x)=\{f_s^*(x)\}$, is an extension of $f\colon A\to\underset{s\in S}{P}X_s$. Since A is dense in X and Y closed in $\underset{s\in S}{P}X_s$, we have

$$f^*(X)=f^*(\bar{A})\subset\overline{f^*(A)}=\overline{f(A)}\subset\bar{Y}=Y$$

and $f^*\colon X\to Y$, i.e. f has an extension from X to Y.

LEMMA 2. *Let A be a dense subspace of an arbitrary topological space X and let $f\colon A\to E$ be a real-valued function defined on A. If for any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of E such that $\bigcap_{i=1}^{\infty}F_i=0$, we have*

$$\bigcap_{i=1}^{\infty}\overline{f^{-1}(F_i)}=0\,,$$

where the bar denotes the closure operation in the space X, then the function f has an extension from X to E.

Proof. Let J_i denote the open interval from $-i$ to i and let

$$F_i=E\setminus J_i\quad\text{and}\quad G_i=X\setminus\overline{f^{-1}(F_i)}\,,\quad\text{for}\quad i=1,2,\ldots$$

Since $\bigcap_{i=1}^{\infty}F_i=\bigcap_{i=1}^{\infty}(E\setminus J_i)=E\setminus\bigcup_{i=1}^{\infty}J_i=0$, we have $\bigcap_{i=1}^{\infty}\overline{f^{-1}(F_i)}=0$, by the assumption. And it follows that

$$\bigcup_{i=1}^{\infty}G_i=\bigcup_{i=1}^{\infty}[X\setminus\overline{f^{-1}(F_i)}]=X\setminus\bigcap_{i=1}^{\infty}\overline{f^{-1}(F_i)}=X\,.$$

Let $f_i=f|A\cap G_i$ for $i=1,2,\ldots$ be the function f reduced to $A\cap G_i$. We have

$$f(A\cap G_i)=f\big(A\setminus\overline{f^{-1}(F_i)}\big)\subset f\big(A\setminus f^{-1}(F_i)\big)\subset E\setminus F_i=J_i\subset\bar{J}_i$$

and $f_i\colon A\cap G_i\to\bar{J}_i$ for $i=1,2,\ldots$ From the assumption of our lemma and Theorem 1 we infer that there exists an extension $f_i^*\colon G_i\to\bar{J}_i$. Since the set $A\cap G_i$ is dense in $G_i\subset G_{i+1}$, we have

$$f_{i+1}^*|G_i=f_i^*\quad\text{for}\quad i=1,2,\ldots$$

The mapping $f^*\colon X\to E$ defined by the equation

$$f^*(x)=f_i^*(x)\,,\quad\text{where}\quad x\in G_i\,,$$

is continuous. It is easy to see that $f^*\colon X\to E$ is the desired extension of f.

Proof of Theorem 2. The "only if" part follows from the inclusion

$$\overline{f^{-1}(F_i)}\subset(f^*)^{-1}(F_i)\,,$$

where $f^*\colon X\to Y$ is an extension of f.

[2] This formulation is as follows

THEOREM 1′. *Let A be a dense subspace of an arbitrary topological space X and let $f\colon A\to Y$ be a mapping of A into the compact space Y. The mapping f has an extension from X to Y if and only if, for every finite open covering $\{U_i\}_{i=1}^{m}$ of Y, there exists a finite open covering $\{V_i\}_{i=1}^{n}$ of X such that the covering $\{A\cap V_i\}_{i=1}^{n}$ of the subspace A is a refinement of $\{f^{-1}(U_i)\}_{m}^{i=1}$.*

[3] The dual formulation of Theorem 2 can be obtained from Theorem 1′ by replacing "compact" by "real-compact", "finite" by "countable", and n and m by ∞.

For the proof of the "if" part, let us notice that the space Y being real-compact, can be regarded as a closed subspace of the product $\underset{s\in S}{P} E_s$, where $E_s = E$ for any $s \in S$. By Lemmas 1 and 2 it suffices to show that for every $s \in S$ and any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of $E_s = E$ such that $\bigcap_{i=1}^{\infty} F_i = 0$ we have $\bigcap_{i=1}^{\infty} \overline{f_s^{-1}(F_i)} = 0$. The last equality follows from the fact that

$$f_s^{-1}(F_i) = (p_s f)^{-1}(F_i) = f^{-1}\big(Y \cap p_s^{-1}(F_i)\big),$$

$p_s^{-1}(F_i)$ is closed in $\underset{s\in S}{P} X_s$, and

$$\bigcap_{i=1}^{\infty} [Y \cap p_s^{-1}(F_i)] = Y \cap p_s^{-1}(\bigcap_{i=1}^{\infty} F_i) = 0\,.$$

Remark 1. If we take for X the space of all ordinal numbers less than or equal to Ω (the first uncountable ordinal number) with the order topology, for $A = Y$ the space $X \setminus \Omega$ and for f the identity map, we infer that the assumption of real-compactness of Y cannot be omitted in Theorem 2.

Remark 2. In the proof of Theorem 2 we have regarded only the sequences $\{F_i\}_{i=1}^{\infty}$, where F_i is of the form $Y \cap p_s^{-1}(F)$ for $F = \overline{F} \subset E$, i.e. the sequences $\{F_i\}_{i=1}^{\infty}$, where F_i is the z-set (4); hence in Theorem 2 one can replace "closed subsets" by "z-sets". This modified formulation of Theorem 2 is false for any space Y which is not real-compact. In fact, the identity map $Y \to Y$ cannot be extented to a mapping from vY to Y though the intersection of closures in vY of z-sets $\{F_i\}_{i=1}^{\infty}$, satisfying the condition $\bigcap_{i=1}^{\infty} F_i = 0$, is empty.

2. Real-compactifications.

By a *real-compactification* of a space X we mean an arbitrary real-compact space containing X as a dense subset. More precisely, a real-compactification of a space X is a pair (r, rX), where rX is a real-compact topological space and $r\colon X \to rX$ a homeomorphism of X onto a dense subspace $r(X)$ of rX. For brevity, we shall denote real-compactifications of a space X by rX, r_1X, r_2X, etc.; the prefix r, r_1, r_2, etc. denotes the embedding of X in rX, r_1X, r_2X, etc., respectively. We can define a partial order $\geqslant$ in the class of all real-compactifications of X. Namely, we say that r_1X is greater then r_2X and write $r_1X \geqslant r_2X$ if there exists a mapping $f\colon r_1X \to r_2X$ such that

(4) By a z-set in the space X we mean the set of the form $g^{-1}(0)$, where $g\colon X \to E$. Every closed G_δ-set in the normal space is a z-set. Evidently the counterimage, by any mapping, of a z-set is also a z-set.

$fr_1 = r_2$. If there exists an f which is a homeomorphism, we shall say that real compactifications r_1X and r_2X are equivalent. It is easy to see that r_1X and r_2X are equivalent if and only if the relations $r_1X \geqslant r_2X$ and $r_2X \geqslant r_1X$ both hold.

Lemma 1 and the possibility of regarding any real-compact space as a closed subspace of a product or real lines imply

THEOREM 3. *The real-compactification r_1X of a space X is greater than the real-compactification r_2X of this space if and only if every function $f\colon X \to E$ which can be extended over r_2X can also be extended over r_1X, i.e. if from the existence of a function $f_2\colon r_2X \to E$ such that $f_2r_2 = f$ follows the existence of a function $f_1\colon r_1X \to E$ satisfying the equality $f_1r_1 = f$.*

By Theorem 2 we have the following two theorems, containing intrinsic criteria for the relation $r_1X \geqslant r_2X$ and for the equivalence of real-compactifications:

THEOREM 4. *The real-compactification r_1X of a space X is greater than the real-compactification r_2X of this space if and only if, for any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of X, we have the implication*

$$\Big(\bigcap_{i=1}^{\infty} \overline{r_2(F_i)} = 0\Big) \Rightarrow \Big(\bigcap_{i=1}^{\infty} \overline{r_1(F_i)} = 0\Big)\,.$$

THEOREM 5. *The real-compactifications r_1X and r_2X of a space X are equivalent if and only if, for any sequence $\{F_i\}_{i=1}^{\infty}$ of closed subsets of X, we have the equivalence*

$$\Big(\bigcap_{i=1}^{\infty} \overline{r_1(F_i)} = 0\Big) \equiv \Big(\bigcap_{i=1}^{\infty} \overline{r_2(F_i)} = 0\Big)\,.$$

In the sequel we shall regard equivalent real-compactifications of a space X as equal. The Hewitt real-compactification vX is the greatest element in the class of all real-compactifications of a space X, partially ordered by the relation $\geqslant$. The question about the existence of a smallest one is answered by the following

THEOREM 6. *The smallest element in the class of all real-compactifications of a space X, partially ordered by the relation $\geqslant$, exists if and only if X is locally compact. The smallest real-compactification of a locally compact space X is the one point compactification of X* (5).

(5) The one point compactification of a locally compact space X is the set $\omega X = X \cup \{\infty\}$, where ∞ is not a member of X, with the topology whose members are open sets of X and the sets $U \cup (\omega X \setminus F)$ such that U is an open and F a compact subset of X. The mapping $\omega\colon X \to \omega X$, where $\omega(x) = x$, is the homeomorphism of X onto the dense subset $\omega(X) = X$ of ωX.

Proof. If X is locally compact, then, for any real-compactification rX of the space X, the set $r(X)$ is open in rX since $\overline{r(X)} = rX$ (cf. [3], p. 45). Hence the mapping $f\colon rX \to \omega X$ defined by the formula

$$f(y) = \begin{cases} r^{-1}(y) & \text{if} \quad y \in r(X), \\ \infty & \text{if} \quad y \in rX \setminus r(X) \end{cases}$$

is continuous because the counterimage of an open set $U \cup (\omega X \setminus F) \subset \omega X$ is open in rX. Since $fr = \omega$, we have $rX \geqslant \omega X$.

Now let us suppose that wX is the smallest real-compactification of the space X. The space wX, as an image of βX, is compact and thus it is the smallest compactification of X. From this fact we infer (cf. [3], p. 150) that X is locally compact and wX is equal to ωX.

Finally, let us notice the following

THEOREM 7. *A space X has a unique real-compactification if and only if it has a unique compactification* ([6]).

Proof. It suffices to prove that a non-compact space X which has exactly one compactification possesses only one real-compactification.

Let X be a non-compact space with unique compactification. We then have $\omega X = \beta X$. From the inclusion $vX \subset \beta X$ it follows that either $vX = X$, or $vX = \beta X$. Since every space X with unique compactification is pseudocompact (cf. [3], p. 95), i.e. every function $f\colon X \to E$ is bounded and every real-compact and pseudocompact space is compact, we infer that $X \neq vX = \beta X = \omega X$, i.e. the smallest and the greatest real-compactifications of X are equal. Thus the space X has a unique real-compactification.

References

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([6]) For various characterizations of such spaces, see [3], pp. 95 and 238.

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