The global dimension of the group rings of abelian groups

by

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Let R be a ring with an identity element; all modules that will be considered are left modules. If A is a R-module then an projective resolution of A is an exact sequence

$$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots$$

of R-modules, $P_0, P_1, \ldots$ being projective modules. If there exists a projective resolution of A such that $P_k = 0$ for $k > n$, but there is no such resolution with $P_n = 0$, then we define the *left projective dimension* of A as $\mathrm{l.dim}_R A = n$; if there exists no such number n then we put $\mathrm{l.dim}_R A = \infty$.

It is known that the (left) projective dimension of A is the supremum of all numbers m such that there exists R-module B satisfying $\mathrm{Ext}_R^m(A, B) \neq 0$.

The (left) *global dimension* of the ring R is defined as

$$\mathrm{l.gl.dim}\, R = \sup_A \mathrm{l.dim}_R A$$

and A varies over all (left) R-modules.

It is known that the global dimension of R is the supremum of all numbers m such that there exist R-modules A, B satisfying $\mathrm{Ext}_R^m(A, B) \neq 0$.

The (left) *weak dimension* of R-module A is defined as the supremum of all numbers m such that there exists a right R-module B such that $\mathrm{Tor}_m^R(B, A) \neq 0$. The (left) weak dimension of A is denoted by $\mathrm{l.w.dim}_R A$.

The weak global dimension of the ring R is defined as the supremum of all numbers m such that there exist a left R-module A and a right R-module B such that $\mathrm{Tor}_m^R(B, A) \neq 0$. The weak global dimension of R is denoted by $\mathrm{w.gl.dim}\, R$.

It is well known (see [5]) that

$$\mathrm{w.gl.dim}\, R \leqslant \mathrm{l.gl.dim}\, R ,$$

and if R is left Noetherian ring then

$$\text{w.gl.dim}\, R = \text{l.gl.dim}\, R\,.$$

Let Π be a group with an operation of multiplication; the group ring $R(\Pi)$ consists of all elements of the form

$$\sum_{\sigma \in \Pi} r_\sigma \cdot \sigma$$

subjected to the conditions $r_\sigma \in R$, $r_\sigma = 0$ for almost all $\sigma \in \Pi$. The addition and multiplication are defined in a natural way.

The structure of abelian groups Π and rings R such that the weak global dimension of the group ring $R(\Pi)$ is finite was determined by A. J. Douglas in [3]. It was proved that $\text{w.gl.dim}\, R(\Pi)$ is finite if and only if the following conditions are satisfied:

(i) $\text{w.gl.dim}\, R < \infty$,

(ii) $\text{rank}\, \Pi < \infty$,

(iii) *if the group Π contains an element of finite order q then $qR = R$.*

If all the above conditions hold then

$$\text{w.gl.dim}\, R(\Pi) = \text{w.gl.dim}\, R + \text{rank}\, \Pi\,.$$

If we put $R = Z$ (the ring of rational integers) then we get as a corollary: *If Π is an abelian group then* $\text{w.gl.dim}\, Z(\Pi)$ *is finite if and only if the group Π is torsion free and of finite rank; moreover*

$$\text{w.gl.dim}\, Z(\Pi) = \text{rank}\, \Pi + 1\,.$$

The group of rational integers Z may be viewed as a $Z(\Pi)$-module if we admit elements of Π as trivial operators on Z. K. Varadarajan has proved in [6] that *if Π is an abelian group then* $\text{l.dim}_{Z(\Pi)} Z < \infty$ *if and only if Π is torsion free and rank $\Pi < \infty$*; moreover

$$\text{l.dim}_{Z(\Pi)} Z = \begin{cases} \text{rank}\, \Pi & \text{if } \Pi \text{ is a finitely generated group}\,, \\ \text{rank}\, \Pi + 1 & \text{in the opposite case}\,. \end{cases}$$

In the present paper we determine the global dimension of the group ring $Z(\Pi)$ of an abelian group Π. We prove that $\text{gl.dim}\, Z(\Pi) < \infty$ if and only if Π is a torsion free group of finite rank; moreover

$$\text{gl.dim}\, Z(\Pi) = \begin{cases} \text{rank}\, \Pi + 1 & \text{if } \Pi \text{ is a finitely generated group}\,, \\ \text{rank}\, \Pi + 2 & \text{in the opposite case}\,. \end{cases}$$

The first of the above equalities follows by the result of A. J. Douglas, because $Z(\Pi)$ is a Noetherian ring.

1. In this section we state some results that will be needed in the sequel.

(1.1) *If Π is a finite cyclic group then* $\text{gl.dim}\, Z(\Pi) = \infty$.

Proof. It is known (see [2], Chapter XII, § 7) that for even n

$$\text{Ext}^n_{Z(\Pi)}(Z, Z) = H^n(\Pi, Z) \neq 0;$$

hence $\text{gl.dim}\, Z(\Pi) = \infty$.

(1.2) *If Π' is a subgroup of an abelian group Π then*

$$\text{gl.dim}\, Z(\Pi') \leqslant \text{gl.dim}\, Z(\Pi)\,.$$

Proof. The ring $Z(\Pi)$ may be considered as a free $Z(\Pi')$-module; then

$$\text{Ext}^p_{Z(\Pi')}\big(Z(\Pi), C\big) = 0$$

for all $p > 0$ and all $Z(\Pi')$-modules C.

The ring $Z(\Pi)$ may be viewed as a left $Z(\Pi)$-module and a right $Z(\Pi')$-module. If A is an arbitrary (left) $Z(\Pi')$-module then $Z(\Pi)\otimes_{Z(\Pi')} A$ becomes a left $Z(\Pi)$-module; if this module is considered as $Z(\Pi')$-module, then it is a direct sum of modules isomorphic with A. If we apply Proposition 4.1.4 (Chapter VI of [2]) then we get for an arbitrary $Z(\Pi')$-module B

$$(1) \qquad \text{Ext}^m_{Z(\Pi)}\big(Z(\Pi)\otimes_{Z(\Pi')} A, {}^{(\varphi)}C\big) \approx \text{Ext}^m_{Z(\Pi')}\big(Z(\Pi)\otimes_{Z(\Pi')} A, C\big)$$

where

$${}^{(\varphi)}C = \text{Hom}_{Z(\Pi')}\big(Z(\Pi), C\big)\,.$$

If $\text{Ext}^m_{Z(\Pi')} \neq 0$ and $\text{Ext}^m_{Z(\Pi')}(A, C) \neq 0$ then by (1) it follows that $\text{Ext}^m_{Z(\Pi)} \neq 0$ and the proof is finished.

(1.3) If $\Pi_1 \subset \Pi_2 \subset \ldots$ *is an increasing sequence of abelian groups and* $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$ *then*

$$\sup_n \text{gl.dim}\, Z(\Pi_n) \leqslant \text{gl.dim}\, Z(\Pi) \leqslant 1 + \sup_n \text{gl.dim}\, Z(\Pi_n)\,.$$

Proof. The first inequality follows by (1.2), and the second one is a consequence of Corollary 1 of [1] when applied to the ring $Z(\Pi)$ as a direct limit of rings $Z(\Pi_n)$.

(1.4) *Let R be a ring with an identity element and let $s_2, s_3, \ldots$ be elements of R which are non units and non zero divisors; if F is a free R-module and $v_1, v_2, \ldots$ are free generators of F then the system of equations*

$$(2) \qquad x_n - s_{n+1} x_{n+1} = v_n \qquad (n = 1, 2, \ldots)$$

admits no solutions in F.

Proof. Let F^* be the complete direct product of modules Rv_n:

$$F^* = \prod_{n=1}^{\infty} Rv_n .$$

The R-module F^* contains F. If we denote

$$p_{nk} = s_{n+1}s_{n+2}\dots s_k \quad \text{for} \quad k > n ,$$
$$p_{nn} = 1 ,$$
$$p_{nk} = 0 \quad \text{for} \quad k < n ,$$

and

$$y_n^* = \{p_{nk}v_k\} \in F^* ,$$

then we have

$$y_n^* - s_{n+1}y_{n+1}^* = \{p_{nk}v_k\} - s_{n+1}\{p_{(n+1)k}v_k\} = v_k .$$

Let us assume that the system of equations (2) admits solutions $x_1, x_2, \dots$ in F; then the elements $u_n = x_n - y_n^*$ belong to F^* and $u_n = s_{n+1}u_{n+1}$. Moreover, $u_1 = s_2 \dots s_n u_n$ and u_1 is of the form $u_1 = \{r_k v_k\}$, $r_k \in R$. Since $x_1 = u_1 + y_1^* \in F$, then $r_k + p_{1k} = 0$ for $k > N$.

Let us compute the coefficient r at v_{N+1} of x_{N+2}. We have $x_{N+2} = u_{N+2} + y_{N+2}^*$ and $y_{N+2}^* = \{p_{N+2,k}v_k\}$. Consequently, $s_2 \dots s_{N+2} r = r_{N+1} = -p_{1,N+1} = -s_2 \dots s_{N+1}$ and $s_{N+2} \cdot r = -1$; thus s_{N+2} is a unit in R on the contrary to our assumption.

2. In this section we determine the global dimension of the group ring of a free abelian group.

LEMMA 1. *If* $\mathrm{l.gl.dim} R = m < \infty$, Π *is an infinite cyclic group then*

$$\mathrm{Ext}_{R(\Pi)}^{m+1}(A, B) \approx \mathrm{Ext}_R^m(A, B)$$

for any R*-modules* A, B *(elements of* Π *operate trivially on* A *and* B*).*

Proof. If σ denotes a generator of Π then the group ring $R(\Pi)$ contains the polynomial ring $R[\sigma]$.

At first we prove

$$\mathrm{Ext}_{R[\sigma]}^{m+1}(A, B) \approx \mathrm{Ext}_R^m(A, B) .$$

It was proved in [4] that if Λ, Γ are K-algebras then in the situation $({}_{\Lambda-\Gamma}A, {}_{\Lambda-\Gamma}B)$ there exists a spectral sequence

(I) $$H^p\big(\Gamma, \mathrm{Ext}^q(A, B)\big) \underset{p}{\Rightarrow} \mathrm{Ext}_{\Lambda\otimes_K\Gamma}^n(A, B) .$$

If we put $K = Z$, $\Lambda = R$, $\Gamma = Z[\sigma]$ then $\Lambda \otimes_K \Gamma = R[\sigma]$ and we obtain

(3) $$H^p\big(Z[\sigma], \mathrm{Ext}_R^q(A, B)\big) \underset{p}{\Rightarrow} \mathrm{Ext}_{R[\sigma]}^n(A, B)$$

for any $R(\Pi)$-modules A, B.

Since $\dim Z[\sigma] = 1$, then the property (Q) of polynomial rings (see [4], pp. 82-83) implies

(4) $$H^p(Z[\sigma], C) = 0 \quad \text{for} \quad p > 1 ,$$
$$H^1(Z[\sigma], C) \approx C$$

for any symmetric $Z[\sigma]^e$-module C.

If we admit elements of Π to operate trivially on both sides of A and B, then $\mathrm{Ext}_R^q(A, B)$ becomes a symmetric $Z[\sigma]^e$-module.

By the "maximum term principle" of spectral sequences, by (3) and (4) we get

$$\mathrm{Ext}_R^m(A, B) \approx H^1\big(Z[\sigma], \mathrm{Ext}_R^m(A, B)\big) \approx \mathrm{Ext}_{R[\sigma]}^{m+1}(A, B) .$$

The elements of Π operate trivially on A; then $R(\Pi) \otimes_{R[\sigma]} A \approx A$. The ring $R(\Pi)$ is a sum of an increasing sequence of free cyclic $R[\sigma]$-modules; then

$$\mathrm{Tor}_p^{R[\sigma]}\big(R(\Pi), A\big) = 0 \quad \text{for all} \quad p > 0$$

and we can apply Proposition 4.1.3 (Chapter VI of [2]) to the inclusion $R[\sigma] \to R(\Pi)$. Thus

$$\mathrm{Ext}_{R[\sigma]}^{m+1}(A, B) \approx \mathrm{Ext}_{R(\Pi)}^{m+1}\big(R(\Pi) \otimes_{R[\sigma]} A, B\big)$$

and by the preceeding formulae we have

$$\mathrm{Ext}_{R(\Pi)}^{m+1}(A, B) \approx \mathrm{Ext}_R^m(A, B) .$$

LEMMA 2. *If* Π *is a free abelian group then*

$$\mathrm{l.gl.dim} R(\Pi) = \mathrm{l.gl.dim} R + \mathrm{rank}\, \Pi .$$

Proof. If $\mathrm{l.gl.dim} R = \infty$ then by (1.2) we have

$$\mathrm{l.gl.dim} R(\Pi) \geqslant \mathrm{l.gl.dim} R(\{1\}) = \mathrm{l.gl.dim} R = \infty$$

and both sides of the equality are infinite.

If $\mathrm{l.gl.dim} R = m < \infty$ and Π is a free abelian group of rank 1 then by formula (3), p. 74, of [4] we get

$$\mathrm{l.gl.dim} R(\Pi) = \mathrm{l.gl.dim} R \otimes_Z Z(\Pi) \leqslant \dim Z(\Pi) + \mathrm{l.gl.dim} R ,$$

$\dim Z(\Pi)$ being the left projective dimension of $Z(\Pi)$ considered as a module over the enveloping algebra of $Z(\Pi)$. By Theorem 6.2 (Chapter X of [2]) we have $\dim Z(\Pi) = \dim_{Z(\Pi)} Z$; then

$$\mathrm{l.gl.dim} R(\Pi) \leqslant \mathrm{l.dim}_{Z(\Pi)} Z + \mathrm{l.gl.dim} R = 1 + m$$

and from the Lemma 1 it follows that $\mathrm{l.gl.dim} R(\Pi) \geqslant m+1$. Consequently,

$$\mathrm{l.gl.dim} R(\Pi) = 1 + \mathrm{l.gl.dim} R = \mathrm{l.gl.dim} R + \mathrm{rank}\,\Pi .$$

If Π is a free abelian groups and $\sigma_1, \ldots, \sigma_r$ are free generators of Π and $\Pi' = \{\sigma_1, \ldots, \sigma_{r-1}\}$, then the group ring $R(\Pi)$ is naturally isomorphic with the group ring of infinite cyclic group $\{\sigma_r\}$ over the ring $R(\Pi')$ and the induction step follows by the preceeding part of the proof.

If Π is a free abelian group of infinite rank then by using (1.2) and the Lemma for groups of finite rank we have

$$\mathrm{l.gl.dim} R(\Pi) \geqslant \mathrm{l.gl.dim} R + r \quad \text{for any} \quad r = 1, 2, \ldots$$

and the lemma follows.

3. In this section we determine the global dimension of the group ring $Z(\Pi)$ of an abelian group of rank 1. This ring is a commutative one, then we may omitt the letter l in all dimensions.

LEMMA 3. *Let Π be a non cyclic torsion free abelian group of rank 1; then $\dim_{Z(\Pi)} Z_r = \mathrm{gl.dim} Z(\Pi) = 3$ (Z_r denotes the cyclic group of order r and Π operates trivially on Z_r).*

Proof. There exists an increasing sequence of infinite cyclic groups $\Pi_1 \subset \Pi_2 \subset \ldots$ such that $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$. Then by (1.3) and Lemma 2 we have

$$2 \leqslant \mathrm{gl.dim} Z(\Pi) \leqslant 3 . \tag{5}$$

Let σ_n ($n = 1, 2, \ldots$) be generators of the groups Π_n such that for some integers $t_n > 1$ we have

$$\sigma_n^{t_n} = \sigma_{n-1} \quad (n = 2, 3, \ldots) .$$

If we denote

$$s_n = 1 + \sigma_n + \sigma_n^2 + \ldots + \sigma_n^{t_n - 1} ,$$

then $1 - \sigma_{n-1} = (1 - \sigma_n) s_n$.

We show next that the exact sequence

$$0 \leftarrow Z \overset{\varepsilon}{\leftarrow} P_0 \overset{d_1'}{\leftarrow} P_1 \overset{d_2'}{\leftarrow} P_2 \leftarrow 0 ,$$

where

$P_0 = Z(\Pi)$,

P_1 is a free $Z(\Pi)$-module on free generators $x_1, x_2, \ldots$,

P_2 is a free $Z(\Pi)$-module on free generators $y_1, y_2, \ldots$,

ε is the unit augmentation,

$d_1'(x_n) = 1 - \sigma_n$,

$d_2'(y_n) = x_n - s_{n+1} x_{n+1}$,

is the shortest projective resolution of $Z(\Pi)$-module Z.

The elements σ_n generate all the group Π; then $\mathrm{Ker}\,\varepsilon = \mathrm{Im}\, d_1'$.

Since $d_1'(x_n - s_{n+1} x_{n+1}) = 1 - \sigma_n - s_{n+1}(1 - \sigma_{n+1}) = 0$, we have $\mathrm{Im}\, d_2' \subset \mathrm{Ker}\, d_1'$. On the other hand, if $r_1 x_1 + \ldots + r_n x_n$ belongs to $\mathrm{Ker}\, d_1'$ then because of the relation

$$x_i = d_2'(y_i) + s_{i+1} x_{i+1}$$

we have

$$r_1 x_1 + \ldots + r_n x_n = d_2'(r_1' y_1 + \ldots + r_n' y_n) + r x_{n+1} ,$$

and $r x_{n+1}$ is in $\mathrm{Ker}\, d_1'$. Consequently, $d_2'(r x_{n+1}) = r(1 - \sigma_{n+1}) = 0$ and $r = 0$, because $Z(\Pi)$ has no zero divisors.

If $\mathrm{Im}\, d_1'$ would be a projective module then the exact sequence

$$0 \leftarrow \mathrm{Im}\, d_1' \leftarrow P_1 \overset{d_2'}{\leftarrow} P_2 \leftarrow 0$$

would split and then we would have a $Z(\Pi)$-homomorphism $\varrho\colon P_1 \to P_2$ such that $\varrho d_2'$ is the identity on P_2. Thus for all $n = 1, 2, \ldots$ we have

$$y_n = \varrho d_2'(y_n) = \varrho(x_n - s_{n+1} x_{n+1}) = \varrho(x_n) - s_{n+1} \varrho(x_{n+1})$$

contradicting (1.4) (s_{n+1} are non units in $Z(\Pi)$).

The group Z_r admits a free Z-resolution

$$0 \leftarrow Z_r \leftarrow R_0 \overset{d_1''}{\leftarrow} R_1 \leftarrow 0$$

where $R_0 = R_1 = Z$, $d_1''(m) = rm$.

Let S be the tensor product of the above resolutions, $P, R, S = P \otimes_Z R$. S is a free, acyclic $Z(\Pi)$-complex. In fact, the spectral sequence of the complex S filtered with respect to the first index is

$$E_{p,q}^0 = P_p \otimes_Z R_q ;$$

since P_p is Z-free, then

$$E_{p,0}^1 = P_p \otimes_Z Z_r ,$$

$$E_{p,q}^1 = 0 \quad \text{for} \quad q > 0 .$$

Moreover $E_{p,0}^2 = 0$ for $p > 0$ because P_p are Z-free and $E_{0,0}^2 = Z \otimes_Z Z_r = Z_r$. Consequently, $Z(\Pi)$-complex S is a projective $Z(\Pi)$-resolution of Z_r.

By inequality (5) it is sufficient to prove that the module

$$M = \mathrm{Im}(S_2 \to S_1)$$

is not a projective one. We have an exact sequence

$$0 \leftarrow M \leftarrow S_2 \overset{d_3}{\leftarrow} S_3 \leftarrow 0 ;$$

if M would be a projective $Z(\Pi)$-module then there would exists a $Z(\Pi)$-homomorphism $\varrho\colon S_2 \to S_3$ such that ϱd_3 is the identity on S_3. A free

$Z(\Pi)$-base of S_3 consists of elements $z_n = y_n \otimes 1$ $(n = 1, 2, \ldots)$. By the definition of differential operator d_3 we have

$$\begin{aligned} z_n = \varrho d_3(z_n) &= \varrho d_3(y_n \otimes 1) \\ &= \varrho[(d_2' y_n) \otimes 1 + y_n \otimes d_1''(1)] \\ &= \varrho[(x_n - s_{n+1} x_{n+1}) \otimes 1 + y_n \otimes r], \end{aligned}$$

and if we denote $\xi_n = \varrho(x_n \otimes 1)$ then

$$z_n = \xi_n - s_{n+1}\xi_{n+1} + \varrho(y_n \otimes 1).$$

If $\bar{S}_3 = S_3/rS_3$ then $\bar{S}_3$ is a free $Z_r(\Pi)$-module and the cosets $\bar{z}_n$ of z_n are free generators of $\bar{S}_3$. Moreover, there are elements $\bar{\xi}_n \in \bar{S}_3$ such that

$$\bar{z}_n = \bar{\xi}_n - \bar{s}_{n+1}\bar{\xi}_{n+1}.$$

It is easy to see that the elements $\bar{s}_n$ $(n > 1)$ of $Z_r(\Pi) = Z(\Pi)/rZ(\Pi)$ are non units and not zero divisors in $Z_r(\Pi)$; this contradicts (1.4) and the lemma is proved.

4. In this section we prove the following theorem.

THEOREM. *If Π is a torsion free abelian group which is not finitely generated then*

$$\operatorname{gl.dim} Z(\Pi) = \dim_{Z(\Pi)} Z_r = \operatorname{rank}\Pi + 2$$

for any non trivial finite cyclic group Z_r (Π operates trivially on Z_r).

If Π is a finitely generated torsion free abelian group then for an arbitrary ring R

$$\operatorname{l.gl.dim} R(\Pi) = \operatorname{l.dim}_{R(\Pi)} A = \operatorname{rank}\Pi + \operatorname{l.gl.dim} R$$

if $\operatorname{l.dim}_R A = \operatorname{l.gl.dim} R$ and Π operates trivially on A.

Proof. The second part of the theorem is a consequence of Lemmas 1 and 2.

Let Π be a torsion free abelian group which is not finitely generated. If $\operatorname{rank}\Pi = \infty$ then the theorem follows by (1.2) and by Lemmas 1 and 2. If $\operatorname{rank}\Pi = 1$ then the theorem follows by Lemma 3.

Let us assume that the theorem is proved for all groups of rank $< r$ $(r > 1)$ and $\operatorname{rank}\Pi = r$. It is easy to prove that the group Π contains a subgroup Π_0 of rank r which is not finitely generated group and is an extension of a group $\Pi_0' \approx Z$ by a torsion free group Π_0'' of rank $r-1$

$$0 \to \Pi_0' \to \Pi_0 \to \Pi_0'' \to 0.$$

Thus the group Π_0'' is not finitely generated.

By (1.3) we can deduce that

$$r+1 \leqslant \operatorname{gl.dim} Z(\Pi) \leqslant r+2. \tag{6}$$

Thus by (1.2) it is sufficient to prove the formulae of the theorem for the group Π_0.

For any $Z(\Pi_0'')$-module A and a $Z(\Pi_0)$-module C we have a spectral sequence

$$\operatorname{Ext}^p_{Z(\Pi_0'')}(A, H^q(\Pi_0', C)) \underset{p}{\Rightarrow} \operatorname{Ext}^n_{Z(\Pi_0)}(A, C)$$

(see [2], Chapter XVI, § 6). If we put $A = Z_r$ and for C we take a $Z(\Pi_0)$-module such that elements of Π_0' operate trivially and $\operatorname{Ext}^{r+1}_{Z(\Pi_0'')}(Z_r, C) \neq 0$, then

$$\begin{aligned} H^q(\Pi_0', C) &= 0 \quad \text{for} \quad q > 1, \\ H^1(\Pi_0', C) &= C \end{aligned}$$

and by the induction hypothesis

$$\operatorname{Ext}^s_{Z(\Pi_0'')} = 0 \quad \text{for} \quad s > r+1.$$

The "maximum term principle" of spectral sequences yields

$$\operatorname{Ext}^{r+2}_{Z(\Pi_0)}(Z_r, C) \approx \operatorname{Ext}^{r+1}_{Z(\Pi_0'')}(Z_r, C) \neq 0;$$

then

$$\operatorname{gl.dim} Z(\Pi_0) \geqslant \dim_{Z(\Pi_0)} Z_r \geqslant r+2$$

and the theorem follows by an application of (6) to the group Π_0.

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