On the most general plane closed point set through which it is possible to pass a pseudo-arc *

by

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Introduction. R. L. Moore and J. R. Kline proved [6] that, in the plane, a closed and compact point set M is a subset of an arc if and only if every component of M is either a single point or an arc t such that no point of t, except its end points, is a limit point of $M-t$. To show that this theorem remains true in the plane if the word "arc" is everywhere replaced by "pseudo-arc" is the primary purpose of this paper. In the case of the pseudo-arc, however, a simpler statement of this theorem is possible since it has been shown by Bing [1] and by Moise [5] that every point of a pseudo-arc is an end point of it. The author's ability to do the work represented by this paper is largely a result of his training by Professor R. L. Moore.

The notation and terminology used throughout this paper is, for the most part, that used by R. L. Moore in [7]. The space under consideration is the plane and S denotes the set of all points in the plane.

Definitions. A *simple chain* is a finite coherent collection C of domains such that there do not exist three domains of C such that, if d is one of them, then $C-d$ is coherent. If C is a simple chain, the domains of the collection C are called the *links* of C and, if d is a link of C such that $C-d$ is coherent, then d is called an *end link* of C. If C is a simple chain, the statement *XYZ in C* means that X, Y, and Z are links of C and $C-Y$ is the sum of two collections U and V, containing X and Z, respectively, such that no domain of U has a point in common with any domain of V. If X and V are end links of the simple chain C and Y and U are two links of C such that (1) either X is Y or XYU in C, and (2) either U is V or YUV in C, then Y is said to *precede U in the order from X to V in C*.

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If X and Y are end links of the simple chain C then C is called the *chain* XY and, if A and B are points of X and Y, respectively, C is said to be a *simple chain from* A *to* B.

If every link of the simple chain D is a subset of some link of the simple chain C, then D is said to be a *refinement* of C, and, if the closure of each link of D is a subset of some link of C, it is said to be a *strong refinement* of C.

If ε is a positive number, an *ε-chain* is a simple chain each link of which has diameter less than ε. If δ is a positive number and C is a simple chain such that, if X and Y are two nonintersecting links of C, then the distance from $\bar{X}$ to $\bar{Y}$ is greater than δ, then C is said to be a *δ-regular chain*. If there is a positive number δ such that the simple chain C is δ-regular, then C is a *regular chain*.

If M is a point set and there exists a sequence $C_1, C_2, \ldots$ such that (1) for each n, C_n is a $1/n$-chain, (2) for each n, C_{n+1} is a strong refinement of C_n, and (3) M is the common part of the point sets $C_1^*, C_2^*, \ldots$, then each such sequence is called a *defining sequence for* M. If every simple chain of a defining sequence α for the point set M is a regular chain, then α is said to be a *regular defining sequence for* M. If a point set has a defining sequence, it is a compact continuum and is called a *chainable* (or *snake-like*) *continuum*. If A is a point of a chainable continuum M and there exists a defining sequence α for M such that A belongs to an end link of each term of α, then A is said to be an *end point* of M. If A and B are two points of a chainable continuum M and there exists a defining sequence α for M such that each term of α is a simple chain from A to B, then M is said to be *chainable from* A *to* B.

The simple chain D is said to be *crooked in the simple chain* C [1] if and only if it is true that D is a strong refinement of C and, if XYU and YUV in C ($= XYUV$ in C) and X' and V' are two links of D intersecting X and V, respectively, then there exist links Y' and U' of D lying wholly in Y and U, respectively, such that $X'U'Y'V'$ in D. If A and B are two points of the continuum M and there exists a defining sequence $C_1, C_2, \ldots$ for M of simple chains from A to B such that, for each n, C_{n+1} is crooked in C_n, then M is called a *pseudo-arc*, [4].

THEOREM 1. *If M is a compact continuum and C is a simple chain covering M, there exists a regular chain covering M which is a strong refinement of C.*

Proof. Let C' denote a subcollection of C covering M such that no proper subcollection of C' covers M. Then C' is a simple chain. Let $L_1, L_2, \ldots, L_n$ denote the links, in that order, of C'. For each positive integer $i < n+1$, let K_i denote the set to which P belongs if and only if P is a point of $M \cdot L_i$ which lies in no other link of C'. Let $D_1, D_2, \ldots, D_n$ denote domains containing $K_1, K_2, \ldots, K_n$, respectively, such that there closures are mutually exclusive. For each $i < n+1$, let H_i denote the set to which P belongs if and only if P is a point of $M \cdot L_i$ not in D_{i-1} nor in D_{i+1}. If i and j are two positive integers $< n+1$ such that $|i-j| > 1$, H_i and H_j are mutually exclusive closed point sets and H_i and L_j are mutually exclusive. The point set $H_1 + H_2 + \ldots + H_n$ is M.

Let δ denote a positive number such that, if $i < n+1$, the distance from H_i to $S - L_i$ is less than 3δ and, for each $i < n+1$, denote by d_i the set of all points at a distance from H_i less than δ. Then E, the collection of all domains d_i, for all $i < n+1$, is a simple chain covering M, for each $i < n+1$, d_i is a subset of L_i, and, if i and j are two positive integers $< n+1$ such that $|i-j| > 1$, the distance from $\bar{d}_i$ to $\bar{d}_j$ is greater than δ.

THEOREM 2. *Suppose that M is a closed and compact point set, n is a positive integer, and C is a regular chain covering M such that either no proper subchain of C covers M or no end link of C intersects M. Then there exists a positive number ε such that, if G is a coherent collection of n domains, each with diameter less than ε, one domain of which contains a point of M, then $\bar{G}^*$ is a subset of some link of C.*

Proof. Let ε_1 denote a positive number such that the distance from M to $S - C^*$ is greater than ε_1. Let M' denote the set to which P belongs if and only if P is a point of M or P is a point such that the distance from P to M is not greater than $\varepsilon_1/2$. For each link L of C which is not a subset of any other link of C, let P_L denote a point which belongs to L but to no other link of C, and let M'_L denote the set to which P belongs if and only if P is either P_L or a point of M' which belongs to L but to no other link of C. Let ε_2 denote a positive number such that, if L is a link of C which is not a subset of any other link of C, then the distance from M'_L to $S - L$ is greater than ε_2. Let ε_3 denote a positive number such that C is an ε_3-regular chain.

Denote by ε a positive number which is less than each of the numbers $\varepsilon_1/2n$, ε_2/n, and ε_3/n, and suppose that G is a coherent collection of n domains, each of diameter less than ε, one of which contains a point of M. Then $\bar{G}^*$ has diameter less than $n\varepsilon$.

Suppose that $\bar{G}^*$ is not a subset of any link of C. Since $\bar{G}^*$ has diameter less than ε_1, it is a subset of C^*, and, since it has diameter less than ε_3, it does not intersect three different links of C. Indeed, there exist two intersecting links X and Y of C such that $X + Y$ contains $\bar{G}^*$. Suppose that $\bar{G}^*$ contains a point of M'_X. Since $\bar{G}^*$ has diameter less than ε_2, it is a subset of X. Similarly, $\bar{G}^*$ contains no point of M'_Y. Since $\bar{G}^*$ has diameter less than $\varepsilon_1/2$ and a point of M, it is a subset of M' and, hence, is a subset of $X \cdot Y \cdot M'$. Then $\bar{G}^*$ is a subset of X.

THEOREM 3. *Suppose that* (1) ε *is a positive number*, (2) C *is an* ε*-regular chain*, (3) *the* $\varepsilon/2$*-chain* D *is a strong refinement of* C, *and* (4) E *is a simple chain which is crooked in* D. *Then* E *is crooked in* C.

Proof. Suppose that $X_1Y_1U_1V_1$ in C and X_3 and V_3 are links of E intersecting X_1 and V_1, respectively, such that there do not exist links Y_3 and U_3 of E lying in Y_1 and U_1, respectively, such that $X_3U_3Y_3V_3$ in E. Let Y_1' and U_1' denote the non-end links of the subchain X_1V_1 of C such that Y_1' intersects X_1 and U_1' intersects V_1. There do not exists links Y_3 and U_3 of E lying in Y_1' and U_1', respectively, such that $X_3U_3Y_3V_3$ in E.

Let X_2 and V_2 denote links of D containing X_3 and V_3, respectively. Let X_2' denote the last link in the order from X_2 to V_2 in the subchain X_2V_2 of D which intersects X_1, and let V_2' denote the first link in the order from X_2' to V_2 in the subchain $X_2'V_2$ of D which intersects V_1. Let Y_2' and U_2' denote the non-end links of the subchain $X_2'V_2'$ of D such that Y_2' intersects X_2' and U_2' intersects V_2'. Since each link of D has diameter less than $\varepsilon/2$ and X_2' intersects X_1 but is not a subset of X_1, and since there is no link Z of C containing X_2' such that ZX_1V_1 in C, $X_2'+Y_2'$ is a subset of Y_1'. Similarly, $U_2'+V_2'$ is a subset of U_1'.

Let X_3' denote the last link in the order from X_3 to V_3 in the subchain X_3V_3 of E which intersects X_2', and let V_3' denote the first link in the order from X_3' to V_3 of the subchain $X_3'V_3$ of E which intersects V_2'. Since E is crooked in D, there exist links Y_3 and U_3 of E lying in Y_2' and U_2', respectively, such that $X_3'U_3Y_3V_3'$ in E. But then $X_3U_3Y_3V_3$ in E, and Y_3 and U_3 are subsets of Y_1' and U_1', respectively.

THEOREM 4. *Suppose that* (1) M *is a pseudo-arc*, (2) D *is a regular chain covering* M *such that either no proper subchain of* D *covers* M *or no end link of* D *intersects* M, *and* (3) $C_1, C_2, \ldots$ *is a defining sequence for* M *such that, for each* n, C_n *properly covers* M *and* C_{n+1} *is crooked in* C_n. *Then there exists a positive integer* k *such that, if* E *is a simple chain which is a refinement of* C_k, *then* E *is crooked in* D.

Proof. Let ε denote a positive number such that (1) if G is a coherent collection of three domains, each with diameter less than ε, one of which intersects M, then $\bar{G}^*$ is a subset of some link of D; and (2) D is 2ε-regular. Let k denote a positive integer $>(1+\varepsilon)/\varepsilon$. Then C_{k-1} is an ε-chain, and, since C_k is crooked in C_{k-1}, C_k is crooked in D. Let E denote a simple chain which is a refinement of C_1, and suppose that (1) $X_1Y_1U_1V_1$ in D, and (2) X_4 and V_4 are links of E intersecting X_1 and V_1, respectively. Let X_3 denote a link of C_k containing X_4, and let X_2 denote a link of C_{k-1} containing X_3. Let V_3 denote a link of C_k containing V_4, and let V_2 denote a link of C_{k-1} containing V_3. Since X_2 intersects X_1, no three link subchain of C_{k+1}, one of whose links is X_2, has a link which intersects a link of D which does not intersect X_1. Similarly, no three link subchain of C_{k-1}, one of whose links is V_2, has a link which intersects a link of D which does not intersect V_1. Therefore, there exist links Y_2 and U_2 of C_{k-1} lying in Y_1 and U_1, respectively, such that $X_2Y_2U_2V_2$ in C_{k-1}. Since C_k is crooked in C_{k-1}, there exist links Y_3 and U_3 of C_k lying in Y_2 and U_2, respectively, such that $X_3U_3Y_3V_3$ in C_k. Then there exist links U_4 and Y_4 of E lying in U_3 and Y_3, respectively, such that $X_4U_4Y_4V_4$ in E. But Y_4 is a subset of Y_1 and U_4 is a subset of U_1. Hence, every simple chain which is a refinement of C_k is crooked in D.

THEOREM 5. *If* M *is a pseudo-arc, there exists a regular defining sequence* $C_1, C_2, \ldots$ *for* M *such that, for each* n, C_{n+1} *is crooked in* C_n.

Proof. Suppose that M is a pseudo-arc, A and B are two points of M, and $D_1, D_2, \ldots$ is a defining sequence for M such that, for each n, D_n is a simple chain from A to B and D_{n+1} is crooked in D_n. For each n, D_n properly covers M. Denote by $C_1, C_2, \ldots$, and $k_1, k_2, \ldots$ sequences such that C_1 is a regular chain covering M no subcollection of which covers M, C_1 is a strong refinement of D_1, and, for each n, (1) k_n is a positive integer such that every simple chain which is a refinement of D_{k_n} is crooked in C_n, and (2) C_{n+1} is a regular chain covering M no subcollection of which covers M such that C_{n+1} is a strong refinement of D_{k_n}. Then $C_1, C_2, \ldots$ is a regular defining sequence for M such that, for each n, C_{n+1} is crooked in C_n.

That every chainable continuum has a regular defining sequence was proven by G. W. Henderson in [3].

DEFINITIONS. If M is a point set and ε is a positive number, an ε*-chain with respect to* M is a finite coherent collection C of domains with respect to M, each with diameter less than ε, such that there do not exist three domains with respect to M of the collection C such that, if d is one of them, $C-d$ is coherent.

An *open-ended simple chain* is a simple chain C such that, if L is an end link of C, then there exists, in the boundary of L, a domain O with respect to the boundary of C^* which does not intersect the boundary of any other link of C. Such a domain O with respect to the boundary of C^* is called an *open end* of C. If C is an open ended simple chain, then the open ends O and O' of C are called *opposite open ends of* C provided O and O' are mutually exclusive and, unless C has only one link, O and O' lie in the boundaries of different end links of C.

A *simple consolidation* of a simple chain C is a simple chain D each link of which is either a link of C or the sum of the links of a subchain of C. The *primary open-ended consolidation* of a simple chain C is a simple consolidation D of C which is an open-ended simple chain such that, if E is any simple consolidation of C which is open-ended, then E is a simple consolidation of D.

THEOREM 6. *If J is a circle of radius ε, then no ε-chain with respect to J covers J and, therefore, no ε-chain covers J.*

Proof. Suppose that J is a circle of radius ε, C is an ε-chain with respect to J covering J, and $L_1, L_2, \ldots, L_n$ are the links, in that order, of C. For each positive integer $i < n+1$, let T_i denote the arc of J with end points in the boundary with respect to J of L_i and which contains L_i. For each $i < n+1$, T_i is a subset of some semicircle of J. Since L_2 contains the end points of T_1, T_2 contains T_1. Indeed, for each $i < n$, T_{i+1} contains T_i. Then T_n is J.

THEOREM 7. *If δ is a positive number and C is a δ-regular chain, then every consolidation of C is a δ-regular chain.*

THEOREM 8. *Suppose that ε is a positive number and C is an ε-chain such that C^* is a simple domain. Then the primary open-ended consolidation of C is a 4ε-chain.*

Proof. Let $L_1, L_2, \ldots, L_n$ denote the links, in that order, of C, and let A and B denote points of L_1 and L_n, respectively. If X is either A or B, denote by J_X the circle with center X and radius ε, by I_X the interior of J_X, and by D_X the component containing X of $I_X \cdot C^*$. Let E denote the simple chain of which e is a link if and only if either (1) e is the sum of the links of C which intersect D_A, (2) e is the sum of the links of C which intersect D_B, or (3) e is a link of C which intersects neither D_A nor D_B. Then E is a 4ε-chain which is open-ended and is a simple consolidation of C. Then the primary open-ended consolidation of C is a 4ε-chain.

THEOREM 9. *Suppose that M is a closed and compact point set no component of which separates S and G is a collection of domains such that each component of M lies wholly in some domain of G. Then there exists a finite collection H of simple domains covering M whose closures are mutually exclusive such that each domain of H is a subset of some domain of the collection G.*

Proof. For each component k of M, let J_k denote a simple closed curve lying in $S-M$ whose interior, I_k, contains k and lies wholly in some domain of the collection G. Let Q denote the collection of all domains I_k for all components k of M, and let Q' denote a finite subcollection of Q covering M. Let β denote the sum of the boundaries of the domains of the collection Q'. For each component D of $S-\beta$ which contains a point of M, let C_D denote a simple domain containing $M \cdot D$ whose closure is a subset of D. Let H denote the collection of all domains C_D for all components D of $S-\beta$ which contain a point of M. Then H is a finite collection of simple domains, covering M, whose closures are mutually exclusive such that each domain of H is a subset of some domain of the collection G.

THEOREM 10. *A closed and compact point set M is a subset of a pseudo-arc if and only if every component of M is either a single point or a pseudo-arc.*

Indeed, if M is such a closed and compact point set, there exists a pseudo-arc T, containing M, no composant of which contains two components of M.

Proof. Moise [4] proved that every non-degenerate subcontinuum of a pseudo-arc is a pseudo-arc. Therefore, every component of a closed subset of a pseudo-arc is either a single point or a pseudo-arc.

Suppose that M is a closed and compact point set every component of which is either a single point or a pseudo-arc. Let A and B denote two points of $S-M$. For each component k of M, denote by $C_{1k}, C_{2k}, \ldots$ a regular defining sequence for k such that, for each n, C_{nk} properly covers k and $C_{n+1,k}$ is crooked in C_{nk}. (In case k is degenerate, for example, each chain of the sequence $C_{1k}, C_{2k}, \ldots$ may be degenerate.) The remainder of the proof will be divided into five parts.

Part I. Denote by G_1 the collection of all domains C_{4k}^* for all components k of M. Denote by H_1 a finite collection of simple domains properly covering M such that (1) each domain of the collection H_1 is a subset of a domain of the collection G_1, and (2) if d_1 and d_2 are two domains of the collection H_1, then $\bar{d}_1$ and $\bar{d}_2$ are mutually exclusive subsets of $S-(A+B)$. Let $h_{11}, h_{12}, \ldots, h_{1n_1}$ denote the domains of the collection H_1. For each i $(1 \leqslant i \leqslant n_1)$, (1) denote by k_{1i} a component of M such that $C_{4k_{1i}}^*$ contains h_{1i}; (2) denote by α_{1i} the chain of which X is a link if and only if for some link Y of $C_{4k_{1i}}$ which intersects h_{1i}, X is $Y \cdot h_{1i}$; (3) denote by α_{1i}' the primary open-ended consolidation of α_{1i}; and (4) denote by O_{1i} and O_{1i}' opposite open ends of α_{1i}'. For each i $(1 \leqslant i \leqslant n_1)$, α_{1i}' is a regular 1-chain.

Let $A_{10}B_{10}, A_{11}B_{11}, \ldots, A_{1n_1}B_{1n_1}$ denote mutually exclusive arcs lying wholly in $S-H_1^*$ such that (1) A_{10} is A and B_{1n_1} is B, and (2) for each i $(1 \leqslant i \leqslant n_1)$, A_{1i} is a point of O_{1i}' and $B_{1,i-1}$ is a point of O_{1i}. Denote by $\beta_{10}, \beta_{11}, \ldots, \beta_{1n_1}$ regular 1-chains the closures of the sums of the links of which are mutually exclusive such that, for each i $(0 \leqslant i \leqslant n_1)$, (1) β_{1i} is a simple chain from A_{1i} to B_{1i} covering the arc $A_{1i}B_{1i}$ and having more than two links; (2) each link of β_{1i} is connected; (3) if X is a link of β_{1i} and Y is a link of one of the chains $\alpha_{11}', \alpha_{12}', \ldots, \alpha_{1n_1}'$ and $\bar{X}$ and $\bar{Y}$ have a point in common, then Y is the only link Z of one of those chains such that $\bar{X}$ and $\bar{Z}$ have a point in common, and X is an end link of β_{1i} which contains neither A nor B; and (4) no link of β_{1i} contains a point of M.

Denote by D_1 the chain of which d is a link if and only if d is a link of one of the chains $\alpha_{11}', \alpha_{12}', \ldots, \alpha_{1n_1}'$ or of one of the chains $\beta_{10}, \beta_{11}, \ldots, \beta_{1n_1}$. Then D_1 is a regular 1-chain from A to B, covering M, and D_1^* is connected. Let E_{1A} and E_{1B} denote the end links of D_1 containing A and B, respectively. The point sets $\bar{E}_{1A}$, $\bar{E}_{1B}$, and M are mutually exclusive. Let ε_1 denote a positive number such that (1) if d is a domain with diameter less than ε_1 which contains a point of M, then $\bar{d}$ is a subset of some link of D_1, and (2) D_1 is ε_1-regular. Then $\varepsilon_1 < 1$.

Part II. For each component k of the point set M, denote by C'_{2k} an $\varepsilon_1/3$-chain of the sequence $C_{1k}, C_{2k}, \ldots$ such that every simple chain which is a refinement of C'_{2k} is crooked in D_1, and denote by $p_2(k)$ a positive integer such the closure of every domain with diameter $4/p_2(k)$ which intersects M is a subset of some link of C'_{2k}. For each i $(1 \leqslant i \leqslant n_1)$, (1) denote by G_{2i} the collection of all domains $C^*_{p_2(k),k}$ for all components k of $M \cdot h_{1i}$; (2) denote by H_{2i} a finite collection of simple domains properly covering $M \cdot h_{1i}$ such that (a) each domain of the collection H_{2i} is a subset of some domain of the collection G_{2i}, and (b) if d_1 and d_2 are two domains of the collection H_{2i}, then $\bar{d}_1$ and $\bar{d}_2$ are mutually exclusive subsets of h_{1i}; and (3) denote by $h_{2i1}, h_{2i2}, \ldots, h_{2in_{2i}}$ the domains of the collection H_{2i}. For each i $(1 \leqslant i \leqslant n_1)$, and each j $(1 \leqslant j \leqslant n_{2i})$, (1) denote by k_{2ik} a component of M such that $C_{p_1(k_{2ij}),k_{2ij}}$ contains h_{2ij}; (2) denote by α_{2ij} the chain of which X is a link if and only if for some link Y of $C_{p_2(k_{2ij}),k_{2ij}}$ which intersects h_{2ij}, X is $Y \cdot h_{2ij}$; (3) denote by α'_{2ij} the primary open-ended consolidation of α_{2ij}; and (4) denote by O_{2ij} and O'_{2ij} opposite open ends of α'_{2ij}. If i is a positive integer $(1 \leqslant i \leqslant n_1)$, and j is a positive integer $(1 \leqslant j \leqslant n_{2i})$, α'_{2ij} is a refinement of $C_{2k_{2ij}}$ and, therefore, is an $\varepsilon_1/3$-chain which is crooked in D_1; and α'_{2ij} is a regular chain.

For each i $(1 \leqslant i \leqslant n_1)$, let $A'_{2i0}B_{2i0}, A_{2i1}B_{2i1}, \ldots, A_{2in_{2i}}B'_{2in_{2i}}$ denote mutually exclusive arcs lying wholly, except for their end points, in $h_{1i} - H^*_{2i}$ such that (1) A'_{2i0} is $B_{1,i-1}$ and $B'_{2in_{2i}}$ is A_{1i}, and (2) for each j $(1 \leqslant j \leqslant n_{2i})$, A_{2ij} is a point of O'_{2ij} and $B_{2,i,j-1}$ is a point of O_{2ij}. For each i $(1 \leqslant i \leqslant n_1)$, denote by Z_{2i} the point set $A'_{2i0}B_{2i0} + A_{2i1}B_{2i1} + \ldots + A_{2in_{2i}}B'_{2in_{2i}} + \bar{h}_{2i1} + \bar{h}_{2i2} + \ldots + \bar{h}_{2in_{2i}}$. For each i $(1 \leqslant i \leqslant n_1)$, Z_{2i} is a compact continuum which does not separate D^*_1.

Let $A_{10}P_{10}Q_{10}B_{10}, A_{11}P_{11}Q_{11}B_{11}, \ldots, A_{1n_1}P_{1n_1}Q_{1n_1}B_{1n_1}$ denote mutually exclusive arcs (different from but having the same end points as $A_{10}B_{10}$, $A_{11}B_{11}, \ldots, A_{1n_1}B_{1n_1}$, respectively) lying wholly, except for their end points in $D^*_1 - (Z_{21} + Z_{22} + \ldots + Z_{2n_1} + A + B)$ such that (1) for each i $(1 \leqslant i \leqslant n_1)$, P_{1i} is a point of E_{1B} and Q_{1i} is a point of E_{1A}; (2) for each i $(0 \leqslant i \leqslant n_1 - 1)$, if X and Y are two links of D_1 such that (a) XYE_{1B} is in D_1, (b) every link of $\alpha'_{1,i+1}$ follows Y in the order from E_{1A} to E_{1B} in D_1, and (c) Z is a point of the subarc $Q_{1i}B_{1i}$ of $A_{1i}P_{1i}Q_{1i}B_{1i}$ lying in X, then there exist points Z' and Z'' of $Q_{1i}B_{1i}$ lying in E_{1B} and Y, respectively, such that Z, Z', Z'', and B_{1i} are in that order on $Q_{1i}B_{1i}$; and (3) for each i $(1 \leqslant i \leqslant n_1)$, if X and Y are two links of D_1 such that (a) $E_{1A}XY$ in D, (b) every link of α'_{1i} precedes X in the order from E_{1A} to E_{1B} in D_1, and (c) Z is a point of the subarc $A_{1i}P_{1i}$ of $A_{1i}P_{1i}Q_{1i}B_{1i}$ lying in Y, then there exist points Z' and Z'' of $A_{1i}P_{1i}$ lying in E_{1A} and X, respectively, such that A_{1i}, Z'', Z', and Z are in that order on $A_{1i}P_{1i}$.

Denote by $\gamma_{10}, \gamma_{11}, \ldots, \gamma_{1n_1}$ regular $\varepsilon_1/3$-chains the closures of the sums of the links of which are mutually exclusive such that, for each i $(1 \leqslant i \leqslant n_1)$, (1) γ_{1i} is a simple chain from A_{1i} to B_{1i} covering the arc $A_{1i}P_{1i}Q_{1i}B_{1i}$; (2) each link of γ_{1i} is a connected domain whose closure is a subset of some link of D_1; and (3) if X is a link of γ_{1i}, j is positive integer $\leqslant n_1$, and $\bar{X}$ intersects Z_{2j}, then j is the only positive integer $p \leqslant n_1$ such that $\bar{X}$ intersects Z_{2p}, X is an end link of γ_{1i} which contains neither A nor B, $\bar{X}$ does not intersect any of the point sets $h_{2j1}, h_{2j2}, \ldots, h_{2jn_{2j}}$, and X intersects only one of the arcs $A'_{2j0}B_{2j0}, A_{2j1}B_{2j1}, \ldots, A_{2jn_{2j}}B'_{2jn_{2j}}$.

For each i $(0 \leqslant i \leqslant n_1)$, let γ'_{1i} denote a regular chain of connected domains from A_{1i} to B_{1i} which is crooked in γ_{1i} such that the closure of no non-end link of γ'_{1i} intersects one of the point sets $Z_{21}, Z_{22}, \ldots, Z_{2n_1}$.

For each i $(1 \leqslant i \leqslant n_1)$, (1) let A_{2i0} denote the last point of $A'_{2i0}B_{2i0} \cdot \overline{\gamma'^{*}_{1,i-1}}$ in the order from A'_{2i0} to B_{2i0} on the arc $A'_{2i0}B_{2i0}$; (2) let $A_{2i0}B_{2i0}$ denote the appropriate subarc of the arc $A'_{2i0}B_{2i0}$; (3) let $B_{2in_{2i}}$ denote the first point of $A_{2in_{2i}}B'_{2in_{2i}} \cdot \overline{\gamma'^{*}_{1i}}$ in the order from $A_{2in_{2i}}$ to $B'_{2in_{2i}}$ on the arc $A_{2in_{2i}}B'_{2in_{2i}}$; and (4) let $A_{2in_{2i}}B_{2in_{2i}}$ denote the appropriate subarc of the arc $A_{2in_{2i}}B'_{2in_{2i}}$.

For each i $(1 \leqslant i \leqslant n)$, denote by $\beta_{2i0}, \beta_{2i1}, \ldots, \beta_{2in_{2i}}$ regular $\varepsilon_1/3$-chains the closures of the sums of the links of which are mutually exclusive such that, for each j $(0 \leqslant j \leqslant n_{2i})$, (1) β_{2ij} is a simple chain from A_{2ij} to B_{2ij} covering the arc $A_{2ij}B_{2ij}$ and having more than two links; (2) each link of β_{2ij} is a connected domain whose closure is a subset of some link of α'_{1i}; (3) if X is a link of β_{2ij} and Y is a link either of one of the simple chains $\alpha'_{2i1}, \alpha'_{2i2}, \ldots, \alpha'_{2in_{2i}}$ or of one of the simple chains $\gamma'_{10}, \gamma'_{11}, \ldots, \gamma'_{1n_1}$ and X and Y have a point in common, then Y is the only link Z of one of those chains such that $\bar{X}$ and $\bar{Z}$ have a point in common and X is an end link of β_{2ij}; and (4) no link of β_{2ij} contains a point of M.

Denote by D_2 the chain of which d is a link if and only if d is a link of one of the chains $\gamma'_{10}, \gamma'_{11}, \ldots, \gamma'_{1n_1}$ or, for some positive integer $i \leqslant n_1$, d is a link of one of the chains $\beta_{2i0}, \beta_{2i1}, \ldots, \beta_{2in_{2i}}$ or of one of the chains $\alpha'_{2i1}, \alpha'_{2i2}, \ldots, \alpha'_{2in_{2i}}$. Then D_2 is a regular $\varepsilon_1/3$-chain and D^*_2 is connected. Denote by E_{2A} and E_{2B} the end links of D_2 containing A and B, respectively. The point sets $\bar{E}_{2A}$, $\bar{E}_{2B}$, and M are mutually exclusive. Let ε_2 denote a positive number such that (1) if d is a domain with diameter less than ε_2 which contains a point of M, then $\bar{d}$ is a subset of some link of D_2, and (2) D_2 is ε_2-regular. Then $\varepsilon_2 < \varepsilon_1/3$.

Let $h_{21}, h_{22}, \ldots, h_{2n_2}$ denote the finite sequence $h_{211}, h_{212}, \ldots, h_{21n_{21}}$, $h_{221}, h_{222}, \ldots, h_{22n_{22}}, \ldots, h_{2n_11}, h_{2n_12}, \ldots, h_{2n_1q}$, where $q = n_{2n_1}$, and let H_2 denote the collection of all simple domains which are terms of that sequence. If each of the symbols p, i, and j denotes a positive integer

($p \leqslant n_2$, $i \leqslant n_1$, and $j \leqslant n_{1i}$), such that h_{2p} is h_{2ij}, then (1) denote a'_{2ij} by a'_{2p}, (2) denote O_{2ij} and O'_{2ij} by O_{2p} and O'_{2p}, respectively, and (3) denote by $B_{2,p-1}$ and A_{2p} points of O_{2p} and O'_{2p}, respectively. Let A_{20} denote A and let B_{2n_2} denote B.

Part III. Repeat the process described in Part II above infinitely many times with appropriate subscripts having r added to them on the r^{th} repeat ($r = 1, 2, \ldots$). Note that, for each n, $\varepsilon_{n+1} < \varepsilon_n/3 < 1/3^{n-1}$ and the closure of each link of D_{n+1} is a subset of some link of D_n.

Denote by T the common part of the point sets $D_1^*, D_2^*, \ldots$ Then T is a chainable continuum containing $M+A+B$.

Part IV. Suppose that $X_1Y_1U_1V_1$ in D_1, and X_5 and V_5 are links of D_5 intersecting X_1 and V_1, respectively, such that there do not exist links Y_5 and U_5 of D_5 lying in Y_1 and U_1, respectively, such that $X_5U_5Y_5V_5$ in D_5. Let R_{11} and S_{11} denote the non-end links of the subchain X_1V_1 of D_1 intersecting X_1 and V_1, respectively. There do not exist links Y_5 and U_5 of D_5 lying in R_{11} and S_{11}, respectively, such that $X_5U_5Y_5V_5$ in D_5.

Let X_4, X_3, and X_2 denote links of D_4, D_3, and D_2, respectively, such that X_4 contains X_5, X_3 contains X_4, and X_2 contains X_3. Let V_4, V_3, and V_2 denote links of D_4, D_3, and D_2, respectively such that V_4 contains V_5, V_3 contains V_4, and V_2 contains V_3. Then X_2 intersects X_1 and V_2 intersects V_1. Let X'_2, X'_3, and X'_4 denote links of D_2, D_3, and D_4, respectively, such that (1) X'_2 is the last link in the order from X_2 to V_2 of the subchain X_2V_2 of D_2 which intersects X_1, and (2) if i is either 3 or 4, X'_i is the last link in the order from X_i to V_i of the subchain X_iV_i of D_i which is a subset of X'_{i-1}. Let V'_2, V'_3, and V'_4 denote links of D_2, D_3, and D_4, respectively, such that (1) V'_2 is the first link in the order from X'_2 to V_2 of the subchain X'_2V_2 of D_2 which intersects V_1, and (2) if i is either 3 or 4, V'_i is the first link in the order from X'_i to V_i of the subchain X'_iV_i of D_i which is a subset of V'_{i-1}. For each i ($2 \leqslant i \leqslant 4$), let R_{i1}, R_{i2}, and R_{i3} denote the second, third, and fourth links, respectively, in the order from X'_i to V'_i in the subchain $X'_iV'_i$ of D_i and let S_{i1}, S_{i2}, and S_{i3} denote the second, third, and fourth links, respectively, *in the order from* V'_i *to* X'_i in the subchain $X'_iV'_i$ of D_i. For each i ($2 \leqslant i \leqslant 4$), $R_{i1}+R_{i2}+R_{i3}$ is a subset of $R_{i-1,1}$ and $S_{i1}+S_{i2}+S_{i3}$ is a subset of $S_{i-1,1}$. It follows, therefore, that, if i is either 3 or 4, there do not exist links Y_i and U_i of D_i lying in $R_{21}+R_{22}+R_{23}$ and $S_{21}+S_{22}+S_{23}$, respectively, such that $X'_iU_iY_iV'_i$ in D_i.

If i is a positive integer $\leqslant n_3$, a'_{3i} is crooked in D_2 and, hence, R_{33} and S_{33} are not both links of a'_{3i}. Also, if i is an integer ($0 \leqslant i \leqslant n_3$), γ'_{3i} is crooked in D_2 and, hence, R_{43} and S_{43} are not both links γ'_{3i}. (Note that if i is a positive integer and j is an integer ($0 \leqslant j \leqslant n_i$), γ_{ij} is a subchain of D_{i+1}.)

Suppose that $0 \leqslant i \leqslant n_3$ and γ'_{3i} is a subchain of the subchain $X'_4V'_4$ of D_4. Since the arc $A_{3i}P_{3i}Q_{3i}B_{3i}$ intersects every link of D_3 and each link of γ_{3i} is a subset of some link of D_3, every non-end link of D_3 contains a link of γ_{3i}. Then every non-end link of D_3 contains a link of γ'_{3i}. Let Z denote a link of γ'_{3i} lying in R_{22} and let Z' denote a link of γ'_{3i} lying in S_{22}. Then it is not true that $X'_4Z'ZV'_4$ in D_4. Now, γ'_{3i} is crooked in D_2 and, therefore, there exist links W and W' of γ'_{3i} lying in R_{23} and S_{23}, respectively, such that $ZW'WZ'$ in γ'_{3i}. Then $X'_4W'WV'_4$ in D_4, a contradiction.

Each of the chains $\gamma'_{30}, \gamma'_{31}, \ldots, \gamma'_{3n_3}$ has more than two links. It follows, then, that either three of the links X'_4, R_{41}, R_{42}, and R_{43} or three of the links V'_4, S_{41}, S_{42}, and S_{43} all belong to one of the chains $\gamma'_{30}, \gamma'_{31}, \ldots, \gamma'_{3n_3}$. Then, either both R_{41} and R_{42} or S_{41} and S_{42} belong to one of those chains.

Suppose that i is an integer ($0 \leqslant i \leqslant n_3$) such that R_{41} and R_{42} both are links of γ'_{3i}. Let X'_5 denote a link of the subchain X_5V_5 which is a subset of R_{41}. Then there exists an integer j ($0 \leqslant j \leqslant n_4$) such that X'_5 is a link of γ'_{4j}. Let d denote a link of γ_{4j} containing X'_5 and let P denote a point of the arc $A_{4j}P_{4j}Q_{4j}B_{4j}$ lying in d. Let L denote the last link of the subchain X'_5V_5 in the order from X'_5 to V_5 which is a link of γ'_{4j} and let P' denote the end point of the arc $A_{4j}P_{4j}Q_{4j}B_{4j}$ lying in L. Let d' denote a link of γ_{4j} containing L. Denote by e_4 the end link of D_4 such that $R_{41}R_{42}e_4$ in D. Then there exist points Z and Z' of the arc $A_{4j}P_{4j}Q_{4j}B_{4j}$ lying in e_4 and R_{42}, respectively, such that P, Z, Z', and P' are in that order on the arc $A_{4j}P_{4j}Q_{4j}B_{4j}$ and, hence, links g and g' of γ_{4j} lying in S_{22} and R_{22}, respectively, such that $dg'gd'$ in γ_{4j}. Therefore, there exist links U_5 and Y_5 of γ'_{4j} lying in g' and g, respectively, such that $X'_5\,UY_5V_5$ in D_5, Y_5 is a subset of R_{11}, and U_5 is a subset of S_{11}. A similar contradiction would be reached if it were assumed that S_{41} and S_{42} both were links of one of the chains $\gamma'_{30}, \gamma'_{31}, \ldots, \gamma'_{3n_3}$. Therefore, D_5 is crooked in D_1.

Similarly, for each n, D_{n+4} is crooked in D_n. Hence, the common part of the domains $D_1^*, D_5^*, \ldots, D_{4i-3}^*, \ldots$, which is T, is a pseudo-arc.

Part V. The continuum T is chainable from A to B. Therefore, by a theorem due to Bing [2], it is irreducible from A to B.

Suppose that k_1 and k_2 are two components of the point set M, and K is a proper subcontinuum of T which intersects both k_1 and k_2. Suppose that Z is one of the two points A and B such that K does not contain Z. Let ε denote a positive number such that no point of $K+k_1+k_2$ is at a distance less than ε from Z. Denote by p a positive integer $> 2/\varepsilon$ such that k_1 and k_2 are not both subsets of the same domain of the collection H_p. Let D'_{p+1} denote a subchain of D_{p+1} properly covering $K+k_1+k_2$ and let X_1 and X_2 denote links of D'_{p+1} intersecting k_1 and k_2, respectively. There exists a positive integer $i < n_p$ such that γ'_{pi} is a subchain of the subchain X_1X_2 of D_{p+1} and, thus, is a subchain of D'_{p+1}. Since the arc $A_{pi}P_{pi}Q_{pi}B_{pi}$ intersects every link of D_p, every non-end link of D_p contains a link of γ'_{pi}. In particular, the non-end link Y of D_p which intersects E_{pZ} contains a link of γ'_{pi} and therefore a point Q of $K+k_1+k_2$. Then Q is at distance from Z less than $2/p<\varepsilon$, a contradiction. Thus, no composant of T contains two components of M.

The following theorem can be proven by an argument which is a simplification of the argument for Theorem 10:

THEOREM 11. *A closed and compact point set M is a subset of a chainable continuum if and only if every component of M is either a single point or a chainable continuum.*

Indeed, if M is such a closed and compact point set, there exists an indecomposable chainable continuum T, containing M, no composant of which contains two components of M.

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Metric characterizations of Banach and Euclidean spaces

by

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Introduction. One of the most important classes of metric spaces, both intrinsically and for its great usefulness in theoretical physics, is formed by the (metrically) complete, normed, linear spaces. This class was axiomatized and studied by Banach in 1922, and in the same year (and quite independently) the class was defined and investigated both by Hahn and by Wiener.

The usual formulation of the abstract Banach space (over the reals) is in terms of three primitive concepts: (1) *addition* (with each (ordered) pair of elements x,y there is associated a unique element $x+y$—the *ordered sum* of x and y); (2) *scalar multiplication* (with each real number λ and each element x there is associated a unique element $\lambda\cdot x$—the *scalar multiple* of x by λ); and (3) *normation* (with each element x there is associated a unique real number $\|x\|$—the *norm* of x). The three primitive notions are subjected to ten postulates, which are stated in another part of this paper. A normed linear space is a Banach space provided it is *complete* (that is, if $\{x_n\}$ is an infinite sequence of elements such that

$$\lim_{i,j\to\infty}\|x_i+(-1)x_j\|\equiv\lim_{i,j\to\infty}\|x_i-x_j\|=0,$$

then an element x exists such that $\lim\limits_{i\to\infty}|x-x_i|=0$).

The concept of *distance* is introduced in a normed linear space by defining the distance xy of two elements x,y to be the norm of their difference,

$$xy=\|x-y\|,$$

and it is easily seen that in terms of this definition, every normed linear space is a *metric* space (that is, (1) $xy\geqslant 0$, (2) $xy=0$ if and only if $x=y$, (3) $xy=yx$, and (4) $xy+yz\geqslant xz$ for each three elements x, y, z of the space). The class of Banach spaces is, therefore, a (proper) subclass of the class of all metric spaces, and the problem arises of characterizing metrically this subclass among the members of the whole class. More precisely, the problem is to obtain conditions, expressed *wholly* and *explicitly* in terms of the metric, in order that an arbitrary metric space