For $C \subset S^2$, denote by p_N, p_S the poles and by S^1 the equator of S^2. Let C' be the union of the set $C \cap S^1$ and all the boundaries of components of $S^2 - C$ that intersect S^1. Then C' cuts S^2 between p_N and p_S. Indeed, by the hypothesis, the poles do not belong to C' and if a continuum $K \subset S^2$ joins them, it must meet $C \cap S^1$ or $S^1 - C$. In the second case, K intersects the common part of S^1 and some component G of $S^2 - C$. Hence K meets the boundary of G because $p_N \in K - G$. Thus K meets C'.

It follows that the projection f of C' onto S^1 along the meridians is an essential mapping (ibidem). But if $p, q \in C'$ and $f(p)$, $f(q)$ are antipodal points on S^1, then the distance $\varrho(p, q)$ is at least $\pi - 2d$, according to the hypothesis and the definition of C'. Since $C' \subset C$, the theorem from § 7 gives $\pi - 2d \leqslant \sigma C' \leqslant \sigma C$.

References

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[2] Jan Mycielski, *Independent sets in topological algebras*, Fund. Math. 55 (1964), pp. 139-147.

Reçu par la Rédaction le 25. 3. 1963

Recursive metric spaces *

by

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Introduction. In [10] the author introduced abstract "notation systems" and used them to give an axiomatic treatment of the theory of recursive real numbers. In this paper we use the same methods to constructivize parts of the theory of metric spaces.

A (classical) metric space consists of a set $\mathcal{M}$ together with a function $\delta(\alpha, \beta)$ from $\mathcal{M}$ into the set $\mathcal{R}$ of real numbers which satisfies the three metric axioms. The natural way to define a "recursive metric space", according to the point of view that we adopted in [10], is to substitute an arbitrary notation system M for the set $\mathcal{M}$ and a "recursive operator" $D(\alpha, \beta)$ from M into R (the notation system for the recursive real numbers [10], (1.5), (1.6)) for the distance function $\delta(\alpha, \beta)$ (Definition 1).

It is found that this concept of a recursive metric space is too weak; before we can prove any of the more interesting results of the theory, we have to postulate a deeper connection between the metric and the recursive structure of the space. We shall consider two conditions (A) and (B) (§§ 1 and 4, respectively) on a recursive metric space M, which seem to be sufficient for this purpose.

A space M satisfies (A) if we can effectively compute the limit of a recursive, recursively Cauchy sequence of points of M, whenever it exists.

In order to state our main result we need the concept of an "S-traced" set. A subset E of M is S-*traced* if we can effectively find an element of E in every sphere that intersects E (Definition 2). *If* M *satisfies* (A), *then every point of a listable subset* L *of* M *can be effectively separated from any given* S-*traced subset* E *of the complement of* L *by an open sphere* (the separation theorem, § 2).

* This paper is Part II of the author's Ph. D. Thesis at the University of Wisconsin written under the direction of Professor S. C. Kleene. Part of the material appeared in [9], written under the direction of Doctor D. L. Kreider (now at Dartmouth College) and was presented at Professor H. Rogers' seminar in logic at MIT in the summer of 1960. I wish to express my sincerest appreciation to all three above-named persons for their help and encouragement.

Although the separation theorem is not hard to prove, it expresses a rather deep relation between the metric and the recursive properties of listable subsets of M. It is used in § 3 to derive some well-known unsolvability results, for example that *the irrational recursive real numbers do not form a listable subset of* R.

In § 4 we introduce Condition (B): M satisfies (B) if every listable subset of M is S-traced. We prove there that *every partial recursive operator from a recursive metric space* M_1 *satisfying* (A) *and* (B) *into a recursive metric space* M_2 *is recursively continuous at all points of its domain* (Theorem 3). This general version of a well-known result of recursive analysis (see [1]) is derived here as an immediate corollary of the separation theorem.

A recursive metric space is *recursively separable* if it contains a dense, recursively enumerable subset. These spaces are studied briefly in § 5. A typical result is that, *if* $P(\alpha_1, \ldots, \alpha_n, \beta)$ *is a listable predicate on* R, *then so is* $(E\beta)P(\alpha_1, \ldots, \alpha_n, \beta)$ (Theorem 5).

Another direct corollary of the separation theorem is Theorem 6, which asserts in part that *no listable ordering of* R *or* F (the notation system for the general recursive functions, [10], (1.9), (1.10)) *is a well-ordering.*

In § 7 we give a recursive version of the Baire category theorem (Theorem 7), which implies that *no listable non-empty subset of* R *or* F *is recursively enumerable.*

Friedberg [2] has constructed a listable subset of F which is not open. In § 8 we construct examples of closed, nowhere dense listable subsets of R and F, and we use them to obtain a counterexample to a plausible strengthening of our Theorem 7.

Finally, in § 9 we show that *a recursively open subset of a recursively separable space satisfying* (A) *is a recursive union of spheres* (Theorem 11). This result together with Theorem 3 yield a stronger version of the recursive continuity theorem for recursively separable spaces due to Čeitin [1].

Our notation and terminology is that of [10]. Although we shall use the concepts introduced there without apologies, the only results of [10] that are relevant are those concerning the notation system R, especially Lemmas 4, 5 and 8.

1. Definitions and examples.

DEFINITION 1. *A recursive metric space* (1) (RMS) is a notation system $M = (M, \sim_M)$ together with a binary recursive operator $D(\alpha, \beta)$ from M into R such that, for all $\alpha, \beta, \gamma \in M$, (a) $D(\alpha, \beta) = 0 \equiv \alpha = \beta$, (b) $D(\alpha, \beta) = D(\beta, \alpha)$ and (c) $D(\alpha, \gamma) \leqslant D(\alpha, \beta) + D(\beta, \gamma)$.

(1) Čeitin defines in [1] *constructive metric spaces* in an essentially equivalent way and proves our Theorem 12 below. Lacombe's *recursive metric spaces* ([7], (4.2)) can be identified as the classical completions of our *recursively separable, recursively complete* RMS's (see below, §§ 5, 7). See also footnote 8.

The first example of a RMS is R with the distance operator $|\alpha-\beta|$, which is easily proved to be recursive.

We define the notation system $R^n = (R^n, \sim_{R^n})$ $(n > 1)$ for the set of n-tuples of recursive real numbers by

(1.1) $$R^n = \{\langle x_1, \ldots, x_n\rangle :\ x_1, \ldots, x_n \in R\}, \ (2)$$

(1.2) $$x \sim_{R^n} y \equiv x \in R^n \ \& \ y \in R^n \ \& \ (i)_{i<n}[(x)_i \sim_R (y)_i].$$

If $\langle a_1, \ldots, a_n\rangle$ is the equivalence class of $\langle a_1, \ldots, a_n\rangle$ in R^n (where for $i = 1, \ldots, n$, a_i is an R-index of the recursive real number α_i), R^n is a RMS with the distance operator

(1.3) $$D^{R^n}(\langle\alpha_1, \ldots, \alpha_n\rangle, \langle\beta_1, \ldots, \beta_n\rangle) = [(\alpha_1-\beta_1)^2 + \ldots + (\alpha_n-\beta_n)^2]^{1/2}.$$

Let us identify an element α of F with the general recursive function it represents, so the notation $\alpha(t)$ ($\alpha \in F$, t a natural number) is meaningful. It can be shown that F is a RMS with the distance operator

(1.4) $$D^F(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ 2^{-\mu t[\alpha(t) \neq \beta(t)]} & \text{otherwise. } (3) \end{cases}$$

Other important examples of classical metric spaces include the space of continuous functions on the closed interval $[0, 1]$ of real numbers and Hilbert space. There are recursive analogs of both these spaces, though we shall not give the definitions here.

We recall the definition of a listable predicate or set in a notation system ([10], Def. 6 and remarks preceding Lemma 10). In order to be able to refer to the listable subsets of a given notation system T in a convenient way, we let $L(T)$ be the set of indices of listable subsets of T,

(1.5) $$n \in L(T) \equiv (x)(y)[x \in T \ \& \ y \in T \ \& \ x \sim_T y \ \& \ \{n\}(x)\downarrow \to \{n\}(y)\downarrow].$$

For $n \in L(T)$, we let L_n^T (or simply L_n, if the particular notation system T to which we are referring is indicated by the context) be the listable subset of T with index n,

(1.6) $$L_n = \{\bar{x}^T :\ \{n\}(x)\downarrow\}.$$

In the sequel $M = (M, \sim_M)$ will always be a RMS with distance operator $D(\alpha, \beta)$, determined by the partial recursive function $d(x, y)$ with Gödel number d. We frequently refer to the elements of M as *points*.

(2) $\langle x_1, \ldots, x_n\rangle = p_0^{x_1} \cdot \ldots \cdot p_{n-1}^{x_n}$, where $p_0, p_1, p_2, \ldots$ is the sequence of primes with $p_0 = 2$.

(3) Although $D^F(\alpha, \beta)$ is a rational number for all $\alpha, \beta \in F$, $D^F(\alpha, \beta)$ is not a recursive operator from F into the rational numbers; i.e. there is no partial recursive function $d(x, y)$ such that, if $x, y \in F$, then $d(x, y)\downarrow$ and $D^F(\bar{x}, \bar{y}) = r(d(x, y))$.

For $x \in M$, we often let $\bar{x}$ (rather than $\bar{x}^{\mathrm{M}}$) be the point of M with M-index x.

For $a_0 \in \mathrm{M}$ and k any natural number, $\mathrm{S}(a_0, k)$ is the *open sphere* with center a_0 and (rational) radius 2^{-k},

$$\mathrm{S}(a_0, k) = \{a \in \mathrm{M}\colon \boldsymbol{D}(a_0, a) < 2^{-k}\}. \tag{1.7}$$

LEMMA 1. *For every $a_0 \in \mathrm{M}$ and every k, $\mathrm{S}(a_0, k)$ is a listable subset of* M. *In fact there is a primitive recursive function* $\mathrm{s}(x, k)$ *(depending on* M) (4) *such that, if $x \in M$, then $\mathrm{s}(x, k) \in L(\mathrm{M})$ and $\mathrm{S}(\bar{x}, k) = \mathrm{L}_{\mathrm{s}(x,k)}$.*

Proof. (5) $\mathrm{s}(x, k) = \Lambda y \text{ less } \big(\{d\}(x, y),\ \varrho(\mathrm{r}^{-1}(2^{-k}))\big)$.

Very little can be proved in the generality of Definition 1 about a RMS M. For example, we cannot except to do any analysis on M unless the limit operations are, in some sense, effective. This restriction on M is formulated as Condition (A) below.

We recall ([10], Def. 10) that a sequence $a_0, a_1, a_2, \ldots$ of elements of a notation system T is *recursive* if there is a general recursive function $\mathrm{f}(x)$ such that, for all x, $\mathrm{f}(x) \in T$ and $a_x = [\mathrm{f}(x)]^{\mathrm{T}}$. We say that $\mathrm{f}(x)$ determines $a_0, a_1, a_2, \ldots$ and we call any Gödel number f of $\mathrm{f}(x)$ an *index* of $a_0, a_2, a_2, \ldots$.

In case of a RMS M, the sequence $a_0, a_1, a_2, \ldots$ is *recursively Cauchy* if there is a general recursive function $\mathrm{g}(t)$ such that, for all t and k, $\boldsymbol{D}(a_{\mathrm{g}(t)}, a_{\mathrm{g}(t)+k}) < 2^{-t}$. We say that $\mathrm{g}(t)$ is a *recursive Cauchy criterion* for $a_0, a_1, a_2, \ldots$ and we call any Gödel number g of $\mathrm{g}(t)$ an *r.c. index* of $a_0, a_1, a_2, \ldots$ (compare [10], Def. 11).

The statement $a = \lim_{x\to\infty} a_x$, where $a, a_0, a_1, a_2, \ldots$ are points of M, has the same meaning as classically, i.e.

$$(t)(En)(x)[x \geqslant n \to \boldsymbol{D}(a, a_x) < 2^{-t}].$$

(A) *There is a partial recursive function $\mathrm{c}^{\mathrm{M}}(f, g)$ (a convergence function for* M) *such that, if f is an index of a recursive sequence $a_0, a_1, a_2, \ldots$ of points of* M *with r.c. index g and if there is an $a \in \mathrm{M}$ such that $a = \lim_{x\to\infty} a_x$, then $\mathrm{c}^{\mathrm{M}}(f, g)\downarrow$, $\mathrm{c}^{\mathrm{M}}(f, g) \in M$ and $a = [\mathrm{c}^{\mathrm{M}}(f, g)]^{\mathrm{M}}$.*

(4) In fact $\mathrm{s}(x, k)$ is primitive recursive in d, an index of the distance operator on M. It can be verified that, whenever we use the phrase "depending on..." for a function that we define, this dependence is (primitive or partial) recursive in the parameters involved. For example, $\mathrm{sep}(n, t, x)$ (Th. 1) is partial recursive in d and c^{M}. Similarly, $\mathrm{rc}(x, k)$ (Th. 3) is partial recursive in d_1, c^{M_1}, tl^{M_1} (a Gödel number of $\mathrm{tl}^{\mathrm{M}_1}(n)$), d_2 and f, etc.

(5) We defined $\mathrm{less}(x, y)$ in [10], Lemma 5. It is partial recursive and such that, for x and $y \in R$, $\mathrm{less}(x, y)\downarrow \equiv \bar{x}^{\mathrm{R}} < \bar{y}^{\mathrm{R}}$. The function $\varrho(x) = \varrho^{\mathrm{R}}(x)$ ([10], (4.5)) provides an R-index of the x'th rational number $\mathrm{r}(x)$ ([10], (1.1)). Here (and in similar cases in the future) we shall not bother to show that $\varrho(\mathrm{r}^{-1}(2^{-k}))$ is (in fact primitive) recursive.

We have already seen that R satisfies (A) ([10], Lemma 8). (6) This implies immediately that R^n $(n > 1)$ satisfies (A). We can prove that F satisfies (A) by setting

$$\mathrm{c}^{\mathrm{F}}(f, g) = \Lambda t\big\{\{f\}\big(\{g\}(t+1)\big)\big\}(t). \tag{1.8}$$

A RMS M is *weakly recursively complete* if every recursive, recursively Cauchy sequence of points in M has a limit; *recursively complete* if it is weakly recursively complete and satisfies (A). The spaces R, R^n and F are all recursively complete. The open interval

$$(0, 1) = \{a \in \mathrm{R}\colon 0 < a < 1\}$$

(as a natural sub-notation system of R ([10], (6.7.a), (6.7.b))) is an example of a RMS which satisfies (A) but is not recursively complete.

2. The separation theorem. Some of the most fundamental results of the classical theory of metric spaces cannot be obtained without the axiom of choice. In order to prove the recursive versions of these results, we have to confine ourselves to spaces (or subsets of spaces) for which the relevant recursive versions of the axiom of choice are true.

Let S be a family of subsets of a notation system T. We say that a subset E of T is *traced with respect to* S, if we can effectively find an element of E in every $\mathrm{A} \in \mathrm{S}$ such that $\mathrm{E} \cap \mathrm{A} \neq \emptyset$. We make this precise for the case where S is the family of spheres with rational radius in a RMS M.

DEFINITION 2. A subset E of a RMS M is *traced with respect to the spheres* (S-*traced*), if there is a partial recursive function $\mathrm{t}(x, k)$ such that, if $x \in M$ and $\mathrm{E} \cap \mathrm{S}(\bar{x}, k) \neq \emptyset$, then $\mathrm{t}(x, k)\downarrow$, $\mathrm{t}(x, k) \in M$ and $[\mathrm{t}(x, k)]^{\mathrm{M}} \in \mathrm{E} \cap \mathrm{S}(\bar{x}, k)$. We say then that $\mathrm{t}(x, k)$ is an S-*tracing function* for E and we call any Gödel number t of $\mathrm{t}(x, k)$ an S-*tracing index* of E.

Let E be a subset of M. A point $a \in \mathrm{M}$ is a (*classical*) *limit point* of E if, for every k, $\mathrm{E} \cap \mathrm{S}(a, k) \neq \emptyset$; a *recursive limit point* of E if there is a recursive sequence $a_0, a_1, a_2, \ldots$ of points of E converging to a. Clearly every recursive limit point of E is a classical limit point of E. The importance of S-traced sets stems from the fact that the converse of this statement holds if E is S-traced. Because if $a = \bar{a}$ is a limit point of E, then $\Lambda k\,\mathrm{t}(a, k)$ is an index of a recursive sequence of points of E which converges to a. A set E is *closed* if it contains all its limit points; *recursively closed* if it contains all its recursive limit points. Thus a recursively closed S-traced set is closed.

(6) In the case of R the definition of "recursively Cauchy" given here agrees with [10], Def. 11.(a).

The next theorem is the basic result of this paper. For the proof, we need the following version of Kleene's second recursion theorem ([4], Th. XXVII). *For each* $n > 0$, *there is a primitive recursive function* $\mathrm{rt}^n(e, x_1, \ldots, x_n)$ *such that, for each* $e, x_1, \ldots, x_n$, *the number* $m = \mathrm{rt}^n(e, x_1, \ldots, x_n)$ *is a Gödel number of* $\{e\}(m, x_1, \ldots, x_n, y)$ *as a function of* y, *i.e. for all* y,

$$(2.1)\qquad \{m\}(y) \simeq \{e\}(m, x_1, \ldots, x_n, y)\,.$$

To prove this we set

$$(2.2)\qquad \mathrm{rt}^n(e, x_1, \ldots, x_n) = S_1^{n+2}(f, e, f, x_1, \ldots, x_n)\,,$$

where f is a Gödel number of

$$\mathrm{f}(e, m, x_1, \ldots, x_n, y) \simeq \{e\}\big(S_1^{n+2}(m, e, m, x_1, \ldots, x_n), x_1, \ldots, x_n, y\big)\,.$$

THEOREM 1. *Let* M *be a* RMS *satisfying* (A). *There is a partial recursive function* $\mathrm{sep}(n, t, x)$ (*depending on* M) (4) *such that, if* $n \in L(\mathrm{M})$, t *is an* S-*tracing index of some set* $\mathrm{E} \subset \mathrm{M}$ *disjoint from* L_n ($E \cap L_n = \emptyset$) *and* x *is an* M-*index of some point* $\bar{x} \in \mathrm{L}_n$, *then* $\mathrm{sep}(n, t, x)\downarrow$ *and* $\mathrm{E} \cap \mathrm{S}\big(\bar{x}, \mathrm{sep}(n, t, x)\big) = \emptyset$. (The separation theorem.)

Intuitively, for every point $\bar{x}$ of a listable set L_n and for every S-traced set E disjoint from L_n, we can effectively find a sphere with center $\bar{x}$ disjoint from E.

Proof. Let c^{M} be a Gödel number of the convergence function $\mathrm{c}^{\mathrm{M}}(f, g)$ for M and id a Gödel number of the identity function

$$(2.3)\qquad \mathrm{id}(x) = x\,.$$

Using the abbreviation

$$(2.4)\qquad P(u, m, n) \equiv T_2\big(c^{\mathrm{M}}, m, id, (u)_0\big)\ \&\ T_1\Big(n, U\big((u)_0\big), (u)_1\Big)\,,$$

we define a partial recursive function $\mathrm{f}(m, n, t, x, y)$ by

$$\mathrm{f}(m, n, t, x, y) \simeq \begin{cases} x & \text{if}\quad (u)_{u \leqslant y}\overline{P}(u, m, n)\,, \\ \{t\}\big(x, \mathrm{k}(m, n)\big) & \text{otherwise}\,, \end{cases}$$

where

$$(2.5)\qquad \mathrm{k}(m, n) \simeq \mu u P(u, m, n)\,.$$

Let f be a Gödel number of $\mathrm{f}(m, n, t, x, y)$, and put

$$(2.6)\qquad m = \mathrm{rt}^3(f, n, t, x)\,.$$

By (2.1), for all y,

$$(2.7)\qquad \{m\}(y) \simeq \mathrm{f}(m, n, t, x, y)\,.$$

We now assume that n, t and x satisfy the three hypothesis. We shall show by contradiction that

$$(2.8)\qquad (Eu)P(u, m, n)\,.$$

Suppose that (2.8) is false. Then, for all y, $\{m\}(y) = x$, and thus m is an index of the recursive sequence $\bar{x}, \bar{x}, \bar{x}, \ldots$ of points of M which converges to $\bar{x}$ with r.c. index id. Thus, by (A),

$$(2.9)\qquad c^{\mathrm{M}}(m, id)\downarrow\ \&\ c^{\mathrm{M}}(m, id) \in M\ \&\ [c^{\mathrm{M}}(m, id)]^{\mathrm{M}} = \bar{x} \in \mathrm{L}_n\,.$$

The last conjunct of (2.9) implies that $\{n\}\big(c^{\mathrm{M}}(m, id)\big)\downarrow$, which together with the first conjunct of (2.9) implies (2.8). Thus (2.8) is proved, and in particular $\mathrm{k}(m, n)\downarrow$.

If $y \geqslant \mathrm{k}(m, n)$, then $\{m\}(y) \simeq \{t\}\big(x, \mathrm{k}(m, n)\big)$. If $\mathrm{E} \cap \mathrm{S}\big(\bar{x}, \mathrm{k}(m, n)\big) \neq \emptyset$, then by the definition of an S-tracing index, $\{t\}\big(x, \mathrm{k}(m, n)\big)\downarrow$. Thus m is an index of the recursive sequence of points $\bar{x}, \ldots (\mathrm{k}(m, n)$ times$) \ldots, \bar{x}$, $\big[\{t\}\big(x, \mathrm{k}(m, n)\big)\big]^{\mathrm{M}}, \big[\{t\}\big(x, \mathrm{k}(m, n)\big)\big]^{\mathrm{M}}, \ldots$ which converges to $\big[\{t\}\big(x, \mathrm{k}(m, n)\big)\big]^{\mathrm{M}}$ with r.c. index id. Hence $\{n\}\big(c^{\mathrm{M}}(m, id)\big)$ is not defined, since by definition, $\big[\{t\}\big(x, \mathrm{k}(m, n)\big)\big]^{\mathrm{M}} \notin \mathrm{L}_n$, contradicting (2.8). It is thus enough to set

$$(2.10)\qquad \mathrm{sep}(n, t, x) \simeq \mathrm{k}(m, n)$$

which by (2.4), (2.5) and (2.6) is a partial recursive function of n, t and x.

3. Listable sets and predicates. The most direct application of the separation theorem is to the case of a listable subset L_n of a RMS M satisfying (A) whose complement $\mathrm{C}_n = \mathrm{M} - \mathrm{L}_n$ is S-traced. Under these assumptions L_n must be *open*, i.e. every point of L_n is contained in some open sphere lying entirely in L_n (cf. §§ 8, 9).

If T is a notation system and $W_e \subset T$, we let $\mathrm{W}_e^{\mathrm{T}}$ (or simply W_e, if T is indicated by the context) be the recursively enumerable subset of T determined by W_e ([10], (2.2) and Def. 9).

LEMMA 2. *Every recursively enumerable subset of a* RMS M *is* S-*traced. In fact there is a primitive recursive function* $\mathrm{tr}_1(e)$ (*depending on* M) (4) *such that, if* W_e *determines a recursively enumerable subset* W_e *of* M, *then* $\mathrm{tr}_1(e)$ *is an* S-*tracing index of* W_e.

Proof. Given a sphere $\mathrm{S}(\bar{x}, k)$ in M, we search through the elements of W_e (which can be effectively enumerated) until we find one that belongs to the listable set $\mathrm{L}_{\mathrm{s}(x,k)} = \mathrm{S}(\bar{x}, k)$. Formally,

$$(3.1)\qquad \mathrm{tr}_1(e) = \Lambda x k\, \Big(\mu u\big[T_1\big(e, (u)_0, (u)_1\big)\ \&\ T_1\big(\mathrm{s}(x, k), (u)_0, (u)_2\big)\big]\Big)_0\,.$$

THEOREM 2. *The complement* $\mathrm{C}_n = \mathrm{M} - \mathrm{L}_n$ *of a listable subset of a* RMS M *which satisfies* (A) *is recursively closed.*

Proof. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be a recursive sequence of points in C_n. The set $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ is recursively enumerable and hence, by the preceding lemma, S-traced. Since it is disjoint from L_n, the separation theorem implies that every point of L_n can be separated from $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ by an open sphere. In particular, no point of L_n is a limit point of $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$.

This purely topological result supplies very easy proofs of the non-listability of several subsets of R of F. For example, the irrational recursive real numbers do not form a listable subset of R, since there are recursive sequences of rational numbers which converge to irrational numbers. We can show in the same way that none of the following subsets of R is listable: the rational numbers, the real algebraic numbers, the recursive real transcedental numbers, the set $\{\alpha\}$ containing a single recursive real number, the closed interval $[\alpha, \beta] = \{\xi \in R\colon \alpha \leqslant \xi \leqslant \beta\}$ (α and β given elements of R, $\alpha < \beta$). Similarly, none of the following subsets of F is listable: $\{\alpha \in F\colon \alpha(t) = n$, for all $t \geqslant k\}$, $\{\alpha \in F\colon \alpha(t) = n$, for infinitely many t's$\}$, the set $\{\alpha\}$ containing a single general recursive function α, etc. By a simple contradiction argument we see that the predicate $\alpha = \beta$ is not listable on either of these two notation systems.

For each of these results there are direct, elementary proofs (for example, see Shapiro's [13]). The separation theorem isolates the simple topological argument that is common to all these proofs. (7)

4. The recursive continuity theorem. Perhaps the most important result of recursive analysis is that asserting the recursive continuity of a partial recursive operator $\boldsymbol{F}(\alpha)$ from R into R (a *partial recursive real function*) at all points of its domain. A weak form of this theorem was anticipated by Markov in [8]. Čeitin [1] gives a general version in the context of a "recursively separable" (§ 5 below) recursively complete RMS.

In this section we deduce the recursive continuity theorem as a direct corollary of Theorem 1 for a RMS that satisfies (A) and the following condition.

(B) *There is a partial recursive function* $\mathrm{tl}^{M}(n)$ *such that, if* $n \in L(M)$, *then* $\mathrm{tl}^{M}(n)\downarrow$ *and* L_n *is S-traced with S-tracing index* $\mathrm{tl}^{M}(n)$.

In § 9 we give Čeitin's slightly stronger result for a recursively separable RMS satisfying (A) (Theorem 12).

Although there are RMS's that satisfy (A) and (B) but are neither recursively separable nor recursively complete, Theorem 3 does not cover any interesting cases not already covered by Čeitin's result. We prefer to state it separately because it shows that the continuity of recursive operators does not depend on the topological assumptions of (recursive) separability and completeness. In this version it is the constructivity assumptions (A) and (B) that are essential. (8)

(7) We can prove ([10], Lemma 9) (which asserts that R has no non-trivial recursive subsets) as follows. We first use the Weierstrass bisection method to show that if $A \subset R$ is recursive and non-trivial, then there is an $\alpha \in R$ which is a limit point of both A and $R - A$. Now Theorem 2 implies that both $\alpha \in A$ and $\alpha \in R - A$, which is absurd.

Lemma 3. *Let M satisfy (A) and (B). There is a partial recursive function* $\mathrm{sep}_1(n, m, x)$ *(depending on M)* (4) *such that, if* $n \in L(M)$, $m \in L(M)$, $L_n \cap L_m = \emptyset$ *and* x *is an M-index of some point* $\bar{x} \in L_n$, *then* $\mathrm{sep}_1(n, m, x)\downarrow$ *and* $L_m \cap S(\bar{x}, \mathrm{sep}_1(n, m, x)) = \emptyset$.

Proof. $\mathrm{sep}_1(n, m, x) \simeq \mathrm{sep}(n, \mathrm{tl}^{M}(m), x)$.

Theorem 3. *Let* M_1 *satisfy (A) and (B), let* M_2 *be a RMS and let* $\boldsymbol{F}(\alpha)$ *be a partial recursive operator from* M_1 *into* M_2. *There is a partial recursive function* $\mathrm{rc}(x, k)$ *(depending on* M_1, M_2 *and* $\boldsymbol{F}(\alpha)$*)* (4) *such that, if* x *is an* M_1*-index of some point in the domain of* $\boldsymbol{F}(\alpha)$ *(i.e.* $\boldsymbol{F}(\bar{x})\downarrow$*), then* $\mathrm{rc}(x, k)\downarrow$ *and for all* $y \in M_1$,

$$\boldsymbol{F}(\bar{y})\downarrow \ \&\ \boldsymbol{D}_1(\bar{x}, \bar{y}) < 2^{-\mathrm{rc}(x,k)} \rightarrow \boldsymbol{D}_2(\boldsymbol{F}(\bar{x}), \boldsymbol{F}(\bar{y})) < 2^{-k}.$$

(The recursive continuity theorem.)

Proof. Let $\boldsymbol{D}_1(\alpha, \beta)$ and $\boldsymbol{D}_2(\alpha, \beta)$ be the recursive distance operators of M_1 and M_2 respectively, determined by the partial recursive functions $d_1(x, y)$ and $d_2(x, y)$, and let $f(x)$ be a partial recursive function which determines $\boldsymbol{F}(\alpha)$. If $x \in M_1$, the partial recursive functions

$$g_1(x, k, y) \simeq \mathrm{less}\big(d_2(f(x), f(y)), \varrho(r^{-1}(2^{-k-1}))\big),$$
$$g_2(x, k, y) \simeq \mathrm{less}\big(\varrho(r^{-1}(2^{-k-1})), d_2(f(x), f(y))\big)$$

determine (as functions of y) the listable subsets of M_1

$$L^1 = \{\alpha \in M_1\colon \boldsymbol{F}(\bar{x})\downarrow \ \&\ \boldsymbol{F}(\alpha)\downarrow \ \&\ \boldsymbol{D}_2(\boldsymbol{F}(\bar{x}), \boldsymbol{F}(\alpha)) < 2^{-k-1}\},$$
$$L^2 = \{\alpha \in M_1\colon \boldsymbol{F}(\bar{x})\downarrow \ \&\ \boldsymbol{F}(\alpha)\downarrow \ \&\ \boldsymbol{D}_2(\boldsymbol{F}(\bar{x}), \boldsymbol{F}(\alpha)) > 2^{-k-1}\}$$

respectively. If $\bar{x}$ is not in the domain of $\boldsymbol{F}(\alpha)$, both these sets are empty. If $\boldsymbol{F}(\bar{x})\downarrow$, at least $L^1 \neq \emptyset$, since $\bar{x} \in L^1$, and in any case $L^1 \cap L^2 = \emptyset$. Set

$$\mathrm{rc}(x, k) \simeq \mathrm{sep}_1(\Lambda y\, g_1(x, k, y), \Lambda y\, g_2(x, k, y), x). \tag{4.1}$$

By Lemma 3, if $x \in M_1$, $y \in M_1$, $\boldsymbol{F}(\bar{x})\downarrow$, $\boldsymbol{F}(\bar{y})\downarrow$ and $\boldsymbol{D}_1(\bar{x}, \bar{y}) < 2^{-\mathrm{rc}(x,k)}$, then $\bar{y} \notin L^2$, i.e. $\boldsymbol{D}_2(\boldsymbol{F}(\bar{x}), \boldsymbol{F}(\bar{y})) \leqslant 2^{-k-1} < 2^{-k}$.

5. Recursive separability. A RMS M is *recursively separable* if it contains a recursively enumerable subset W_p which is *dense*, i.e. which intersects every sphere.

(8) The author became aware of Čeitin's [1] from Shepherdson's review [14], after he had given in [9] a proof of Theorem 3.

LEMMA 4. R, R^n $(n > 1)$ *and* F *are recursively separable.*

Proof. In the case of R (R^n), the rational points (the points with rational coordinates) form a dense recursively enumerable subset. In the case of F, the set

$$\{a \in \mathrm{F}\colon (Ek)(x)[x \geqslant k \rightarrow a(x) = 0]\}$$

of general recursive functions that are *eventually zero* can be easily seen to be recursively enumerable and dense.

The basic result for recursively separable spaces is:

THEOREM 4. *Let* M *be a recursively separable* RMS *satisfying* (A) *with dense recursively enumerable subset* W_p. *For every listable subset* L_n *of* M,

$$\mathrm{L}_n = \emptyset \equiv \mathrm{L}_n \cap \mathrm{W}_p = \emptyset .$$

Proof. Assume that there is a listable subset L_n of M which does not intersect W_p and a point $a_0 \in \mathrm{L}_n$. Since W_p is S-traced (by Lemma 2), Theorem 1 implies that there is an open sphere with center a_0 disjoint from W_p, contradicting our assumption that W_p is dense.

COROLLARY 4.1. *If* M *satisfies* (A) *and is recursively separable, then it satisfies* (B).

Proof. The intersection of a listable subset L_n and a sphere $\mathrm{S}(\bar{x}, k)$ is a listable subset of M, and is thus empty unless it intersects W_p. A point in $\mathrm{L}_n \cap \mathrm{S}(\bar{x}, k) \cap \mathrm{W}_p$ can be found (if one exists) by testing simultaneously, in a uniform way, all members of W_p for membership in both L_n and $\mathrm{S}(\bar{x}, k)$. Formally,

$$(5.1)\qquad \mathrm{tl}^{\mathrm{M}}(n) = \Lambda xk \Big(\mu u\big[T_1(p, (u_0), (u)_1) \; \& \; T_1(n, (u)_0, (u)_2) \; \& \; T_1(\mathrm{s}(x, k), (u)_0, (u)_3)\big]\Big)_0 .$$

Another direct corollary of Theorem 4 is the next result, stated here only for the case of R.

COROLLARY 4.2. *If* $\boldsymbol{F}(a)$ *and* $\boldsymbol{G}(a)$ *are partial recursive operators from* R *into* R *(partial recursive real functions) such that* $\boldsymbol{F}(a) = \boldsymbol{G}(a)$ *for all rational numbers* a *in the intersection of their domains, then* $\boldsymbol{F}(a) = \boldsymbol{G}(a)$ *for all* $a \in \mathrm{R}$ *in the intersection of their domains.*

Proof. The set $\{a \in \mathrm{R}\colon \boldsymbol{F}(a)\downarrow \;\&\; \boldsymbol{G}(a)\downarrow \;\&\; \boldsymbol{F}(a) \neq \boldsymbol{G}(a)\}$ is listable. Hence either it is empty or it contains rational numbers.

It is easy to verify that the set of listable predicates on a notation system T is closed under the operations of conjunction and disjunction. For example, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are partial recursive functions that determine the listable predicates $P(a)$ and $Q(a)$, then $\mathrm{f}(x) + \mathrm{g}(x)$ determines $P(a) \;\&\; Q(a)$. In case M is a recursively separable RMS satisfying (A) we can assert something much stronger.

THEOREM 5. *Let* M *be a recursively separable* RMS *satisfying* (A). *If* $P(a_1, \ldots, a_n, \beta)$ *is a listable predicate on* M, $(E\beta)P(a_1, \ldots, a_n, \beta)$ *is also listable on* M.

Proof. For each fixed n-tuple $a_1, \ldots, a_n$ of points,

$$(E\beta)P(a_1, \ldots, a_n, \beta) \equiv \{\beta \in \mathrm{M}\colon P(a_1, \ldots, a_n, \beta)\} \neq \emptyset .$$

But the set on the right is listable, and thus (by Theorem 4) it is non-empty if and only if it intersects W_p, the dense recursively enumerable subset of M. It follows that, if $\mathrm{f}(x_1, \ldots, x_n, y)$ with Gödel number f determines $P(a_1, \ldots, a_n, \beta)$ as a listable predicate on M, then

$$\mathrm{g}(x_1, \ldots, x_n) \simeq \mu u\big[T_1(p, (u)_0, (u)_1) \;\&\; T_{n+1}(f, x_1, \ldots, x_n, (u)_0, (u)_2)\big]$$

determines $(E\beta)P(a_1, \ldots, a_n, \beta)$.

The set of listable predicates on a recursively separable RMS M satisfying (A) resembles the set of number-theoretic predicates of the form $(Ey)R(x_1, \ldots, x_n, y)$ with R recursive in that it is closed under conjunction, disjunction and existential quantification. (In general it is not closed under negation, for example in the case of $\mathrm{N} = (N, \lambda xy\, x = y)$ or R by [10], Lemma 9.) It would be interesting to study what other conditions must be imposed on M before a nontrivial hierarchy theory can be developed starting with the set of listable predicates.

6. A theorem on listable orderings.

A *listable ordering* on a notation system T ([10], Def. 7) is a listable predicate $a \precsim \beta$ defined on T which is a classical linear, irreflexive ordering. The next theorem is a standard example of how the separation theorem and its corollaries can be used to provide almost routine proofs of plausible statements.

A point a of a RMS is *isolated* if some open sphere with center a contains no points other than a. A RMS with no isolated points is *perfect*.

THEOREM 6. *A perfect* RMS M *satisfying* (A) *and* (B) *cannot be listably well-ordered. In fact, for every listable ordering* $a \precsim \beta$ *of* M *we can define a recursive sequence* $a_0, a_1, a_2, \ldots$ *of points of* M *which is an infinite descending chain, i.e.* $a_0 \succsim a_1 \succsim a_2 \ldots$

Proof. Let $\mathrm{less}^{\mathrm{M}}(x, y)$ be a partial recursive function with Gödel number $less^{\mathrm{M}}$ which determines the given listable ordering $a \precsim \beta$ of M. We define partial recursive functions $\min^{\mathrm{M}}(x, y)$ and $\max^{\mathrm{M}}(x, y)$ which determine

$$\mathbf{MIN}^{\mathrm{M}}(a, \beta) = \begin{cases} a & \text{if} \quad a \precsim \beta , \\ \beta & \text{if} \quad \beta \precsim a , \\ \text{undefined} & \text{if} \quad a = \beta \end{cases}$$

and

$$\mathbf{MAX}^{\mathrm{M}}(a, \beta) = \begin{cases} \beta & \text{if} \quad a \precsim \beta , \\ a & \text{if} \quad \beta \precsim a , \\ \text{undefined} & \text{if} \quad a = \beta , \end{cases}$$

respectively as partial recursive operators on M, by

$$\min^{M}(x, y) \simeq \big(\mu u\big[[T_2(less^{M}, x, y, (u)_0) \,\&\, (u)_1 = x] \vee [T_2(less^{M}, y, x, (u)_0) \,\&\, (u)_1 = y]\big]\big)_1$$

and

$$\max^{M}(x, y) \simeq x + y - \min^{M}(x, y) .$$

Let t_0 be an R-index of the recursive real number **0**. For each $x, y \in M$,

$$h_1(x, y, z) \simeq \operatorname{less}(t_0, d(x, z)) + \operatorname{less}(t_0, d(y, z))$$

determines (as a function of z) the listable set

$$L^{x,y} = \{\alpha \in M:\ \alpha \neq \bar{x} \,\&\, \alpha \neq \bar{y}\} .$$

Since M is perfect, every sphere with center $\bar{x}$ and radius 2^{-k} must contain more than two points; hence it must contain points of $L^{x,y}$. We can find one such point by setting

$$h_2(x, y, k) \simeq \{\mathrm{tl}^{M}(\Lambda z\, h_1(x, y, z))\}(x, k) .$$

It is easy to verify now that for every $x, y \in M$ such that $\bar{x} \neq \bar{y}$,

$$(6.1)\qquad f_1(x, y, k) \simeq \max^{M}(x, h_2(x, y, k))$$

and

$$(6.2)\qquad g_1(x, y, k) \simeq \min^{M}(x, h_2(x, y, k))$$

are M-indices of points $\alpha = \alpha(x, y, k)$ and $\beta = \beta(x, y, k)$ respectively such that

$$(6.3)\qquad \alpha, \beta \in S(\bar{x}, k) \,\&\, \alpha \neq \bar{y} \,\&\, \beta \neq \bar{y} \,\&\, \beta \precsim \alpha .$$

Let

$$(6.4)\qquad k(x, y) \simeq \operatorname{sep}_1(\Lambda z\,\mathrm{less}^{M}(z, y), \Lambda z\,\mathrm{less}^{M}(y, z), x) .$$

Lemma 3 implies that if $\bar{x} \precsim \bar{y}$, then, for each $\alpha \in M$,

$$(6.5)\qquad \alpha \in S(\bar{x}, k(x, y)) \rightarrow \alpha = \bar{y} \vee \alpha \precsim \bar{y} .$$

We now choose two points $\bar{x}_0$ and $\bar{y}_0$ of M such that $\bar{x}_0 \precsim \bar{y}_0$. By (6.5), $S(\bar{x}_0, k(x_0, y_0))$ contains no points $\alpha \precsim \bar{y}_0$. Hence, by (6.3), if $\alpha = \alpha(x_0, y_0, k(x_0, y_0))$ and $\beta = \beta(x_0, y_0, k(x_0, y_0))$, then $\beta \precsim \alpha \precsim \bar{y}_0$. We now repeat the construction starting with the points β and α instead of $\bar{x}_0$ and $\bar{y}_0$.

Formally, we define three functions $f(t)$, $g(t)$ and $k(t)$ by simultaneous induction,

$$f(0) = y_0 ,\quad g(0) = x_0 ,\quad k(0) = k(x_0, y_0) ,$$
$$f(t+1) = f_1(g(t), f(t), k(t)) ,$$
$$g(t+1) = g_1(g(t), f(t), k(t)) ,$$
$$k(t+1) = k(g(t+1), f(t+1)) .$$

It is easy to show that $f(t)$, $g(t)$ and $k(t)$ are all general recursive and that $f(t)$ determines a recursive infinite descending chain, i.e. for all t, $f(t) \in M$ and $[f(t)]^{M} \succsim [f(t+1)]^{M}$.

It should be pointed out here that, in view of the complicated structure of listable sets and predicates, even in the simple cases of R and F (see § 9), the assertion of Theorem 8 is not as self-evident as it perhaps looks on first sight.

For the case of R and R^n, this result can be strengthened: The only listable orderings of R are the natural ordering $\alpha < \beta$ and its reverse $\alpha > \beta$. There is no listable ordering of R^n $(n > 1)$ [11].

7. Recursive completeness. One of the most basic theorems of the classical theory of metric spaces is the *Baire category theorem* ([3], Ch. XVI, Sec. 9, Th. 33) which asserts that *a complete metric space is not a denumerable union of closed, nowhere dense sets.* (9) In this section we give a recursive version of this result which implies in particular that no non-empty listable subset of R or F is recursively enumerable.

In the remarks following Definition 2 we mentioned that the classical closure of a subset E of a RMS M need not coincide with the recursive closure of E, unless E is S-traced. It thus appears that in a constructive version of the category theorem we should require the closed sets in the definition of "first category" to be S-traced. This restriction is not enough, as we shall see in § 8. In addition to requiring the sets in question to be "constructively closed", we must also require that their complements be "constructively open".

A *listable-S-traced* (*l-S-t*) subset of a RMS M is a listable set L_n whose complement $C_n = M - L_n$ is S-traced. If t is an S-tracing index of C_n, $\langle n, t\rangle = 2^n 3^t$ is an *l-S-t index* of L_n. The complement C_n of an *l-S-t* subset L_n is an *S-t-l* set. We note that, by Theorem 2 and the remarks following Definition 2, an S-t-l subset of a RMS satisfying (A) is closed.

A subset E of a RMS M is *of the first category*, if it is a recursive union of nowhere dense S-t-l sets, i.e. if

$$E = \bigcup_{t \geqslant 0} C_{(f(t))_0} ,$$

where $f(t)$ is general recursive and, for all t, $f(t)$ is an l-S-t index of some set $L_{(f(t))_0}$ dense in M. We say then that $f(t)$ *determines* E and we call any Gödel number f of $f(t)$ an *index* of E (as a subset of M of the first category).

(9) A closed subset of a metric space is *nowhere dense* if it contains no sphere.

THEOREM 7. *Let* M *be a recursively complete* RMS *satisfying* (A) *and* (B). *Then, no sphere in* M *is of the first category. In fact there is a partial recursive function* $b(a, k_0, f)$ *(depending on* M) (4) *such that, if* $a \in \alpha \in$ M *and if* f *is an index of some set* E *of the first category, then* $b(a, k_0, f)\downarrow$, $b(a, k_0, f) \in M$ *and* $[b(a, k_0, f)]^M \in S(\alpha, k_0) - E$. (The Baire category theorem.)

Proof. Under our definitions we can constructivize directly the classical proof of the Baire theorem.

Let us assume that a and f satisfy the hypothesis. For this proof, we let $L^t = L_{(\{f\}(t))_0}$ and $C^t = M - L^t$.

For each $x \in M$ and each t and k, $L^t \cap S(\bar{x}, k) \neq \emptyset$, since L^t is dense. We can find effectively an index of some element in $L^t \cap S(\bar{x}, k)$ by setting

$$(7.1) \qquad g_1(x, t, k) \simeq \{tl^M((\{f\}(t))_0)\}(x, k) .$$

For each t, $(\{f\}(t))_1$ is an S-tracing index of C^t. Hence Theorem 1 implies that if $\bar{x} \in L^t$, then

$$(7.2) \qquad k_1(x, t) \simeq \operatorname{sep}\big((\{f\}(t))_0, (\{f\}(t))_1, x\big)\downarrow$$

and

$$(7.3) \qquad S(\bar{x}, k_1(x, t)) \subset L^t .$$

Now let k_0 be given. We define two recursive functions $g(t)$ and $k(t)$ by the simultaneous induction:

$$g(0) \simeq g_1(a, 0, k_0+2) ,$$
$$k(0) \simeq \max\big(k_1(g(0), 0), k_0+2\big) ,$$
$$g(t+1) \simeq g_1\big(g(t), t+1, k(t)+2\big) ,$$
$$k(t+1) \simeq \max\big(k_1(g(t+1), t+1), k(t)+2\big) .$$

It follows easily from (7.1) and (7.2) that, for each t,

$$g(t)\downarrow \;\&\; k(t)\downarrow \;\&\; g(t) \in M \;\&\; \beta_t = [g(t)]^M \in L^t ,$$

and

$$S(\beta_t, k(t)) \subset S\big(\beta_t, k_1(g(t), t)\big) \subset L^t .$$

We observe that

$$(7.4) \qquad S(\alpha, k_0+1) \supset S(\beta_0, k(0)) .$$

Because by definition $D(\alpha, \beta_0) < 2^{-k_0-2}$ and $k(0) \geqslant k_0+2$, and hence $\gamma \in S(\beta_0, k(0)) \rightarrow D(\gamma, \alpha) \leqslant D(\gamma, \beta_0) + D(\beta_0, \alpha) < 2^{-k_0-2} + 2^{-k_0-2} = 2^{-k_0-1}$.

Similarly, for each t,

$$(7.5) \qquad S(\beta_t, k(t)+1) \supset S(\beta_{t+1}, k(t+1)) .$$

Because, by definition, $D(\beta_{t+1}, \beta_t) < 2^{-k(t)-2}$ and $k(t+1) \geqslant k(t)+2$, and hence $\gamma \in S(\beta_{t+1}, k(t+1)) \rightarrow D(\gamma, \beta_t) \leqslant D(\gamma, \beta_{t+1}) + D(\beta_{t+1}, \beta_t) < 2^{-k(t)-2} + 2^{-k(t)-2} = 2^{-k(t)-1}$. In particular, (7.5) implies that, for each t and u,

$$(7.6) \qquad D(\beta_t, \beta_{t+u}) < 2^{-k(t)-1} < 2^{-t} ,$$

since trivially $k(t) > t$. Thus $\beta_0, \beta_1, \beta_2, \ldots$ is recursively Cauchy with r.c. index *id* (cf. (2.3)). Hence

$$(7.7) \qquad \begin{gathered} b(a, k_0, f) \simeq c^M(\Lambda t\, g(t), id)\downarrow , \\ b(a, k_0, f) \in M , \quad \beta = [b(a, k_0, f)]^M = \lim_{t\to\infty} \beta_t , \end{gathered}$$

and by (7.4) and (7.5),

$$(7.8) \qquad D(\alpha, \beta) \leqslant 2^{-k_0-1} < 2^{-k_0} .$$

It remains to show that, for each t, $\beta \notin C^t$. To prove this, we notice that from (7.3) and the definitions of $g(t)$ and $k(t)$,

$$(7.9) \qquad \gamma \in C^t \rightarrow D(\gamma, \beta_t) \geqslant 2^{-k(t)} .$$

Hence (7.6) implies that, for each u, $D(\gamma, \beta_{t+u}) \geqslant D(\gamma, \beta_t) - D(\beta_t, \beta_{t+u}) > 2^{-k(t)} - 2^{-k(t)-1} = 2^{-k(t)-1}$. Thus $D(\gamma, \beta) \geqslant 2^{-k(t)-1}$ which implies $\gamma \neq \beta$.

Classically every denumerable subset of a perfect space is of the first category. The next lemma is a recursive version of this fact.

LEMMA 5. *Every recursively enumerable subset of a perfect* RMS M *is of the first category. In fact there is a primitive recursive function* $b_1(e)$ *(depending on* M) (4) *such that, if* W_e *determines a recursively enumerable subset* W_e *of* M, *then* W_e *is of the first category with index* $b_1(e)$. (10)

Proof. The whole space M is an l-S-t subset of itself with l-S-t index $m_0 = \langle id, \Lambda xk\, \mu u[u \neq u]\rangle$. Thus the empty set is a nowhere dense S-t-l set. Similarly, if M is perfect and $x \in M$, then $M - \{\bar{x}\}$ is an l-S-t

(10) We can prove ([10], Lemma 7) by setting $tr(e) = b(t_0, 0, b_1(e))$. Then $tr(e)$ is primitive recursive, since $b(x, k, f) \simeq c^M(\Lambda t\, g(t), id)$ by (7.4) and $c^R(f, g)$ is primitive recursive ([10], Lemma 8). We can see directly that $tr(e)$ can be taken to be primitive recursive (without tracing the definitions) by setting $tr(e) = \Lambda t\,\{b(t_0, 0, b_1(e))\}(t)$ instead of the above function. The same trick can be used to show that every recursive operator $F(\alpha)$ on any notation system T into R or F is determined by a primitive recursive function. For, if $f(x)$ determines $F(\alpha)$, $g(x) = \Lambda t\,\{f(x)\}(t)$ also determined $F(\alpha)$ and is primitive recursive.

subset of M with 1-S-t index $\mathrm{f}(x) = \langle \Lambda z\, \mathrm{less}\big(t_0, \mathrm{d}(x, z)\big), \Lambda zk\, x\rangle$. (Here t_0 is an R-index of **0**, as in the proof of Theorem 6.) Thus $\{\bar{x}\}$ is a nowhere dense S-t-1 set. To prove the lemma, we simply write W_e as a recursive union of empty sets and the unit sets of its members,

$$\mathrm{b}_1(e) = \Lambda t\, \mu u\Big[\big[\overline{T}_1\big(e, (t)_0, (t)_1\big)\ \&\ u = m_0\big] \vee \big[T_1\big(e, (t)_0, (t)_1\big)\ \&\ u = \mathrm{f}\big((t)_0\big)\big]\Big].$$

THEOREM 8. *Let* M *be recursively complete satisfying* (A) *and* (B): *No non-empty listable subset of* M *is of the first category. In particular, if* M *is perfect, no non-empty listable subset of* M *is recursively enumerable.*

Proof. Theorem 7 implies directly that the complement of a set E of the first category with index f is not only dense but S-traced, with S-tracing index $\Lambda xk\, \mathrm{b}(x, k, f)$. Since a listable set with an S-traced complement is (by the separation theorem) open, it cannot have a dense complement unless it is empty.

8. A listable, closed, nowhere dense set. The simplest listable subsets of a RMS M that one can construct (see § 9) are open. This, together with the separation theorem, suggest the conjecture that under suitable assumptions on M, every listable set will be open. Friedberg first constructed in [2] a listable subset of F which contains the constant function **0** but no sphere with **0** as center (he was motivated in [2] by a different problem). In this section we shall construct listable subsets of R and F that are closed and nowhere dense. These extremely thin listable "Cantor" sets will give us a counterexample to an apparently plausible strengthening of Theorem 7.

Throughout this section M shall be a recursively separable RMS satisfying (A) (and hence by Corollary 4.1 (B)) with dense recursively enumerable subset W_p. We shall construct some closed listable subsets of M which, under an additional "thickness" hypothesis (satisfied by both R and F), will be nowhere dense.

We first define a natural notation system to represent the "recursive completion" of M. In order to simplify notation, we put $x_u \simeq \{x\}(u)$, and $\boldsymbol{x}_u = [x_u]^{\mathrm{M}}$ in case $x_u \in M$.

(8.1) $$x \in M_k \equiv (u)(v)_{u\leqslant v\leqslant k}[x_u\downarrow\ \&\ x_u \in W_p\ \&\ x_v\downarrow\ \&\ x_v \in W_p\ \&\ \&\ \boldsymbol{D}(\boldsymbol{x}_u, \boldsymbol{x}_v) < 2^{-u}].$$

(8.2) $$x \sim_k y \equiv x \in M_k\ \&\ y \in M_k\ \&\ \boldsymbol{D}(\boldsymbol{x}_k, \boldsymbol{y}_k) < 2^{-k+2}.$$

(8.3) $$x \in M_c \equiv (k)\, x \in M_k.$$

(8.4) $$x \sim_{\mathrm{M_c}} y \equiv (k)\, x \sim_k y.$$

It can easily be verified that $x \in M_k$ and $x \sim_k y$ are recursively enumerable predicates of x, y and k.

The relations $\sim_k$ $(k = 0, 1, 2, \ldots)$ are not equivalence relations. They satisfy the following two properties, which imply that $\sim_{\mathrm{M_c}}$ is an equivalence relation on M_c.

(8.5.a) $$x \sim_{k+1} y \rightarrow x \sim_k y.$$

For $\boldsymbol{D}(\boldsymbol{x}_k, \boldsymbol{y}_k) \leqslant \boldsymbol{D}(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}) + \boldsymbol{D}(\boldsymbol{x}_{k+1}, \boldsymbol{y}_{k+1}) + \boldsymbol{D}(\boldsymbol{y}_{k+1}, \boldsymbol{y}_k) < 2^{-k} + 2^{-k+1} + 2^{-k} = 2^{-k+2}$.

(8.5.b) $$x \sim_{k+2} y\ \&\ y \sim_{k+2} z \rightarrow x \sim_k z.$$

For $\boldsymbol{D}(\boldsymbol{x}_k, \boldsymbol{z}_k) \leqslant \boldsymbol{D}(\boldsymbol{x}_k, \boldsymbol{x}_{k+2}) + \boldsymbol{D}(\boldsymbol{x}_{k+2}, \boldsymbol{y}_{k+2}) + \boldsymbol{D}(\boldsymbol{y}_{k+2}, \boldsymbol{z}_{k+2}) + \boldsymbol{D}(\boldsymbol{z}_{k+2}, \boldsymbol{z}_k) < 2^{-k} + 2^{-k} + 2^{-k} + 2^{-k} = 2^{-k+2}$.

LEMMA 6. *The recursive function*

(8.6) $$x^* = \mathrm{f_c}(x) = \Lambda t\, \{\mathrm{tr}_1(p)\}(x, t+1)$$

determines a recursive operator $\boldsymbol{F}_c(\alpha)$ *from* M *into the notation system* $\mathrm{M_c} = (M_c, \sim_{\mathrm{M_c}})$.

Proof. Since W_p is dense, if $x \in M$, then $(u)[x_u^*\downarrow\ \&\ x_u^* \in W_p]$. [11] Also if $u \leqslant v$, $\boldsymbol{D}(\boldsymbol{x}_u^*, \boldsymbol{x}_v^*) \leqslant \boldsymbol{D}(\boldsymbol{x}_u^*, \bar{x}) + \boldsymbol{D}(\bar{x}, \boldsymbol{x}_v^*) < 2^{-u-1} + 2^{-v-1} \leqslant 2^{-u}$. Thus $x^* \in M_c$. Now if $x \sim_{\mathrm{M}} y$, then, for each k, $\boldsymbol{D}(\boldsymbol{x}_k^*, \boldsymbol{y}_k^*) \leqslant D(\boldsymbol{x}_k^*, \bar{x}) + \boldsymbol{D}(\bar{x}, \bar{y}) + \boldsymbol{D}(\bar{y}, \boldsymbol{y}_k^*) < 2^{-k-1} + 2^{-k-1} < 2^{-k+2}$, hence $x^* \sim_k y^*$.

Because of the lemma, each listable subset $\mathrm{L^c}$ of $\mathrm{M_c}$ defines a listable subset L of M by

(8.7) $$\alpha \in \mathrm{L} \equiv \boldsymbol{F}_c(\alpha) \in \mathrm{L^c}.$$

Let x_0 be some fixed member of M_c, and $\mathrm{g}(t)$ some fixed general recursive function with Gödel number g such that, for each t,

(8.8) $$\mathrm{g}(t+1) \geqslant \mathrm{g}(t) + 3.$$

We define a sequence $A_0, A_1, A_2, \ldots$ of sets of natural numbers by the induction

(8.9.a) $$x \in A_0 \equiv x = x_0,$$

(8.9.b) $$x \in A_{n+1} \equiv (Ey)[y \in A_n\ \&\ x \sim_{\mathrm{g}(x)} y].$$

It is easy to verify that each A_n is recursively enumerable and that $A = \bigcup_n A_n = W_e$, where

(8.10) $$e = \mathrm{ce}(x_0, g)$$

with $\mathrm{ce}(x_0, g)$ primitive recursive.

---

[11] In the statement of Lemma 2 it is only asserted that if $x \in M$, $W_e \subset M$ and $W_e \cap \mathrm{S}(\bar{x}, k) \neq \emptyset$, then $[\{\mathrm{tr}_1(e)\}(x, k)]^{\mathrm{M}} \in W_e \cap \mathrm{S}(\bar{x}, k)$. However it is obvious from (3.1) that under these assumptions $\{\mathrm{tr}_1(e)\}(x, k) \in W_e$.

The inductive step (8.9.b) of the definition of A implies trivially that

$$(8.11)\qquad x \in A \ \&\ x \sim_{M_c} y \rightarrow y \in A .$$

Thus $e \in L(M_c)$. Let $L^c = L_e$ be the listable subset of M_c with index e and put $L = \boldsymbol{F}_c^{-1}(L^c)$, the corresponding listable subset of M.

To show that L is closed, we observe that, if $\alpha = \bar{a} \in M$ is a limit point of L, then there must be some $\beta = \bar{b} \in L$ with $\boldsymbol{D}(\alpha, \beta) < 2^{-g(a^*)}$. Since $\beta \in L$, $b^* \in A$, i.e. $b^* \in A_n$ for some n. Now

$$\boldsymbol{D}(\boldsymbol{a}^*_{g(a^*)}, \boldsymbol{b}^*_{g(a^*)}) \leqslant \boldsymbol{D}(\boldsymbol{a}^*_{g(a^*)}, \alpha) + \boldsymbol{D}(\alpha, \beta) + \boldsymbol{D}(\beta, \boldsymbol{b}^*_{g(a^*)})$$
$$< 2^{-g(a^*)} + 2^{-g(a^*)} + 2^{-g(a^*)} < 2^{-g(a^*)+2},$$

hence $a^* \sim_{g(a^*)} b^*$ and $a^* \in A_{n+1}$ which implies $\alpha \in L$.

We also note that if $x_0 = x^*$ for some $x \in M$, then $\bar{x} \in L$. In this case L is a non-empty listable closed subset of M.

Lemma 7. *For fixed k, let $t^0 = x_0$, and let $t^1, \ldots, t^m$ $(m \leqslant k)$ be the finitely many members of A that are $\leqslant k$. If $x \in A \cap M_{g(k)}$, then there is some $i \leqslant m$ such that $x \sim_{g(k)} t^i$.*

Proof. By induction on the number n such that $x \in A_n$.

Basis: $x \in A_0$. In this case $x = x_0$, and hence $x \sim_{g(k)} x_0 = t^0$.

Induction step: $x \in A_{n+1}$. If x is one of the t^i's there is nothing to prove. Otherwise, there is some $x^1 \in A_n$ such that $x \sim_{g(x)} x^1$. Proceeding in the same manner, we find finitely many distinct elements $x^1, \ldots, x^s$ $(s \leqslant n)$ of A and some t^i $(i \leqslant m)$ such that

$$(8.12.a)\qquad (j)_{1 \leqslant j \leqslant s}\, x^j \geqslant k+1$$

and

$$(8.12.b)\qquad x \sim_{g(x)} x^1 \sim_{g(x^1)} x^2 \ldots x^{s-1} \sim_{g(x^{s-1})} x^s \sim_{g(x^s)} t^i .$$

Thus,

$$\begin{aligned}\boldsymbol{D}(\boldsymbol{x}_{g(k)}, \boldsymbol{t}^i_{g(k)}) \leqslant\ & \boldsymbol{D}(\boldsymbol{x}_{g(k)}, \boldsymbol{x}_{g(x)}) && + \boldsymbol{D}(\boldsymbol{x}_{g(x)}, \boldsymbol{x}^1_{g(x)}) \\ & + \boldsymbol{D}(\boldsymbol{x}^1_{g(x)}, \boldsymbol{x}^1_{g(x^1)}) && + \boldsymbol{D}(\boldsymbol{x}^1_{g(x^1)}, \boldsymbol{x}^2_{g(x^1)}) \\ & + \quad \cdots && + \boldsymbol{D}(\boldsymbol{x}^{s-1}_{g(x^{s-1})}, \boldsymbol{x}^s_{g(x^{s-1})}) \\ & + \boldsymbol{D}(\boldsymbol{x}^s_{g(x^{s-1})}, \boldsymbol{x}^s_{g(x^s)}) && + \boldsymbol{D}(\boldsymbol{x}^s_{g(x^s)}, \boldsymbol{t}^i_{g(x^s)}) \\ & + \boldsymbol{D}(\boldsymbol{t}^i_{g(x^s)}, \boldsymbol{t}^i_{g(k)}) .\end{aligned}$$

Now (8.1) implies that, if $z \in M_{\max(u,v)}$, then $\boldsymbol{D}(\boldsymbol{z}_u, \boldsymbol{z}_v) < 2^{-\min(u,v)} < 2^{-u} + 2^{-v}$. Using this, (8.12.a) and (8.12.b),

$$\boldsymbol{D}(\boldsymbol{x}_{g(k)}, \boldsymbol{t}^i_{g(k)}) < 2^{-g(k)} + 2\sum_{r \geqslant k+1} 2^{-g(r)} + \sum_{r \geqslant k+1} 2^{-g(r)+2} + 2^{-g(k)} .$$

(Here $2\sum_{r \geqslant k+1} 2^{-g(r)}$ is an upper bound of the first column above, except for the top and bottom entries, and $\sum_{r \geqslant k+1} 2^{-g(r)+2}$ is an upper bound of the second column. We are using the fact that the x^j's are distinct.) Thus

$$\boldsymbol{D}(\boldsymbol{x}_{g(k)}, \boldsymbol{t}^i_{g(k)}) < 2^{-g(k)+1} + 2\sum_{r \geqslant g(k+1)} 2^{-r} + 4\sum_{r \geqslant g(k+1)} 2^{-r}$$
$$= 2^{-g(k)+1} + 6 \cdot 2^{-g(k+1)+1} < 2^{-g(k)+1} + 2^{-g(k+1)+4} .$$

Since (8.8) implies that $2^{-g(k+1)+4} \leqslant 2^{-g(k)+1}$,

$$\boldsymbol{D}(\boldsymbol{x}_{g(k)}, \boldsymbol{t}^i_{g(k)}) < 2^{-g(k)+2}, \qquad \text{i.e.} \qquad x \sim_{g(k)} t^i .$$

Let $g(t)$ be some function satisfying (8.8). We say that M is *somewhere thin* (*with respect to* g) if, for some $\alpha \in M$ and some k, $S(\alpha, k)$ can be covered by $\leqslant k+1$ spheres of radius $2^{-g(k+1)}$. If M is not somewhere thin, it is *nowhere thin*.

Theorem 9. (a) *If* M *is nowhere thin with respect to* g, *then the listable closed subset* L *of* M *defined in this section is nowhere dense.*

(b) *If* $g(t) = 3t$, *then both* F *and* R *are nowhere thin.*

Proof. To prove (a) by contradiction, assume that M is nowhere thin, but that, for some $\alpha \in M$ and some k, $S(\alpha, k) \subset L$. For this k, choose $t^0, t^1, \ldots, t^m$ $(m \leqslant k)$ as in Lemma 7. If $\bar{x} \in L$, then $x^* \in A \cap M_c$, and thus, for some $i \leqslant m$, $x \sim_{g(k)} t^i$. Hence $\boldsymbol{D}(\bar{x}, \boldsymbol{t}^i_{g(k)}) \leqslant \boldsymbol{D}(\bar{x}, \boldsymbol{x}^*_{g(k)}) + \boldsymbol{D}(\boldsymbol{x}^*_{g(k)}, \boldsymbol{t}^i_{g(k)}) < 2^{-g(k)} + 2^{-g(k)+2} < 2^{-g(k)+3} \leqslant 2^{-g(k+1)}$. Thus each $\bar{x} \in L$, and in particular each $\bar{x} \in S(\alpha, k)$, is contained in some sphere $S(\boldsymbol{t}^i_{g(k)}, g(k+1))$. Since there are at most $k+1$ such spheres, this contradicts our assumption that M is nowhere thin.

(b) is trivial in the case of F, since a sphere of radius 2^{-k} cannot be covered by a finite union of spheres of radius $2^{-g(k+1)} = 2^{-3k-3} < 2^{-k-1}$. In the case of R, a sphere of radius 2^{-k} is simply an open interval of length $2 \cdot 2^{-k}$. If an interval of this length were to be covered by $k+1$ intervals of length 2^{-3k-3}, we would have $2^{-k+1} \leqslant (k+1)2^{-3k-3}$, which is absurd.

Remark. In the case of F, the construction of L and the proof of Theorem 9 are much easier, essentially because the relations $\sim_k$ are equivalence relations.

We asserted in (8.10) that $A = W_{ce(x_0, g)}$ where $ce(x_0, g)$ is primitive recursive. The definition of A together with Lemma 6 and (8.11) imply at once that

$$(8.13)\qquad x \sim_M y \rightarrow W_{ce(x^*, g)} = W_{ce(y^*, g)} .$$

Thus, if $a \in \alpha \in \mathrm{M}$, the listable set $\mathrm{L}^{\alpha} = \boldsymbol{F}_c^{-1}(\mathrm{L}_{\mathrm{ce}(a^*, g)})$ does not depend on the particular index a of α. The binary predicate $\alpha \in \mathrm{L}^{\beta}$ is listable on M, since

$$\alpha \in \mathrm{L}^{\beta} \equiv \{\mathrm{ce}(b^*, g)\}(a^*)\downarrow \qquad (a \in \alpha \in \mathrm{M},\ b \in \beta \in \mathrm{M}) . \tag{8.14}$$

Thus for fixed α, $\{\beta\colon \alpha \in \mathrm{L}^{\beta}\}$ is listable. This set is also non-empty, since $\alpha \in \mathrm{L}^{\alpha}$. Thus, by Theorem 4, for each $\alpha \in \mathrm{M}$ there is a $\beta \in W_p$ such that $\alpha \in \mathrm{L}^{\beta}$, i.e.

$$\mathrm{M} = \bigcup_{\beta \in W_p} \mathrm{L}^{\beta} . \tag{8.15}$$

Let $\mathrm{h}(t)$ be a general recursive function which enumerates W_p. We can rewrite (8.15) as

$$\mathrm{M} = \bigcup_t \mathrm{L}_{\mathrm{f}(t)} \tag{8.16}$$

where

$$\mathrm{f}(t) = \Lambda x \left\{\mathrm{ce}\left(\mathrm{h}(t)^*, g\right)\right\}(x^*) . \tag{8.17}$$

THEOREM 10. *There is a general recursive function* $\mathrm{f}(t)$ *such that, for each* t, $\mathrm{f}(t) \in L(\mathrm{R})$, $\mathrm{L}_{\mathrm{f}(t)}$ *is a listable closed nowhere dense set and* $\mathrm{R} = \bigcup_t \mathrm{L}_{\mathrm{f}(t)}$. *Similarly with* F.

We note that each $\mathrm{L}_{\mathrm{f}(t)}$ is S-traced, by Lemma 4 and Corollary 4.1.

9. Recursively open sets. A listable subset L of a RMS M is *recursively open*, if there is a partial recursive function $\mathrm{op}(x)$ such that, if $a \in \alpha \in \mathrm{M}$, then

$$\mathrm{op}(a)\downarrow \equiv \alpha \in \mathrm{L} \tag{9.1.a}$$

and

$$\mathrm{op}(a) \to \mathrm{S}\left(\alpha, \mathrm{op}(a)\right) \subset \mathrm{L} . \tag{9.1.b}$$

We say then that $\mathrm{op}(x)$ *determines* L, and we call any Gödel number op of $\mathrm{op}(x)$ an *index* of L (as a recursively open subset of M).

We saw (Theorem 1) that 1-S-t subsets of M are recursively open, if M satisfies (A). We also constructed listable sets which contain no sphere at all and are closed (Theorem 9). In this section we classify the recursively open subsets of a recursively separable M that satisfies (A). This classification, together with Theorem 3, will yield Čeitin's version of the recursive continuity theorem for recursively separable spaces.

The simplest example of a recursively open subset of M is a sphere $\mathrm{S}(\alpha, k)$. For, if $a \in \alpha$ and

$$\begin{aligned} \mathrm{f}(a, k, x) \simeq \Big(\mu u\Big[& T_2\left(d, a, x, (u)_0\right) \ \& \\ & T_2\Big(less,\ U\left((u)_0\right),\ \varrho\left(\mathrm{r}^{-1}(2^{-k})\right),\ (u)_1\Big) \ \& \\ & T_2\Big(less,\ \varrho\left(\mathrm{r}^{-1}(2^{-(u)_2})\right),\ \mathrm{f}_+\Big(\varrho\left(\mathrm{r}^{-1}(2^{-k})\right),\ \mathrm{f}_-\left(U\left((u)_0\right)\right)\Big),\ (u)_3\Big)\Big]\Big)_2, \end{aligned} \tag{9.2}$$

then

$$\mathrm{sro}(a, k) = \Lambda x\, \mathrm{f}(a, k, x) \tag{9.3}$$

is an index of $\mathrm{S}(\alpha, k)$ as a recursively open set. (12)

The next most obvious examples of recursively open sets are recursive unions of spheres. Let $\mathrm{f}(t)$ and $\mathrm{g}(t)$ be partial recursive functions such that, for each t, if $\mathrm{f}(t)\downarrow$, then $\mathrm{f}(t) \in M$ and $\mathrm{g}(t)\downarrow$. The set

$$\mathrm{L} = \bigcup_{t, \mathrm{f}(t)\downarrow} \mathrm{S}\left([\mathrm{f}(t)]^{\mathrm{M}}, \mathrm{g}(t)\right)$$

is a *Lacombe set* with index $\langle f, g\rangle$, where f and g are Gödel numbers of $\mathrm{f}(t)$ and $\mathrm{g}(t)$, respectively. (Subsets of the classical real numbers defined in this way were called "recursively open" by Lacombe [6].)

LEMMA 8. *Every Lacombe subset of a* RMS M *is recursively open. In fact there is a primitive recursive function* $\mathrm{lro}(x)$ (*depending on* M) (4) *such that, if* $\langle f, g\rangle$ *is a Lacombe index of* L, *then* $\mathrm{lro}(\langle f, g\rangle)$ *is an index of* L *as a recursively open set.*

Proof.

$$\mathrm{lro}(x) = \Lambda y \left\{\mathrm{sro}\left(\{(x)_0\}\left(\mathrm{k}(y)\right),\ \{(x)_1\}\left(\mathrm{k}(y)\right)\right)\right\}(y) ,$$

where

$$\begin{aligned} \mathrm{k}(y) \simeq \Big(\mu u\Big[& T_1\left((x)_0, (u)_0, (u)_1\right) \ \& \ T_1\left((x)_1, (u)_0, (u)_2\right) \ \& \\ & T_1\Big(\mathrm{s}\left(U\left((u)_1\right),\ U\left((u)_2\right)\right), y, (u)_3\Big)\Big]\Big)_0 . \end{aligned}$$

THEOREM 11. *Every recursively open subset of a recursively separable* RMS M *satisfying* (A) *is a Lacombe set. In fact there is a primitive recursive function* $\mathrm{rol}(x)$ (*depending on* M) (4) *such that, if* op *is an index of* L *as a recursively open subset of* M, *then* $\mathrm{rol}(op)$ *is a Lacombe index of* L.

Proof. We use the method of proof of Theorem 1. Let

$$\begin{aligned} P(m, op, x, u) \equiv\ & T_2\left(c^{\mathrm{M}}, m, id, (u)_0\right) \ \& \\ & T_1\left(op,\ U\left((u)_0\right), (u)_1\right) \ \& \ T_1\Big(\mathrm{s}\left(U\left((u)_0\right),\ U\left((u)_1\right)\right), x, (u)_2\Big) . \end{aligned} \tag{9.4}$$

We set

$$\mathrm{h}(m, op, x, t) \simeq \begin{cases} \{\mathrm{tr}_1(p)\}(x, t+1) & \text{if} \quad (u)_{u\leqslant t}\overline{P}(m, op, x, u) , \\ \{\mathrm{tr}_1(p)\}\left(x, \mathrm{s}(m, op, x)\right) & \text{otherwise} , \end{cases} \tag{9.5}$$

where

$$\mathrm{s}(m, op, x) \simeq \mu u P(m, op, x, u) + 1 . \tag{9.6}$$

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(12) The functions $\mathrm{f}_+(x, y)$ and $\mathrm{f}_-(x)$ ([10], Lemma 4) are primitive recursive and determine $\alpha+\beta$ and $-\alpha$ as recursive operators on R to R.

By the recursion theorem ((2.1) and (2.2)), if h is a Gödel number of $\mathrm{h}(m, op, x, t)$ and

$$m = \mathrm{rt}^2(h, op, x)\,, \tag{9.7}$$

then, for each t,

$$\{m\}(t) \simeq \mathrm{h}(m, op, x, t)\,. \tag{9.8}$$

Let us now assume that op is an index of L as a recursively open set and that $x \in M$ and $\bar{x} \in \mathrm{L}$. We show by contradiction that

$$(Eu)P(m, op, x, u)\,. \tag{9.9}$$

Suppose (9.9) is false. Then, for each t, $\{m\}(t) \simeq \{\mathrm{tr}_1(p)\}(x, t+1)$. It can be verified that m is then an index of a recursive sequence of points of M which converges to $\bar{x}$ with r.c. index id. Thus

$$\mathrm{c}^{\mathrm{M}}(m, id)\downarrow \;\&\; \mathrm{c}^{\mathrm{M}}(m, id) \in M \;\&\; [\mathrm{c}^{\mathrm{M}}(m, id)]^{\mathrm{M}} = \bar{x} \in \mathrm{L}\,, \tag{9.10}$$

and hence

$$\{op\}\big(\mathrm{c}^{\mathrm{M}}(m, id)\big)\downarrow \;\& \quad \bar{x} = [\mathrm{c}^{\mathrm{M}}(m, id)]^{\mathrm{M}} \in \mathrm{S}\Big([\mathrm{c}^{\mathrm{M}}(m, id)]^{\mathrm{M}}, \{op\}\big(\mathrm{c}^{\mathrm{M}}(m, id)\big)\Big) \subset \mathrm{L}\,. \tag{9.11}$$

This is a contradiction, since (9.10) and (9.11) together imply (9.9).

It is clear now from (9.4), (9.5) and (9.9) that, if $x \in M$, $\bar{x} \in \mathrm{L}$ and m is defined by (9.7), then

$$\mathrm{c}^{\mathrm{M}}(m, id)\downarrow \;\&\; \mathrm{c}^{\mathrm{M}}(m, id) \in M \;\&\; [\mathrm{c}^{\mathrm{M}}(m, id)]^{\mathrm{M}} \in \mathrm{W}_p \;\& \quad \bar{x} \in \mathrm{S}\Big([\mathrm{c}^{\mathrm{M}}(m, id)]^{\mathrm{M}}, \{op\}\big(\mathrm{c}^{\mathrm{M}}(m, id)\big)\Big) \subset \mathrm{L}\,. \tag{9.12}$$

Thus, each point of L is contained in some sphere of the form $\mathrm{S}\big(\bar{a}, \{op\}(a)\big)$, where $\bar{a} \in \mathrm{W}_p \cap \mathrm{L}$. The set $\{a\colon \bar{a} \in \mathrm{W}_p \cap \mathrm{L}\}$ is not, in general, recursively enumerable. What we must do is find a recursively enumerable subset of if which contains $\mathrm{c}^{\mathrm{M}}(m, id)$ for each $x \in M$, $\bar{x} \in \mathrm{L}$.

Let

$$\begin{aligned} Q(op, x) \equiv (Eu)\Big[P(m, op, x, u)\;\& \\ (t)(z)_{t\leqslant z\leqslant u}\big[\{\mathrm{tr}_1(p)\}(x, t+1)\downarrow \;\&\; \{\mathrm{tr}_1(p)\}(x, t+1) \in W_p \;\& \\ \{\mathrm{tr}_1(p)\}(x, z+1)\downarrow \;\&\; \{\mathrm{tr}_1(p)\}(x, z+1) \in W_p \;\& \\ \boldsymbol{D}\big([\{\mathrm{tr}_1(p)\}(x, t+1)]^{\mathrm{M}}, [\{\mathrm{tr}_1(p)\}(x, z+1)]^{\mathrm{M}}\big) < 2^{-t}\big]\Big]\,, \end{aligned} \tag{9.13}$$

where m is defined by (9.7). It is not hard to verify that $Q(op, x)$ is recursively enumerable, i.e.

$$Q(op, x) \equiv (Ey)R(op, x, y) \tag{9.14}$$

with R recursive. We set

$$\mathrm{rol}(op) = \langle f, g\rangle\,, \tag{9.15}$$

where

$$f = \Lambda x\, \mathrm{c}^{\mathrm{M}}(m, id) + 0\cdot \mu y R(op, x, y)\,, \tag{9.16}$$

$$g = \Lambda x\, \{op\}\big(\{f\}(x)\big)\,. \tag{9.17}$$

Now if $\{f\}(x)\downarrow$, then (9.13), (9.4) and (9.5) imply that

$$\{f\}(x) = \mathrm{c}^{\mathrm{M}}(m, id) \in M \;\&\; [\{f\}(x)]^{\mathrm{M}} \in \mathrm{L}\,. \tag{9.18}$$

Thus $\{g\}(x) \simeq \{op\}\big(\{f\}(x)\big)\downarrow$ and $\mathrm{S}\big([\{f\}(x)]^{\mathrm{M}}, \{g\}(x)\big) \subset \mathrm{L}$. On the other hand, if $x \in M$ and $\bar{x} \in \mathrm{L}$, then (9.12) implies that $\{f\}(x)\downarrow$ and $\bar{x} \in \mathrm{S}\big([\{f\}(x)]^{\mathrm{M}}, \{g\}(x)\big)$. Thus $\mathrm{L} = \bigcup_{t,\{f\}(t)\downarrow} \mathrm{S}\big([\{f\}(t)]^{\mathrm{M}}, \{g\}(t)\big)$.

THEOREM 12. *Let* M_1 *be a recursively separable* RMS *satisfying* (A), *let* M_2 *be a* RMS *and let* $\boldsymbol{F}(\alpha)$ *be a recursive operator from* M_1 *into* M_2. *Then for each* $\alpha \in \mathrm{M}_2$ *and each* k, $\boldsymbol{F}^{-1}\big(\mathrm{S}(\alpha, k)\big)$ *is a Lacombe subset of* M_1. *In fact there is a primitive recursive function* $\mathrm{lc}(a, k)$ (*depending on* M_1, M_2 *and* $\boldsymbol{F}(\alpha)$)(4) *such that, if* $a \in \alpha \in \mathrm{M}_2$, *then* $\mathrm{lc}(a, k)$ *is a Lacombe index of* $\boldsymbol{F}^{-1}\big(\mathrm{S}(\alpha, k)\big)$.

Proof. Let $\mathrm{f}(x)$ with Gödel number f determine $\boldsymbol{F}(\alpha)$ and let $\mathrm{d}_2(x, y)$ with Gödel number d_2 determine the distance operator on M_2. We first seek a partial recursive function $\mathrm{g}(a, k, x)$ such that, for each $a \in \alpha \in \mathrm{M}_2$ and each $x \in M_1$,

$$\mathrm{g}(a, k, x)\downarrow \;\equiv \boldsymbol{F}(\bar{x}) \in \mathrm{S}(\alpha, k)\,, \tag{9.19}$$

and

$$\mathrm{g}(a, k, x)\downarrow \to \mathrm{S}\big(\boldsymbol{F}(\bar{x}), \mathrm{g}(a, k, x)\big) \subset \mathrm{S}(\alpha, k)\,. \tag{9.20}$$

To insure that (9.20) holds if $\mathrm{g}(a, k, x)\downarrow$, it is enough to guarantee that $2^{-\mathrm{g}(a,k,x)} < 2^{-k} - \boldsymbol{D}_2\big(\alpha, \boldsymbol{F}(\bar{x})\big)$. Thus we set

$$\begin{aligned} \mathrm{g}(a, k, x) \simeq \Big(\mu u\Big[T_1\big(f, x, (u)_0\big) \;\&\; T_2\big(d_2, a, U\big((u)_0\big), (u)_1\big)\;\& \\ T_2\Big(less, \varrho\big(\mathrm{r}^{-1}(2^{-(u)_2})\big), \mathrm{f}_+\Big(\varrho\big(\mathrm{r}^{-1}(2^{-k})\big), \mathrm{f}_-\Big(U\big((u)_1\big)\Big)\Big)\Big)\Big]\Big)_2\,. \end{aligned} \tag{9.21}$$ (12)

Theorem 3 implies now that

$$\boldsymbol{F}\Big(\mathrm{S}\big(\bar{x}, \mathrm{rc}(x, \mathrm{g}(a, k, x))\big)\Big) \subset \mathrm{S}\big(\boldsymbol{F}(\bar{x}), \mathrm{g}(a, k, x)\big) \subset \mathrm{S}(\alpha, k)\,.$$

Thus $\Lambda x\, \mathrm{rc}\big(x, \mathrm{g}(a, k, x)\big)$ is an index of $\boldsymbol{F}^{-1}\big(\mathrm{S}(\alpha, k)\big)$ as a recursively open subset of M_1, and hence by Theorem 11

$$\mathrm{lc}(a, k) = \mathrm{rol}\big(\Lambda x\, \mathrm{rc}(x, \mathrm{g}(a, k, x))\big) \tag{9.22}$$

is a Lacombe index of $\boldsymbol{F}^{-1}\big(\mathrm{S}(\alpha, k)\big)$.

Remark. Theorem 12 is the version of the recursive continuity theorem given in Čeitin's [1]. As Čeitin mentions there, it implies directly that a recursive operator on a recursively separable subspace F_1 of F into $N = (N, \lambda xy\ x = y)$ is the restriction of some *partial recursive functional* (in the sense of Kleene [4], § 63) whose domain contains F_1 to F_1. This problem was proposed by Myhill and Shepherdson in [12] and was solved independently of Čeitin by Kreisel, Lacombe and Schoenfield in [5].

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Reçu par la Rédaction le 16. 4. 1963

A remark on Sikorski's extension theorem for homomorphisms in the theory of Boolean algebras

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1. Introduction. In [1], Sikorski proved the following important extension theorem for Boolean homomorphisms.

THEOREM (R. Sikorski). *Let $\mathfrak{B}_0$ be a subalgebra of a Boolean algebra $\mathfrak{B}$, and let $\mathfrak{B}'$ be a complete Boolean algebra. Then every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}'$.*

Sikorski's proof of this theorem consists of two parts: (i) First the following fundamental lemma is proved.

LEMMA. *Let $\mathfrak{B}_0$ be a subalgebra of a Boolean algebra $\mathfrak{B}$, and let $\mathfrak{B}'$ be a complete Boolean algebra. If $a_1, \ldots, a_n$ are a finite number of elements of $\mathfrak{B}$ and if $\mathfrak{B}_n$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{B}_0$ and the elements $a_1, \ldots, a_n$, then every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to a homomorphism of $\mathfrak{B}_n$.*

(ii) Using Zorn's lemma or transfinite induction in conjunction with the preceding Lemma, it is shown, in a standard fashion, that every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to all of $\mathfrak{B}$.

By specialization we obtain that Sikorski's theorem implies the prime ideal theorem for Boolean algebras (see p. 114 in [2]), i.e., every proper ideal in a Boolean algebra can be extended to a prime (= maximal) ideal. It was shown, however, by J. D. Halpern (see [3]) that the axiom of choice is independent from the Boolean prime ideal theorem in a set theory which will be made more explicit in due course. It seems therefore natural to ask whether may be Sikorski's extension theorem follows already from the Boolean prime ideal theorem rather than from the axiom of choice?

In the present paper we shall report on some results which were obtained in trying to settle this question. The present investigations seem to indicate that Sikorski's theorem is independent from the Boolean

* Work on this paper was supported in part by National Science Foundation Grant G-19914.