Disjoint mappings and the span of spaces

by

A. Lelek (Wrocław)

Mappings $f, g\colon X \to Y$ will be called *disjoint* provided that $f(x) \neq g(x)$ for every $x \in X$. This is to say that f and g considered as subsets of the Cartesian product $X \times Y$ are disjoint. The first question arises: how many disjoint mappings may exist of a given space X onto a given space Y?

Observe that if $f\colon X \to Y$ is a mapping and $g, g'\colon Y \to Z$ are disjoint mappings, then the superpositions gf and $g'f$ are also disjoint mappings. Hence, for instance, for those Y which are locally connected continua the above question reduces to the existence of a large family of disjoint mappings $I \to Y$ of the unit segment I onto Y. This can be interpreted as such a motion of some number of points on the domain Y that in the same period of time each of these points runs over the whole of Y and no moving point at any time collides with another.

If we consider mappings $X \to Y$ as points of the functional space Y^X, the presence of a metric structure in Y^X suggests the second question: how large may be the distance between two disjoint mappings of X onto Y? This leads to the notion of the span of a space (see § 5), which also has a natural interpretation. Roughly speaking, the span of a country Y is a maximal number d such that two men can pass over the same part of Y, keeping the distance at least d from one to another.

While in constructions of disjoint mappings (see §§ 1-4) onto a locally connected continuum C the mappings $S \to C$ of the circumference S will be used, the tool for investigating the span (see §§ 5-8) of a compactum C will be the mappings $C \to S$. We shall prove (see § 4) that every polyhedron in which every arc has an empty interior can be transformed onto itself under uncountably many disjoint mappings. However, we start (see § 1) with infinitely many disjoint mappings, which is a simplier case.

§ 1. Infinite families of disjoint mappings. We shall use the term *Peano parametrization* of C to denote an arbitrary mapping of the segment $I = \{t\colon 0 \leqslant t \leqslant 1\}$ onto the space C. By a *triod* we understand the union $A_0 \cup A_1 \cup A_2$ of three arcs that have a common end point v, called the *vertex* of the triod, and are mutually disjoint outside v.

LEMMA. *If $T = A_0 \cup A_1 \cup A_2$ is a triod lying in a locally connected continuum C and L_0, L_1, L_2 are arcs such that*

$$L_i \subset A_i - \{v\} \quad \text{for} \quad i = 0, 1, 2,$$

where v is the vertex of T, then there exist locally connected continua C_0, C_1, C_2 satisfying

$$C = C_0 \cup C_1 \cup C_2 \quad \text{and}$$
$$v \in C_i \subset C - (L_{i+1} \cup L_{i+2}) \quad \text{for} \quad i = 0, 1, 2,$$

where the indices are taken modulo 3.

Proof. Denote by R_i the component of $C - (L_{i+1} \cup L_{i+2})$ that contains v. For every $c \in C$, there is an arc $vc \subset C$. Take the last point p on vc which belongs to T. Then $p \in A_i$ for some $i = 0, 1, 2$ and the continuum $A_i \cup pc$ does not meet $L_{i+1} \cup L_{i+2}$. It follows that $C = R_0 \cup R_1 \cup R_2$.

Therefore there exist locally connected continua K_0, K_1, K_2 such that $C = K_0 \cup K_1 \cup K_2$ and $K_i \subset R_i$ for $i = 0, 1, 2$ (see [1], p. 193). In order to obtain C_i it is enough to add to K_i an arc in R_i, joining v and K_i.

THEOREM. *A locally connected continuum is an arc if and only if it does not admit infinitely many mutually disjoint Peano parametrizations.*

Proof. Evidently, no arc has two disjoint Peano parametrizations. Suppose a locally connected continuum C is not an arc. If C is a simple closed curve, it clearly has infinitely many disjoint Peano parametrizations. We can thus assume that C is not a simple closed curve, which gives the existence of a triod $T = A_0 \cup A_1 \cup A_2$ in C. Let h_i be a homeomorphism of I onto A_i which maps 0 into v, the vertex of T, for $i = 0, 1, 2$. Consider a sequence $g_1, g_2, \ldots$ of mappings $I \to T$, defined by the formula

$$g_n(t) = \begin{cases} h_0(1/3^{n+1} - t), & 0 \leqslant t \leqslant 1/3^{n+1}, \\ h_1(t - 1/3^{n+1}), & 1/3^{n+1} \leqslant t \leqslant 1/3, \\ h_1(2/3 - 1/3^{n+1} - t), & 1/3 \leqslant t \leqslant 2/3 - 1/3^{n+1}, \\ h_2(t - 2/3 + 1/3^{n+1}), & 2/3 - 1/3^{n+1} \leqslant t \leqslant 2/3, \\ h_2(2/3 + 1/3^{n+1} - t), & 2/3 \leqslant t \leqslant 2/3 + 1/3^{n+1}, \\ h_0(t - 2/3 - 1/3^{n+1}), & 2/3 + 1/3^{n+1} \leqslant t \leqslant 1, \end{cases}$$

for $n = 1, 2, \ldots$ It is not difficult to see that these mappings are mutually disjoint and we only need to change them so that they will be Peano parametrizations of C. This can be done in the following way.

Take the segments:

$$I_0^n = \{t\colon 2/3 - 1/3^{n+1} - 1/3^{n+2} \leqslant t \leqslant 2/3 - 1/3^{n+1} + 1/3^{n+2}\},$$
$$I_1^n = \{t\colon 2/3 + 1/3^{n+1} - 1/3^{n+2} \leqslant t \leqslant 2/3 + 1/3^{n+1} + 1/3^{n+2}\},$$
$$I_2^n = \{t\colon 1/3^{n+1} - 1/3^{n+2} \leqslant t \leqslant 1/3^{n+1} + 1/3^{n+2}\},$$

which are all mutually disjoint for $n = 1, 2, \ldots$ Every function g_n will be changed merely on $I_0^n \cup I_1^n \cup I_2^n$. Let us observe that on each of these three segments all the other functions g_m $(m \neq n)$ take values lying in only two of the arcs A_0, A_1, A_2. Namely, denoting

$$L_n = \{t\colon 1/3^{n+2} \leqslant t \leqslant 1\},$$

we have the inclusion

$$g_m(I_i^n) \subset h_{i+1}(L_n) \cup h_{i+2}(L_n),$$

where $i = 0, 1, 2$; $m, n = 1, 2, \ldots$; $m \neq n$, and the indices $i+1$, $i+2$ are understood to be reduced modulo 3.

By the above lemma, applied for $L_i = h_i(L_n)$, we can choose locally connected continua C_i^n such that the point v belongs to each of them, every triple C_0^n, C_1^n, C_2^n fills up C and

$$C_i^n \cap \bigcup_{m \neq n} g_m(I_i^n) = 0$$

for $i = 0, 1, 2$; $n = 1, 2, \ldots$ The vertex v being an interior point of each arc $g_n(I_i^n)$, let us modify the function g_n to a mapping f_n which agrees with g_n outside the segments I_0^n, I_1^n, I_2^n and satisfies

$$f_n(I_i^n) = C_i^n \cup g_n(I_i^n)$$

for $i = 0, 1, 2$. Then $f_1, f_2, \ldots$ are mutually disjoint Peano parametrizations of C.

§ 2. Measures of some distance sets. Let R be the real line and S the unit circumference $|z| = 1$ on the complex plane. The symbol μX will denote the linear Lebesgue measure of the set X, for $X \subset R$ or $X \subset S$. Then $\mu S = 2\pi$.

For arbitrary sets $X, Y \subset S$ we shall denote by $D(X, Y)$ the set of all distances from X to Y, i.e.

$$D(X, Y) = \{|x - y|\colon x \in X,\ y \in Y\},$$

and write shortly $D(X) = D(X, X)$.

Let f be a mapping of X. We shall denote by W_f the whole subset of X on which f is not 1-1, i.e.

$$W_f = \{x\colon \{x\} \neq f^{-1}f(x)\},$$

and by $D(f)$ the set of all distances of points in X which have the same images under f, i.e.

$$D(f) = \{|x-y|:\ f(x) = f(y)\}\,.$$

Further, a mapping f of S will be called *linearly finite* provided that there exists an integer $k(f)$ satisfying the following condition: for every $\varepsilon > 0$ the circumference S can be covered by finite sequence $A_1, \ldots, A_m$ of arcs in S such that all of them have diameters $\delta A_i < \varepsilon$ and each image $f(A_i)$ meets at most $k(f)$ of the images $f(A_1), \ldots, f(A_m)$ for $i = 1, \ldots, m$.

LEMMA. *If f: $S \to C$ is a linearly finite mapping and $\mu W_f = 0$, then $\mu D(f) = 0$.*

Proof. Denote by V_n the union of all sets $f^{-1}(c)$, where $c \in C$, such that $1/n \leqslant \delta f^{-1}(c)$ $(n = 1, 2, \ldots)$. Thus each V_n is a compact subset of W_f and, in view of the equality $\mu W_f = 0$, we can find, for every given index n and an arbitrary number $\nu > 0$, a set $Z \subset S$ such that

$$\mu Z < \nu\,,$$

the interior of Z contains V_n, and Z is the union of finitely many arcs. Let ε be the minimal distance between two different components of Z. Hence $\varepsilon > 0$ and we can cover S by arcs $A_1, \ldots, A_m$ with diameters less than ε so that each $f(A_i)$ intersects at most $k(f)$ of sets from the sequence $f(A_1), \ldots, f(A_m)$.

Let us take only those arcs A_i that intersect V_n. Consequently, for these A_i, all the sets $A_i \cap Z$ are arcs and their union contains V_n. After shortening some of them and, eventually, omitting others, we obtain new arcs $L_1, \ldots, L_h$ such that

$$V_n \subset L_1 \cup \ldots \cup L_h\,,$$

the interiors of L_j are mutually disjoint and

$$L_j \subset A_{i_j} \cap Z\,,$$

where $i_j \neq i_{j'}$ for $j, j' = 1, \ldots, h$; $j \neq j'$. It follows that each set $f(L_j)$ meets at most $k(f)$ of the sets $f(L_1), \ldots, f(L_h)$. Let U be the union of all the distance sets $D(L_i, L_j)$ such that $f(L_i) \cap f(L_j) \neq 0$; $i, j = 1, \ldots, h$.

Then, obviously,

$$D(f|\,V_n) \subset U\,,$$

where $f|\,V_n$ is the function f restricted to V_n, and

$$\mu U \leqslant \sum_{\{i,j\}} \mu D(L_i, L_j)\,,$$

where the non-ordered pair $\{i, j\}$ of indices runs over all such distinct couples of values $1, \ldots, h$ that $f(L_i) \cap f(L_j) \neq 0$. Thus, in the last sum each of the numbers $1, \ldots, h$ appears at most $k(f)$ times as i or j. But since $L_1, \ldots, L_h$ are arcs, we have

$$\mu D(L_i, L_j) \leqslant \mu L_i + \mu L_j$$

for $i, j = 1, \ldots, h$, and therefore

$$\mu U \leqslant k(f)\cdot \sum_{j=1}^{h} \mu L_j = k(f)\cdot \mu\Big(\bigcup_{j=1}^{h} L_j\Big) \leqslant k(f)\cdot \mu Z < k(f)\cdot \nu\,,$$

the equality being a consequence of the fact that the arcs L_j have mutually disjoint interiors.

We get $\mu D(f|\,V_n) = 0$ for $n = 1, 2, \ldots$ Now, the set $D(f)$ can clearly be represented as the union

$$D(f) = \{0\} \cup \bigcup_{n=1}^{\infty} D(f|\,V_n)\,,$$

which gives $\mu D(f) = 0$.

§ 3. Uncountable families of disjoint mappings. Recall that a mapping g: $S \to C$ is said to be *irreducible* if $g(X) = g(S)$ implies $X = S$ for every closed set $X \subset S$, or what is the same, if the set $S - W_g$ is dense in S.

THEOREM. *If there exists a linearly finite irreducible mapping of the circumference S onto a compactum C, then there exists an uncountable family $\boldsymbol{F}$ of mutually disjoint mappings of S onto C. Moreover, for every $s \in S$, the set $\{f(s)$: $f \in \boldsymbol{F}\}$ can be represented as a countable union of non-empty perfect sets such that every open subset of C contains at least one of them.*

Proof. Let g: $S \to C$ be a linearly finite irreducible mapping of S onto C. Since W_g is obviously a F_σ-set and $S - W_g$ is a dense subset of S, there exists a decomposition

$$W_g = X_1 \cup X_2 \cup \ldots$$

of W_g into mutually disjoint 0-dimensional compacta X_n.

We shall construct inductively a sequence $h_1, h_2, \ldots$ of homeomorphisms of S onto itself and a non-decreasing sequence $M_1 \subset M_2 \subset \ldots$ of finite subsets of $S - W_g$. Namely, take h_1 such that $\mu h_1(X_1) = 0$ and put $M_1 = 0$. Suppose h_n and M_n are defined $(n = 1, 2, \ldots)$. Choose M_{n+1} in $S - W_g$ so that $M_n \subset M_{n+1}$ and all the components of the set $S - M_{n+1}$ as well as of the set $S - h_n(M_{n+1})$ have diameters less than $3/2^n$. Then the 0-dimensional compacta

$$Y = h_n(X_1 \cup \ldots \cup X_n \cup M_{n+1}) \quad \text{and} \quad Z = h_n(X_{n+1})$$

do not intersect, which gives the existence of a homeomorphism h': $S \to S$ such that

$$\mu h'(Z) = 0\,,$$

h' maps each component of $S-Y$ onto itself, and is the identity mapping on Y. Putting $h_{n+1} = h'h_n$ we get

$$\mu h_{n+1}(X_{n+1}) = 0 \quad \text{and} \quad |h_n(x) - h_{n+1}(x)| < 3/2^n$$

for every $x \in S$. Thus, the sequence $h_1, h_2, \ldots$ so obtained is uniformly convergent.

The limit mapping $h = \lim h_n$ is a homeomorphism. In fact, if $x, y \in S$ and $x \neq y$, a positive integer n exists such that x and y belong to disjoint arcs $x'x''$ and $y'y''$, respectively, in S with end point staken from M_{n+1}, according to the definition of the sequence $M_1, M_2, \ldots$ But since $h_i(z) = h_n(z)$ for $i > n$ and $z \in M_{n+1}$ by virtue of the definition of the sequence $h_1, h_2, \ldots$, we obtain

$$h_i(x'x'') = h_n(x'x'') \quad \text{and} \quad h_i(y'y'') = h_n(y'y'')$$

for $i > n$, whence the points $h(x)$ and $h(y)$ belong to disjoint arcs $h_n(x'x'')$ and $h_n(y'y'')$, respectively.

Similarly, we have $h_i|X_n = h_n|X_n$ for $i > n = 1, 2, \ldots$ Therefore $h(X_n) = h_n(X_n)$ and so

$$\mu h(X_n) = 0$$

for $n = 1, 2, \ldots$ Hence $\mu h(W_g) = 0$.

Now, h being a homeomorphism, the mapping $g' = gh^{-1}$ is linearly finite and $W_{g'} = h(W_g)$. Applying the lemma from § 2, we infer that $\mu D(g') = 0$. Manifestly, the set $D(g')$ is closed in R, which implies the existence of non-empty perfect sets $P_i \subset S$ such that each open set in S contains at least one of them and

$$D(P) \subset [R - D(g')] \cup \{0\}$$

for $P = P_1 \cup P_2 \cup \ldots$ (see [2], p. 146).

Let r_p, for $p \in S$, be the rotation $z \to p \cdot z$ of S onto itself. The family

$$F = \{g'r_p \colon p \in P\}$$

has all the required properties. Really, if $s \in S$ is an arbitrary point and $p, p' \in P$, $p \neq p'$, then

$$|r_p(s) - r_{p'}(s)| = |p \cdot s - p' \cdot s| = |p - p'| \in R - D(g'),$$

and the inequality $g'r_p(s) \neq g'r_{p'}(s)$ follows. Moreover, the set $\{f(s) \colon f \in F\}$ is the union of the perfect sets $\{g'(p \cdot s) \colon p \in P_i\}$, homeomorphic to P_i, respectively ($i = 1, 2, \ldots$). If G is an open set in C, then $r_s^{-1}g'^{-1}(G)$ contains some P_i and, consequently, $p \in P_i$ implies $g'(p \cdot s) = g'r_s(p) \in G$.

§ 4. Mappings onto polyhedra.

Concerning disjoint mappings, there is a difference between the behaviour of 1-dimensional spaces and that of n-dimensional ones, where $n > 1$. This appears for the cells: a pair of disjoint mappings of n-cell onto itself can easily be found for $n > 1$, but it cannot for $n = 1$. However, as we shall confirm in the sequel, the contrast is more expressive here [1].

Let us first denote by $U_0^n(m)$, for every $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$, the finite collection of congruent n-dimensional cubes that have the length of edge equal to $1/3^m$ and arise by cutting the cube

$$I^n = \{(x_1, \ldots, x_n) \colon 0 \leqslant x_i \leqslant 1,\ i = 1, \ldots, n\}$$

with the $(n-1)$-dimensional hyperplanes of the form $x_i = j/3^m$, where $i = 1, \ldots, n$ and $j = 0, \ldots, 3^m$. Next, establish for a moment an index $i = 1, \ldots, n$. By dividing each cube from $U_0^n(m)$ into two congruent parts, along the hyperplanes of the form $x_i = (2j+1)/2 \cdot 3^m$ ($j = 0, \ldots, 3^m - 1$), we get a finite collection of interiorly disjoint parallelepipeds, which will be denoted by $U_i^n(m)$.

Observe that each element of $U_i^n(m)$ meets at most $2 \cdot 3^n$ elements of $U_j^n(m)$ ($i, j = 0, \ldots, n$; $m = 0, 1, \ldots$; $n = 1, 2, \ldots$).

THEOREM. *Every polyhedron whose dimension at each point is greater than* 1 *admits uncountably many mutually disjoint mappings onto itself.*

Proof. The theorem clearly reduces to connected polyhedra and therefore it is sufficient to prove that the hypothesis of the theorem from § 3 holds, i.e. we must only show that there always exists a linearly finite irreducible mapping of the circumference S onto a connected polyhedron P whose dimension at each point is at least 2.

Let T' be a triangulation of P. Denote by $\Delta_0', \ldots, \Delta_l'$ all the simplexes from T' which have non-empty interiors in P. We shall prove, by induction on l, that there exists a simplicial subdivision T of T' such that all proper faces of Δ_i' belong to T ($i = 0, \ldots, l$) and T possesses a cyclic order property which can be described as follows. There are: a decomposition

$$S = A_0 \cup \ldots \cup A_k$$

of S into interiorly disjoint arcs, a 1-1 correspondence $A_i \leftrightarrow \Delta_i$ between these arcs and all the simplexes $\Delta_0, \ldots, \Delta_k$ of T having non-empty interiors in P, and a function f which maps the set of end points of arcs $A_0, \ldots, A_k$ onto the set of vertices of T' so that if p, p' are end points of A_i, then $f(p), f(p')$ are distinct vertices of Δ_i ($i = 0, \ldots, k$).

In fact, for $l = 0$ we have $P = \Delta_0' = q_0 \ldots q_k$, where $k > 1$ according to the hypothesis. Let q be the centre of gravity of Δ_0'. We define T as the collection of simplexes

$$\Delta_i = qq_0 \ldots \hat{q}_i \ldots q_k$$

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[1] I am indebted to H. Steinhaus who suggested to me that the 2-cell admits uncountably many disjoint Peano parametrizations.

and all their faces ($i = 0, \ldots, k$). Here the sign $\hat{}$ conventionally means that the vertex under it ought to be omitted. Take arcs $A_0, \ldots, A_k$ on S such that $A_i \cap A_{i+1}$ is a single point for $i = 0, \ldots, k$ and for $i+1$ been reduced modulo $k+1$. For every end point p of the form $A_i \cap A_{i+1}$ put $f(p) = q_{i+2}$ (mod $k+1$). This is a good definition, since we have

$$q_{i+2} \in \Delta_i \cap \Delta_{i+1} \pmod{k+1}$$

for $i = 0, \ldots, k$, by virtue of the inequality $k > 1$.

Further, the existence of $\boldsymbol{T}$ for l yields also the existence of $\boldsymbol{T}$ for $l+1$. Indeed, P being connected, there always exists among the simplexes $\Delta_0', \ldots, \Delta_{l+1}'$ one, say Δ_0', such that the union of the rest of these simplexes, i.e.

$$P_1 = \Delta_1' \cup \ldots \cup \Delta_{l+1}',$$

is a connected polyhedron. Take a common vertex v of the polyhedron $P_0 = \Delta_0'$ and P_1, and consider the union $S_0 \cup S_1$ of two circumferences which have only a point s in common. Applying the induction hypothesis, we can find suitable simplicial subdivisions $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ of the triangulation $\boldsymbol{T}'$ restricted to P_0 and P_1, respectively. Since all proper faces of Δ_0' belong to $\boldsymbol{T}_0$ and those of Δ_i' ($i = 1, \ldots, l+1$) belong to $\boldsymbol{T}_1$, the union $\boldsymbol{T} = \boldsymbol{T}_0 \cup \boldsymbol{T}_1$ is a subdivision of $\boldsymbol{T}'$. Moreover, we can find suitable decompositions of S_0 and S_1 into arcs corresponding to simplexes of $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$, as well as mappings f_0 and f_1 of the end points of these arcs into the vertices of $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$, respectively, so that s is an end point of some arcs in S_0 and in S_1, and $f_0(s) = f_1(s) = v$. Let $S = L_0 \cup L_1$ be a decomposition of S into two interiorly disjoint arcs. It is now sufficient to map L_0 and L_1 onto S_0 and S_1, respectively, by sending $L_0 \cap L_1$ into s, and to choose decompositions of L_0 and L_1 into arcs which are mapped onto the decompositions of S_0 and S_1, respectively. The function f appears: if an end point p of an arc obtained from such a decomposition of S is mapped into p^* belonging to S_0 or S_1, take $f(p)$ to be $f_0(p^*)$ or $f_1(p^*)$, respectively.

Denote by n the number of vertices of the triangulation $\boldsymbol{T}$. Then $\boldsymbol{T}$ can be regarded as a complex consisting of some faces of the unit simplex

$$\Delta^{n-1} = \{(x_1, \ldots, x_n) \colon x_1 + \ldots + x_n = 1,\ x_i \geqslant 0,\ i = 1, \ldots, n\}$$

in the n-dimensional Euclidean space E^n. Thus $P \subset \Delta^{n-1}$. Let J^{n-1} be the union of all the proper faces of the cube I^n that contain the point $(1, 1, \ldots, 1)$. The central projection h of Δ^{n-1} onto J^{n-1} in E^n with the centre $(0, 0, \ldots, 0)$ is a homeomorphism, each face of Δ^{n-1} is mapped under h onto the union of some proper faces of I^n and the vertices of Δ^{n-1} are fixed points of h.

Consider an arbitrary simplex Δ_i from the sequence $\Delta_0, \ldots, \Delta_k$ of all the simplexes of $\boldsymbol{T}$ that have non-empty interiors in P. Let d be the dimension of Δ_i. Then

$$h(\Delta_i) = F_0 \cup \ldots \cup F_d,$$

where all F_j are d-dimensional faces of I^n ($j = 0, \ldots, d$). Let p and p' be the end points of the arc A_i. It follows that $f(p)$ and $f(p')$ are distinct vertices of Δ_i and belong to some faces F_j. We can assume that $f(p) \in F_0$.

Let

$$A_i = B_0 \cup \ldots \cup B_{2d}$$

be a decomposition of A_i into arcs such that $p \in B_0$ and $B_{j-1} \cap B_j$ is a single point for $j = 1, \ldots, 2d$. We shall show that each of the faces $F_1, \ldots, F_d$ can be divided into two congruent parts and the d-dimensional parallelepipeds so obtained can be ordered in a sequence $\Pi_1, \ldots, \Pi_{2d}$ in such a way that (after putting $\Pi_0 = F_0$) the function f is extendable on end points of all the arcs B_j in the following sense: if b, b' are end points of B_j, then $f(b), f(b')$ are diametrically opposite vertices of Π_j ($j = 0, \ldots, 2d$).

Really, an isometry corresponding to a permutation of coordinates tn E^n clearly reduces our situation to one where the vertices of Δ_i are he following points:

$$\begin{aligned} f(p) &= (\overbrace{1, 0, \ldots, 0}^{d+1}, 0, \ldots, 0),\\ &\quad (0, 1, \ldots, 0, 0, \ldots, 0),\\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\\ f(p') &= (0, 0, \ldots, 1, 0, \ldots, 0). \end{aligned}$$

Thus the face F_0 is the common part of I and the hyperplane with equations: $x_1 = 1$, $x_i = 0$ ($i = d+2, \ldots, n$). Of course, we have also $x_i = 0$ ($i = d+2, \ldots, n$) for every point of the faces $F_1, \ldots, F_d$. Consequently, we can change their numeration so that F_j be the face determined by equations: $x_{j+1} = 1$, $x_i = 0$ ($i = d+2, \ldots, n$). Now, we cut the cube F_j ($j = 1, \ldots, d$) with the hyperplane $x_j = \frac{1}{2}$ into two parallelepipeds Π_{2j-1} and Π_{2j}, the indices being taken so that Π_{2j} has a point in common with the hyperplane $x_j = 0$. We extend the function f on end points of the arcs B_j as follows: if $b \prec b' \prec b''$ are end points of B_{2j-1} and B_{2j} in the ordering $\prec$ from p to p' on A_i, then:

$$\begin{aligned} f(b) &= (\overbrace{0, 0, \ldots, 0, 1, 1, 1, \ldots, 1}^{d+1}, 0, \ldots, 0),\\ f(b') &= (1, 1, \ldots, 1, \tfrac{1}{2}, 1, 0, \ldots, 0, 0, \ldots, 0),\\ f(b'') &= (\underbrace{0, 0, \ldots, 0, 0}_{j}, 1, 1, \ldots, 1, 0, \ldots, 0). \end{aligned}$$

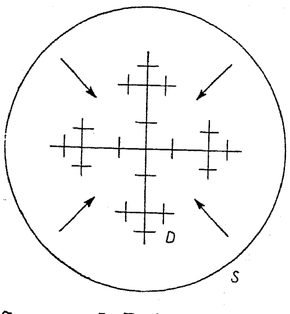
In this way we get the decomposition

$$h(\Delta_i) = \Pi_0 \cup \Pi_1 \cup \ldots \cup \Pi_{2d}$$

into d-dimensional parallelepipeds, where Π_0 is the cube F_0 and $\Pi_1, \ldots, \Pi_{2d}$ are all congruent to the half of F_0. Let us do the same for every simplex Δ_i and, respectively, for every arc A_i $(i = 0, \ldots, k)$. The decomposition of S into some arcs B so obtained corresponds to a decomposition of $h(P)$ into parallelepipeds Π and the function f maps the end points of each B onto a pair of diametrically opposite vertices of Π. We once more extend f, this time on the whole circumference S, by taking a linear mapping of each B onto the diagonal of Π.

Finally, let us change f on each B to the standard Peano mapping of B onto Π, induced by ternary decompositions of B into congruent arcs and those of Π into congruent parallelepipeds. We thus obtain a mapping g of S onto $h(P)$ which is irreducible because the standard Peano parametrizations of parallelepipeds are irreducible and the common part of every two distinct parallelepipeds Π and Π' from our decomposition of $h(P)$ is a nowhere dense subset of both Π and Π'. The last statement follows from the fact that all the simplexes $\Delta_0, \ldots, \Delta_k$ have non-empty interiors in P.

The mapping g is also linearly finite. Indeed, for every m-th successive ternary subdivision of arcs B into arcs, say Ω, given in the m-th step of the standard construction of Peano mapping, the images $g(\Omega)$ are the intersections of some faces of I^n with some parallelepipeds from the collections $U_i^n(m)$, $i = 0, \ldots, n$ (see the beginning of this paragraph). Hence each from these sets $g(\Omega)$ meets at most $2^{2^n} \cdot (n+1) \cdot 2 \cdot 3^n$ others.



Remarks. There are, however, 1-dimensional compacta which can be mapped onto themselves under uncountably many mutually disjoint mappings. For instance, consider a dendrite D in the plane, in which the set of end points is dense and which has the ramification at each point equal to 1, 2 or 4. Let the circumference S surround D (see the figure). If we shall strain S on D, the resulting mapping will be irreducible and linearly finite. Hence, by § 3, D admits uncountably many mutually disjoint Peano parametrizations.

Although, as we have seen in § 1, every dendrite which is not an arc, e.g. the triod, carries an infinite family of mutually disjoint Peano parametrizations, an uncountable family need not exist for some of such

dendrites. Namely, one can show more generally that *if pq is an arc in a space C such that*

$$pq \cap \overline{C - pq} = \{p\},$$

then each family of mutually disjoint Peano parametrizations of C is at most countable. In fact, for every $f\colon I \to C$ from this family, let I_f be a component of $f^{-1}(pq)$ such that it overlaps the set $f^{-1}(q)$. Then all I_f are mutually disjoint segments, and so the family cannot be uncountable.

§ 5. The span.

Let X be a metric space with a distance ϱ. We shall denote by p_1 and p_2 the projections of the Cartesian product $X \times X$ onto its axes, i.e. $p_i(x_1, x_2) = x_i$ for $(x_1, x_2) \in X \times X$; $i = 1, 2$.

We define the *span* σX of the space X to be the least upper bound of numbers ε for which there is a connected subset Z_ε of $X \times X$ such that

$$(1)\ p_1(Z_\varepsilon) = p_2(Z_\varepsilon) \qquad \text{and} \qquad (2)\ \varrho(x_1, x_2) \geqslant \varepsilon \text{ for } (x_1, x_2) \in Z_\varepsilon .$$

Of course, the sets Z_ε in the above definition can be assumed to be closed in $X \times X$. The span σ is a monotone function, i.e. $\sigma X \leqslant \sigma Y$ for $X \subset Y$. Further, if A is a connected space and $f_1, f_2\colon A \to X$ are mappings such that $f_1(A) = f_2(A)$, then for the number

$$\varepsilon = \inf_{a \in A} \varrho\big(f_1(a), f_2(a)\big),$$

the inequality $\sigma X \geqslant \varepsilon$ holds. Indeed, it is enough to take as Z_ε the set of all pairs $\big(f_1(a), f_2(a)\big)$, where $a \in A$. Thus we can write

$$(*) \qquad \sigma X = \sup_{A, f_i} \inf_{a \in A} \varrho\big(f_1(a), f_2(a)\big),$$

where A ranges over all connected spaces, and f_i $(i = 1, 2)$ over all mappings of A into X with $f_1(A) = f_2(A)$.

In the case when the span σX is infinite, let us understand by $\sigma X - \eta$ an arbitrarily large number.

THEOREM. *If $\eta > 0$ and $f\colon X \to Y$ is a mapping, then there exist points $x, x' \in X$ such that*

$$\varrho(x, x') > \sigma X - \eta \quad \textit{and} \quad \varrho\big(f(x), f(x')\big) < \sigma Y + \eta .$$

Proof. Let ε be a number such that $\sigma X - \eta < \varepsilon \leqslant \sigma X$, and such that there is a connected set Z_ε in $X \times X$ satisfying conditions (1) and (2). Putting $f_i = fp_i$ for $i = 1, 2$, we have $f_1(Z_\varepsilon) = f_2(Z_\varepsilon)$ and, by $(*)$, there is a point (x, x') of Z_ε such that the distance between its images under f_1 and f_2 is less than $\sigma Y + \eta$.

COROLLARY. *If $f\colon X\to Y$ is a mapping of a compactum X, then there exist points $x, x'\in X$ such that*

$$\varrho(x,x')\geqslant\sigma X \quad \text{and} \quad \varrho\big(f(x),f(x')\big)\leqslant\sigma Y.$$

Thus, if $\sigma Y=0$, there is a point $y\in Y$ such that $\sigma X\leqslant\delta f^{-1}(y)$.

Since each bounded subset of the real line is clearly of span zero, the above corollary implies that every compactum, on which there exist real continuous functions with arbitrarily small point inverses, is of span zero. Consequently, the span is equal to zero for all the chainable continua, for instance. This statement is, however, equivalent to the known fact that no chainable continuum can be represented as a continuous image of any continuum under a pair of disjoint mappings.

§ 6. A uniformization of real functions.

If $a<b$ and c are real numbers, we shall denote by $[ab]$ the segment $\{t\colon a\leqslant t\leqslant b\}$ and by $[ab,c]$ and $[c,ab]$ the segments

$$\{(x,y)\colon a\leqslant x\leqslant b,\ y=c\} \quad \text{and} \quad \{(x,y)\colon x=c,\ a\leqslant y\leqslant b\}$$

in the plane, respectively.

LEMMA. *If $f_0, f_1\colon R\to R$ are real continuous functions and $a<b<c<d$ are numbers such that*

$$f_j(a)<f_{j+1}(t) \quad \text{for} \quad b\leqslant t\leqslant d,$$
$$f_j(t)<f_{j+1}(d) \quad \text{for} \quad a\leqslant t\leqslant c,$$

where $j=0,1$ and $j+1$ is taken modulo 2, then there is a continuum K and continuous functions $g_0, g_1\colon K\to R$ such that

$$[bc]\subset g_0(K)\cap g_1(K) \quad \text{and} \quad f_0g_0(p)=f_1g_1(p)$$

for every $p\in K$.

Proof. Let Q be the square on the plane which has the points (a,a) and (d,d) as opposite vertices. Denote by A the set of points $(x,y)\in Q$ such that $f_0(x)=f_1(y)$. If

$$(p,q)\in[a,bd]\cup[ac,d] \quad \text{and} \quad (p',q')\in[bd,a]\cup[d,ac],$$

then $f_0(p)<f_1(q)$ and $f_0(p')>f_1(q')$ according to the hypothesis. It follows that every continuum $C\subset Q$ joining (p,q) and (p',q') meets the set A. Hence A cuts the square Q between the arcs

$$[a,bd]\cup[ac,d] \quad \text{and} \quad [bd,a]\cup[d,ac],$$

lying on the boundary of Q. Therefore A contains a continuum K which joins the complementary arcs

$$[a,ab]\cup[ab,a] \quad \text{and} \quad [cd,d]\cup[d,cd]$$

of this boundary. It is sufficient to define $g_0(p)$ and $g_1(p)$ as the abscissa and the ordinate of p, respectively, for $p\in K$.

§ 7. An estimation of the span.

Suppose that $X_1, X_2, \ldots$ are subsets of a compactum X. It readily follows from the definition of the span that

$$\limsup_{i\to\infty}\sigma X_i\leqslant\sigma\operatorname*{Lim\,sup}_{i\to\infty}X_i,$$

and this property allows us to make use of polyhedral approximation of a set for obtaining some estimation of its span. We shall write

$$\varrho(A,B)=\inf_{a\in A,\,b\in B}\varrho(a,b)$$

for subsets A and B of X.

THEOREM. *If $C\subset I^{\omega}$ is a closed subset of the Hilbert cube I^{ω} and $f\colon C\to S$ is an essential mapping of C into the circumference S, then*

$$\inf_{s\in S}\varrho\big(f^{-1}(s),f^{-1}(-s)\big)\leqslant\sigma C.$$

Proof. Let us denote by ε the number on the left side in the last inequality. The mapping f has an extension $\bar f\colon U\to S$ onto a neighbourhood U of C in I^{ω}. For an arbitrary number $\eta>0$ and neighbourhood V of C in I^{ω}, let $W\subset U\cap V$ be a neighbourhood of C in I^{ω} such that

(i) $$\varepsilon-\eta\leqslant\varrho\big(\bar f^{-1}(s)\cap W,\bar f^{-1}(-s)\cap W\big)$$

for every $s\in S$. Choosing a finite cover $\boldsymbol{C}$ of C with sufficiently small open sets we can find a mapping f' of C into the nerve N of $\boldsymbol{C}$ such that $N\subset W$ and

$$|\bar f f'(c)-f(c)|<2$$

for every $c\in C$ (see [1], p. 18). Thus $\bar f f'$ is homotopic to f and hence it is essential. Consequently, $\bar f|N$ is essential too.

Observe that if $\gamma\colon \Delta^n\to S$ is a mapping of the simplex Δ^n into S, $n>1$, $a<b<c$, and

$$h\colon (\mathrm{bd}\,\Delta^n)\times[ab]\to S$$

is a homotopy joining $h_a=\gamma|\mathrm{bd}\,\Delta^n$, i.e. γ on the boundary of Δ^n (²), to the constant mapping $h_b=1$, then h is extendable to such a homotopy

$$\bar h\colon \Delta^n\times[ac]\to S$$

joining $\bar h_a=\gamma$ to $\bar h_c=1$, that $\bar h|(\mathrm{bd}\,\Delta^n)\times[bc]=1$. In fact, h can first be extended to

$$h'\colon \Delta^n\times[ab]\to S$$

(²) We write $h_t(x)$ instead of $h(x,t)$.

such that $h'_a = \gamma$ (ibidem, p. 262). Let χ be the transformation of Δ^n onto the n-dimensional sphere S^n, defined by identifying the boundary $\mathrm{bd}\,\Delta^n$ with a point $q \in S^n$. Since

$$h'_b|\mathrm{bd}\,\Delta^n = h_b = 1\,,$$

the superposition $h'_b\chi^{-1}$ is a mapping of S^n into $S = S^1$. Therefore, for $n > 1$, a homotopy

$$h'': S^n \times [bc] \to S$$

exists such that $h''_b = h'_b\chi^{-1}$, $h''_c = 1$ and $h''(q, t) = 1$ for $b \leqslant t \leqslant c$ (ibidem, p. 333). It is sufficient to define $\bar{h}(x, t)$ as equal, for every $x \in \Delta^n$, to $h'(x, t)$ if $a \leqslant t \leqslant b$, and to $h''(\chi(x), t)$ if $b \leqslant t \leqslant c$.

Let N^n be the n-dimensional skeleton of the polyhedron N in some triangulation. If the mapping $\bar{f}|N^1$ were homotopic to 1, we could thus successively extend the homotopy on all simplexes of $N^2, N^3, \ldots$, and, finally, on the whole polyhedron N. But, $\bar{f}|N$ being an essential mapping $\bar{f}|N^1$ is the same, i.e. we can write

$$\bar{f}|N^1 \,non\!\sim 1$$

(ibidem, p. 326). It follows that there is a continuum $Y \subset N^1$ satisfying

$$\bar{f}|Y \,irr\,non\!\sim 1$$

(ibidem, p. 325). Hence Y is locally connected and consequently it is a simple closed curve (ibidem, p. 322).

Let us take a homeomorphism u of S onto Y and consider the mapping v: $R \to S$ defined by

(ii) $$v(t) = \bar{f}u(e^{2\pi i\cdot t})$$

for $t \in R$. Then a continuous function φ: $R \to R$ covers v, i.e.

(iii) $$v(t) = e^{2\pi i\cdot\varphi(t)}$$

holds for each $t \in R$. Moreover, we have $\varphi(1) = \varphi(0) + k$, where k is the degree of $\bar{f}u$: $S \to S$, and thus φ satisfies

(iv) $$\varphi(t) = \begin{cases} \varphi(t+[-t]+1) - k\cdot([-t]+1), & t \leqslant 0\,, \\ \varphi(t-[t]) + k\cdot[t], & t \geqslant 0\,, \end{cases}$$

where $[t]$ denotes the integer part of the number t. Formula (iv) is a consequence of the uniqueness of the covering function φ for given v and given initial condition, which determines the value of φ at some $t = t_0$ (ibidem, p. 309).

Since $\bar{f}u$ is an essential mapping of S onto itself, we have $k \neq 0$. We conclude from (iv) that the functions f_0, f_1: $R \to R$, defined by

$$f_j(t) = k\cdot(\varphi(t) + j/2)\,,$$

for $j = 0, 1$; $t \in R$, satisfy the hypotheses of the lemma from § 6 for $b = 0$ and $c = 1$ provided that $a < 0$ is sufficiently small and $d > 1$ is sufficiently large. So we have a continuum K and mappings g_0, g_1: $K \to R$ such that

$$[01] \subset g_0(K) \cap g_1(K) \quad \text{and} \quad \varphi g_0(p) = \varphi g_1(p) + \tfrac{1}{2}$$

for $p \in K$. Setting

$$w_j(p) = u(e^{2\pi i\cdot g_j(p)})\,,$$

for $j = 0, 1$; $p \in K$, we get mappings w_0 and w_1 of K onto Y such that

$$\bar{f}w_0(p) = vg_0(p) = e^{2\pi i\cdot\varphi g_0(p)} = e^{2\pi i\cdot\varphi g_1(p)+\pi i}$$
$$= -e^{2\pi i\cdot\varphi g_1(p)} = -vg_1(p) = -\bar{f}w_1(p)\,,$$

according to (ii) and (iii). Therefore $\varepsilon - \eta \leqslant \varrho(w_0(p), w_1(p))$ for every $p \in K$, by (i) and the inclusions $Y \subset N \subset W$. It follows from formula (*) in § 5 that $\varepsilon - \eta \leqslant \sigma W \leqslant \sigma V$. The number $\eta > 0$ and the neighbourhood V of C being arbitrary, we obtain the inequality $\varepsilon \leqslant \sigma C$.

COROLLARY. *Every compactum of span zero is contractible relative to the circumference.* Consequently (see [1], p. 271), *compacta of span zero are at most* 1-*dimensional.*

It follows directly from the definition of the span (see § 5) that each compactum of span zero is atriodic. Thus, by the above corollary, each locally connected continuum of span zero is an arc. This is, however, a weaker form of a previous theorem (see § 1).

§ 8. Spans of plane and spherical sets. Let G be a circle in the plane with a diameter d and suppose that a compactum C cuts the plane between G and the point at infinity. Then each real continuous function on C takes the same value at a pair of points of C whose distance is at least d. This is a consequence of the following theorem and the corollary from § 5, for instance. Moreover, the real continuous function can be here replaced by an arbitrary mapping of C into a space of span zero.

The sphere S^2 will be considered with a distance ϱ taken to be the angle $\alpha \leqslant \pi$ between the rays from the centre of S^2 to points on S^2.

THEOREM. *Let C be a compactum. If $C \subset E^2$ and some bounded component of $E^2 - C$ contains a circle of a diameter d, then $d \leqslant \sigma C$.*

If $C \subset S^2$ and all components of $S^2 - C$ have diameters not greater than $d < \frac{1}{2}\pi$, then $\pi - 2d \leqslant \sigma C$.

Proof. For $C \subset E^2$, let f be the central projection of C onto the boundary of this circle. Then f is an essential mapping (see [1], p. 345) and the theorem from § 7 implies the inequality $d \leqslant \sigma C$.

For $C \subset S^2$, denote by p_N, p_S the poles and by S^1 the equator of S^2. Let C' be the union of the set $C \cap S^1$ and all the boundaries of components of S^2-C that intersect S^1. Then C' cuts S^2 between p_N and p_S. Indeed, by the hypothesis, the poles do not belong to C' and if a continuum $K \subset S^2$ joins them, it must meet $C \cap S^1$ or S^1-C. In the second case, K intersects the common part of S^1 and some component G of S^2-C. Hence K meets the boundary of G because $p_N \in K-G$. Thus K meets C'.

It follows that the projection f of C' onto S^1 along the meridians is an essential mapping (ibidem). But if $p, q \in C'$ and $f(p)$, $f(q)$ are antipodal points on S^1, then the distance $\varrho(p, q)$ is at least $\pi-2d$, according to the hypothesis and the definition of C'. Since $C' \subset C$, the theorem from § 7 gives $\pi-2d \leqslant \sigma C' \leqslant \sigma C$.

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Recursive metric spaces *

by

Y. N. Moschovakis (Madison, Wisc.)

Introduction. In [10] the author introduced abstract "notation systems" and used them to give an axiomatic treatment of the theory of recursive real numbers. In this paper we use the same methods to constructivize parts of the theory of metric spaces.

A (classical) metric space consists of a set $\mathcal{M}$ together with a function $\delta(\alpha, \beta)$ from $\mathcal{M}$ into the set $\mathcal{R}$ of real numbers which satisfies the three metric axioms. The natural way to define a "recursive metric space", according to the point of view that we adopted in [10], is to substitute an arbitrary notation system M for the set $\mathcal{M}$ and a "recursive operator" $D(\alpha, \beta)$ from M into R (the notation system for the recursive real numbers [10], (1.5), (1.6)) for the distance function $\delta(\alpha, \beta)$ (Definition 1).

It is found that this concept of a recursive metric space is too weak; before we can prove any of the more interesting results of the theory, we have to postulate a deeper connection between the metric and the recursive structure of the space. We shall consider two conditions (A) and (B) (§§ 1 and 4, respectively) on a recursive metric space M, which seem to be sufficient for this purpose.

A space M satisfies (A) if we can effectively compute the limit of a recursive, recursively Cauchy sequence of points of M, whenever it exists.

In order to state our main result we need the concept of an "S-traced" set. A subset E of M is S-*traced* if we can effectively find an element of E in every sphere that intersects E (Definition 2). *If* M *satisfies* (A), *then every point of a listable subset* L *of* M *can be effectively separated from any given* S-*traced subset* E *of the complement of* L *by an open sphere* (the separation theorem, § 2).

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