Mappings on spheres

by

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1. Introduction. The following problem appears in Colloquium Mathematicum 9 (1962), p. 165. Let F be a closed subset of the n-dimensional sphere S^n $(n > 1)$, $\dim F \leqslant n-1$, $S^n - F$ connected and f a continuous mapping of S^n into S^n such that the partial mapping $f/S^n - F$ is a local homeomorphism. Must then $f/S^n - F$ be a homeomorphism and $f(S^n) = S^n$? The question is open even if F is a one point set. In the paper [1] the conjecture is proven true if $F = \{p\}$ and one assumes $f(S^n - \{p\}) \subset S^n - \{p\}$. In this paper some examples are given to show that the conjecture is not valid as stated and some conditions are given under which it is true. These will generalize the solution given in [1].

2. Examples. Consider the mapping $f_1(x, y) = (x^2 - y^2, 2xy)$ of the unit circle C in Euclidean 2-space E^2 onto itself. This is equivalent to the mapping z^2 of the complex unit circle onto itself and as such is a local homeomorphism. We may assume that C is in the xy plane in E^3. Consider the union A^2 of all line segments joining $p = (0, 0, 1)$ to each point of C and those joining $p' = (0, 0, -1)$ to each point of C. This set A^2 is a topological 2-sphere and the mapping f_1 can be extended to a mapping f_2 of A^2 onto A^2 as follows. For a point $q = (x, y, z)$ on A^2, with $z \geqslant 0$ and $q \neq p$, the line pq determines a unique point q' on C. Take $f_2(q)$ to be that point on the line segment $pf_1(q')$ which has 3rd coordinate z. For a point $q = (x, y, z)$ with $z \leqslant 0$ and $q \neq p'$ the line $p'q$ determines a unique point q' on C. Take $f_2(q)$ to be that point on the line segment $p'f_1(q)$ with 3rd coordinate z. Let $f_2(p) = p$ and $f_2(p') = p'$. One can show that f_2 is a continuous mapping of A^2 onto A^2 and that f is a local homeomorphism of each point of $A^2 - \{p, p'\}$ and not at the points p and p'. Thus the conjecture fails for a two point set $F = \{p, p'\}$ and $n = 2$. The definition of f_2 could be given for any continuous mapping f_1 of C onto C. In this latter case f_2 is a local homeomorphism at $p\,(p')$ if and only if f_1 is a homeomorphism. If f_1 is a homeomorphism then so is f_2.

Now considering A^2 as a subset of E^4, i.e. retain the first 3-coordinates of each point of A^2 and make the 4th coordinate 0, and using the points $p_1 = (0, 0, 0, 1)$ and $p_1' = (0, 0, 0, -1)$, let A^3 be the union of the line segments joining each point of A^2 to p_1 and p_1', respectively. The set A^3 is a topological 3-sphere and f_2 can be extended to a continuous mapping f_3 if A^3 onto A^3 in a manner similar to that of extending f_1. The mapping f_3 is an open mapping and is a local homeomorphism except on the set F consisting of the line segments p_1p, p_1p', $p_1'p_1$ and $p_1'p'$. F is a topological one sphere, A^3-F is connected, and f/A^3-F is a local homeomorphism but not a homeomorphism. Repeating this procedure gives the result that for $n \geqslant 2$, there exists a finite to one open continuous mapping f of S^n onto S^n with a set $F \subset S^n$, F a topological S^{n-2} sphere, $F = f^{-1}f(F)$, f a local homeomorphism of the connected set S^n-F onto S^n-F, f/F a homeomorphism, and f/S^n-F not a homeomorphism.

3. In this section we consider the conjecture when F is a totally disconnected set and $f(F)$ separates no region. First observe that f is a light mapping for suppose $f^{-1}(y)$ has a non-degenerate component E for some $y \in f(S^n)$. Then at least one point x of E is not in F and f must be a local homeomorphism at x. Since this is not possible, f is a light mapping. By Brouwer's theorem on invariance of domain and theorem 2.3, p. 82, of [3], f is a light open mapping of S^n into S^n and consequently $f(S^n) = S^n$. On the set $S^n-f(F)$ define $n(y)$ to be the number of points in $f^{-1}(y)$ if this is finite and to be ∞ otherwise. For $y \in S^n-f(F)$, $f^{-1}(y)$ is closed and in $S^n-f^{-1}f(F)$ and hence must be finite since f is a local homeomorphism at each point of $f^{-1}(y)$. Suppose now that $f^{-1}(y) = \{x_1, x_2, \ldots, x_r\}$ and $V_1, V_2, \ldots, V_r$ are regions about $x_1, \ldots, x_r$, respectively with $\overline{V}_i \cap \overline{V}_j = \emptyset$ if $i \neq j$ and f/V_i a homeomorphism for $i = 1, 2, \ldots, r$. Let U be any region about y with $\overline{U} \subset \bigcap_{i=1}^{r} f(V_i)$. Each component E of $f^{-1}(U)$ has $f(E) \cap f(U) = f(\overline{E}) \cap f(U)$ implying $f(E)$ is open and closed in $f(U)$ and, therefore, $f(E) = U$. Also if E contains x_i, then $E \subset V_i$ for $E \cap (\overline{V}_i - V_i) \neq \emptyset$ leads to a contradiction. Thus $n(y) = r$ on the region U and this proves that $n(y)$ is continuous and integer valued on the connected set S^n-F, or $n(y) = r$ for $y \in S^n-f(F)$. No point $y \in S^n$ has more than r points in its inverse for if such a point existed then by openness of f there would exist at least $r+1$ open sets, $U_1, U_2, \ldots, U_{r+1}$, pairwise disjoint and having the same image. This image being open would contain a point z of $S^n-f(F)$ and $f^{-1}(z)$ would necessarily intersect each U_i and this is not possible. By theorem 3.5, p. 85, of [3], the image set $f(F)$ can be expressed as a countable union of closed zero dimensional subsets and as such is known [4] to be of dimension zero. It follows that $f^{-1}f(F)$ is also zero dimensional. We have now established the following theorem.

THEOREM 1. *Let F be a closed zero dimensional subset of S^n $(n > 1)$, f a continuous mapping of S^n into S^n such that the partial mapping f/S^n-F is a local homeomorphism. If $f(F)$ separates no region then*

(i) *f is an open mapping and each point inverse has at most r points for some integer r;*

(ii) $f(S^n) = S^n$;

(iii) *$f(F)$ and $f^{-1}f(F)$ are of dimension zero.*

From the properties of f, the fact that a zero dimensional set does not separate S^n $(n > 1)$ and theorem 1 of [1], it follows that $(S^n-f^{-1}f(F), f)$ is a covering of $S^n-f(F)$. If $S^n-f(F)$ were simply connected, then it would follow that $f/S^n-f^{-1}f(F)$ is a homeomorphism. Once this is established it would then follow that f is necessarily a homeomorphism. In case F is a countable point set in S^n $(n \geqslant 3)$ it follows from theorem 3E, p. 293 of [5] that $S^n-f(F)$ is simply connected and thus $f/S^n-f^{-1}f(F)$ is a homeomorphism. Now again it is easily established that f is a homeomorphism.

COROLLARY. *Let F be a countable closed subset of S^n $(n \geqslant 3)$, f a continuous mapping of S^n into S^n such that the partial mapping f/S^n-F is a local homeomorphism. Then f is homeomorphism.*

References

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