that in general $\chi_t$ takes $(E_u, K_{u+1})$ onto $(Q - C_{u+1})$, $I_u$ onto $A$, and $E_{u+1}$ onto $D_{u+1}$ if $2^u \leq 2^t$. The limit of the $\chi_t$'s is the required extension taking $Q$ onto itself.

References

Brevy par la Rédaction le 12. 5. 1963

On a special metric and dimension *

by

J. Nagata (Warszawa)

Once we have characterized [3] a metric space of covering dimension $\leq n$ by means of a special metric as follows:

A metric space $R$ has dim $\leq n$ if and only if we can introduce a metric $g$ in $R$ which satisfies the following condition:

For every $\epsilon > 0$ and for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of $R$ satisfying

$$g(S_{x,y_i}(x), y_i) < \epsilon, \quad i = 1, \ldots, n+2$$

there is a pair of indices $i, j$ such that

$$g(y_i, y_j) < \epsilon \quad (i \neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows:

A separable metric space $R$ has dim $\leq n$ if and only if we can introduce a totally bounded metric $g$ in $R$ which satisfies the following condition:

For every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in $R$, there is a triplet of indices $i, j, k$ such that

$$g(y_i, y_j) \leq g(x, y_k) \quad (i \neq j).$$

The former theorem is not so smart though it is valid for every metric space. The problem of generalizing the latter theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

THEOREM. A metric space $R$ has dim $\leq n$ if and only if we can introduce a metric $g$ in $R$ which satisfies the following condition:

For every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in $R$, there is a pair of indices $i, j$ such that

$$g(y_i, y_j) \leq g(x, y_i) \quad (i \neq j).$$

* The content of this paper is a development in detail of our brief note On a special metric characterizing a metric space of dim $\leq n$, Proc. of Japan Acad. 29 (1953).
(1) $S_{x,y}(x) = (g(g(x, y) < \epsilon)/2).
Fundamenta Mathematicae, T. LV

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Proof. Sufficiency. We shall prove that the following weaker condition is sufficient for $R$ to have $\dim \leq n$.

We can introduce a metric $\rho$ in $R$ such that for a definite number $\delta > 0$ and for every $n + 3$ points $x, y, y_1, ..., y_{n+2}$ in $R$ with $\rho(x, y) < \delta$, $j = 1, ..., n - 2$, there is a pair of indices $i, j$ such that

\[
\rho(y_i, y_j) < \rho(x, y), \quad (i \neq j).
\]

For $n = 0$, the condition for $\rho$ implies that we can introduce a non-Archimedean metric in $R$. Hence, by de Groot’s theorem [1], $R$ has $\dim \leq 0$.

To prove our assertion by induction with respect to the number $n$ we assume its validity and suppose $\rho$ is a metric satisfying the condition for $\delta$ and for every $n + 1$ points $x, y, y_1, ..., y_{n+2}$ in $R$. Let $F$ be a given closed set of $F_1$, then for an arbitrary positive number $\epsilon < \delta$ we consider the open neighborhood $S_{i}(F) = \{p \in F \mid \rho(p, F) < \epsilon\}$ of $F$. To assert $\dim R \leq n + 1$ it suffices to show

\[
\dim B_{S}(F) \leq n,
\]

where $B_{S}(F)$ denotes the boundary of $S_{i}(F)$. For if we can prove the assertion, then for given disjoint closed sets $F, G$ and for an $m_0$ with $1/m_0 < \delta$,

\[
S = \bigcup_{m_0} [S_{i}(F) - S_{i}(G)]
\]

is an open set such that

\[
F \subseteq S \subseteq R - G,
\]

\[
B(S) \subseteq \bigcup_{m_0} B_{S}(F) \cup \bigcup_{m_0} B_{S}(G);
\]

hence by the sum-theorem we obtain $\dim B(S) \leq n$ proving $\dim R \leq n + 1$.

If we denied the assertion, then by the inductive assumption there would be $n + 3$ points $x, y, y_1, ..., y_{n+2}$ in $B_{S_{i}}(F)$ such that

\[
\rho(x, y) < \epsilon, \quad \rho(y_i, y_j) > \rho(x, y)
\]

for every pair $i, j$ with $i \neq j$. We choose a small neighborhood $U(x)$ of $x$ such that for every point $x'$ of $U(x)$,

\[
\rho(x', y) < \epsilon
\]

and

\[
\rho(y_i, y_j) > \rho(x', y), \quad i \neq j
\]

hold. Then there exists a point $y_{n+2}$ of $F$ satisfying

\[
S_{i}(y_{n+2}) \cap U(x) \neq \emptyset.
\]

Take a point $x' \in S_{i}(y_{n+2}) \cap U(x)$; then

\[
\rho(x', y) < \epsilon < \delta, \quad j = 1, ..., n + 3,
\]

\[
\rho(y_i, y_j) > \rho(x', y), \quad i \neq j, 1 \leq i, j \leq n + 2,
\]

\[
\rho(y_i, y_{n+2}) > \epsilon > \rho(x', y), \quad i = 1, ..., n + 2,
\]

\[
\rho(y_{n+2}, y_j) > \epsilon > \rho(x', y), \quad j = 1, ..., n + 2.
\]

But this contradicts the property of $\rho$. Therefore we can conclude that

\[
\dim B_{S_{i}}(F) \leq n
\]

and accordingly

\[
\dim R \leq n + 1.
\]

To carry out the proof of necessity we need the following terminology which is a slight modification of the concept ‘rank’ of a collection of sets established in [5] or [6].

Definition. Let $\mathcal{S}$ be a collection of subsets of $R$. We call the Rank of $\mathcal{S}$ not greater than $n$ and denote it by

\[
\text{Rank} \mathcal{S} \leq n
\]

if $\mathcal{S}$ has the following property:

If $U_1, ..., U_k \in \mathcal{S}, \quad U_1 \cap ... \cap U_k \neq \emptyset, \quad U_i \not\subset U_j$ for every pair $i, j$ with $i \neq j$,

then $k \leq n$.

Incidentally, we call a collection $(U_1, ..., U_k)$ of subsets or the subsets themselves independent if $U_i \not\subset U_j$ for every pair $i, j$ with $i \neq j$.

Proof of necessity. Let $R$ be a metric space of dimension $\leq n$; then we shall explain how to define a metric $\rho$ of $R$ which satisfies the desired condition. By use of the decomposition theorem we decompose $R$ as

\[
R = \bigcup_{i=1}^{n} A_i
\]

for 0-dimensional spaces $A_i, i = 1, ..., n + 1$.

The point of the proof is to define a sequence

\[
B_1 > B_2^* > B_3 > B_4^* > ... \quad (1)
\]

of locally finite open coverings such that

\[
\text{mesh} B_n \rightarrow \sup \{d(V) \mid V \in B_n\} < 1/m
\]

(1) As for terminologies and notations about coverings, see J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940.
and a locally finite open covering $S'_{m,n,p} = (S'_{m,n,p}(V)|V \in S_m)$ for each finite sequence $m_1, m_2, \ldots, m_p$ of integers with $1 \leq m_1 < m_2 < \ldots < m_p$

such that

$$S'_m \in S_m \{S'_m(V) = V \in \forall \in S_m\},$$

if $2^{-m_1} + \cdots + 2^{-m_p} > 2^{-m} + \cdots + 2^{-m_p}$, then $S'_{m,n,p} \supset S'_{m-1,n,p}$.

(6) $\text{Rank} \cup \{S'_{m,n,p}|1 \leq m_1 < m_2 < \ldots < m_p\} \leq n + 1$.

We shall define, by induction on the number $m$, locally finite open coverings $S_1, \ldots, S_m$ and $\{S_{m,n,p}|1 \leq m_1 < m_2 < \ldots < m_p \leq m\}$ satisfying the following condition besides (1), (2), (3), (4). If we put, for brevity,

$$\{S'_{m,n,p}|1 \leq m_1 < \ldots < m_p \leq m\} = \{S_1, \ldots, S_{m-1}\},$$

then

$$U, U' \in S_i \text{ implies either } U \subset U' \text{ or } U = U',$$

$$U', U' \in S_i \cup \ldots \cup S_{m-1} \text{ and } U' \supset U' \implies U' \subset U,$$

$$\text{Rank} \{S_i, \ldots, S_{m-1}\} \leq n + 1,$$

$$\text{ord}_p B(S_i, \ldots, S_{m-1}) \leq n + 1 \text{ for } p \in A_i,$$

where for a collection $S$ of subsets and a point $p$ of $R$, $B(S)$ denotes the collection $\{B(U)|U \in S, p \in U\}$ and $\text{ord}_p S$ denotes the number of the members of $S$ which contain $p$.

For $m = 1$ we construct a locally finite open covering $S_1 = (F_1|a \in A_1)$ with

$$\text{ord}_{F_1} \leq n + 1, \text{ mesh} F_1 < 1,$$

where for a collection $S$ of subsets $S$ denotes the collection $\{F|V \in S\}$.

Then there is an open covering $S'_1 = (F'_1|a \in A_1)$ for which $F'_1 \subset F_1$.

Then, as we have seen in [1], Lemma 2.1, we can construct open sets $V''_a, a \in A_1$, such that $V''_a \subset V'_a \subset V_1$.

$$\text{ord}_p (B(V''_a)|a \in A_1) \leq i - 1 \text{ for } p \in A_1.$$ We choose from $\{V''_a|a \in A_1\}$ the members $V''_a$ for which $V''_a \subset V''_a(\beta \in A_1)$ implies $V''_a = V''_a$ and make a collection $Y_a$ out of them. Then it is easy to see that $S_1 = \{S_i\}$ is a locally finite open covering satisfying all the required conditions. Now, let us assume that we have already defined $S_1, \ldots, S_m$ and $\{S_{m-1}, \ldots, S_{m+1}\}$ to define $S_{m+1}$ and

$$\{S_{m+1}, \ldots, S_{m+1}\} = \{S_{m,n,p} |1 \leq m_1 < \ldots < m_p \leq m + 1\}.$$

(1) We often call a collection of subsets merely a collection. $B(U)$ denotes the boundary of $U$.

First, we construct a locally finite open covering $S$ with

$$\text{mesh } S < 1(m + 1), \text{ mesh } S'^* < S_m$$

such that

$$(10) \text{ if } U_1, \ldots, U_1 \in S_1 \ldots \cup S_{m+1} \text{ and } U_1 \cap \ldots \cap U_1 = 0, \text{ then }$$

$$(11) \text{ for each } p \in R, S(p, S) \text{ meets only finitely many members of } S_1 \ldots \cup S_{m+1};$$

$$(12) \text{ if } U_1, U_2 \in S_1 \ldots \cup S_{m+1} \text{ and } U \subset U_1 \text{ then } S(U, S) \subset S(U, S);$$

$$(13) \text{ if } U', U \in S_1 \ldots \cup S_{m+1} \text{ and } U \supset U', \text{ then }$$

$S(U, S) \supset U'$$.

Since $S_1 \ldots \cup S_{m+1}$ is locally finite, we can choose such a $S$ as follows.

By use of the local finiteness of $S_1 \ldots \cup S_{m+1}$, we can easily choose a sufficiently refined $S$ to satisfy (10) and (12) besides mesh $S < 1(m + 1)$ and $\text{mesh } S'^* < S_m$.

We shall show how to define $S$ to satisfy (10) and (13), too. For each point $p$ of $R$, we define a set $S(p)$ by

$S(p) = \cap \{R - U|U \subset U, U \cap S_1 \cup \ldots \cup S_{m+1}\}$

where $S(p) = S(p), p \in R$ is an open covering since $S_1 \cup \ldots \cup S_{m+1}$ is locally finite. If we choose $S$ such that $S'^* < S_1$, then $S$ clearly satisfies (10).

For each pair $U, U' \in S_1 \cup \ldots \cup S_{m+1}$ satisfying $U \supset U'$, we assign a point $p(U, U') \in U' - U$.

For a definite member $U_1$ of $S_1 \cup \ldots \cup S_{m+1}$

$$X(U) = R - \cup \{p(U, U')|U \subset U', \text{ and } X(U') \subset X(U)\}$$

is an open neighborhood of $U$ by virtue of the local finiteness of $S_1 \cup \ldots \cup S_{m+1}$. We choose an open neighborhood $X(U)$ of $U$ such that

$$X(U) \subset X(U');$$

then $S$ clearly satisfies (13). We note that if $S$ satisfies one of (10), (11), (12) or (13), then every refinement of $S$ also satisfies the same condition. Thus we can construct $S$ satisfying all the desired conditions.
Let $\mathcal{B} = \{V_a | a \in \mathcal{A}\}$; then we can construct an open covering $W' = \{W'_a | a \in \mathcal{A}\}$ satisfying $W'_a \subset V_a$.

Since by (11) each $S(V_a, \mathcal{B})$ meets at most finitely many of $U \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_m$, for each of those $U$ we can define an open set $V'_a(U)$ such that

$W'_a \subset V'_a(U) \subset V_a(U)$.

Then by (12) $\exists \mathcal{B} \subset V'_a(U) \subset V_a(U)$.

If $U \neq U'$, then either $V_a(U) \neq V_a(U')$ or $V'_a(U) \neq V'_a(U')$.

If $U \in \mathcal{S}_m$, $U' \in \mathcal{S}_{m-1}$, $2^{-m} + \ldots + 2^{-m'} < 2^{-h} + \ldots + 2^{-h'}$, then $V_a(U) \subset V_a(U')$.

By virtue of (9) we can choose $V_a(U)$ satisfying the following condition, too,

$\exists B' = \{V'_a(U) | a \in \mathcal{A}, U \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_m\}$

where $B' \subset \mathcal{S}_m$.

Suppose that $S_{m,m-1} = \bigcup \{V'_a(U) | a \in \mathcal{A}, S(V_a, \mathcal{B}) \neq 0\}$.

$S_{m,m-1} = \bigcup \{V'_a(U) | a \in \mathcal{A}\}$.

By (11), $S_{m,m-1}$ is a locally finite open covering.

We choose only those members of $S_{m,m-1}$ which are not contained in any other member and denote the collection of those members also by $S_{m,m-1}$.

Adding those locally finite open coverings $S_{m,m-1}$, $1 \leq m < \ldots < m_2 < m_1 < \ldots < m_2 < m_1$, to the collection

$\Sigma = \{S_1, \ldots, S_{m_1}\}$

we obtain a new collection

$\Sigma' = \{S_1, \ldots, S_{m_1}, S_{m_1-1}, \ldots, S_{m_2-1}\}$.

Then we can see that this collection $\Sigma'$ of coverings satisfies the conditions (6), (7), (8), (9).

As for (6) we have just altered $S_{m,m-1}$ so that $\Sigma'$ satisfies that condition. To see (7) let

$U, U' \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1-1}$ and $U \subsetneq U'$.

If $U \in \mathcal{S}_{m_1-1} \cup \ldots \cup \mathcal{S}_{m_1-1}$, $U' \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$,

then

$U = \bigcup \{V'_a(U'_a) | a \in \mathcal{A}, S(V_a, \mathcal{B}) \neq 0\}$

for some $U'_a \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$. We shall denote such a set $U$ by $S(U'_a, \mathcal{B})$ in the rest of the proof.

Hence

$U \subset U' \subset \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$.

Therefore it follows from (12) that

$U \subset S(U'_a, \mathcal{B}) \subset U'$. If

$U \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$

then

$U' = S(U'_a, \mathcal{B})$.

for some $U'_a \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$. If we assume that $U \subset U'$, then by (13)

$S(U'_a, \mathcal{B}) \neq U'$

and hence $U \subset U'$, which is a contradiction. Thus we obtain $U \subset U'$ which implies

$U \subset U \subset S(U'_a, \mathcal{B}) = U'$.

If

$U, U' \in \mathcal{S}_{m_1-1} \cup \ldots \cup \mathcal{S}_{m_2-1}$

then

$U = S(U'_a, \mathcal{B})$, $U' = S(U'_a, \mathcal{B})$.

for some $U'_a \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$ with $U_a \neq U'_a$. Hence, by (13), we obtain $U \subset U'$.

If $U, U'$ belong to the same covering $\mathcal{S}_n$, then $U$ and $U'$ also belong to the same covering which is impossible since $U \subset U'$. Hence we suppose that $U$ and $U'$ belong to the distinct coverings $S_{m,m}$ and $S_{m,m}$, respectively. If

$2^{-m} + \ldots + 2^{-m'} < 2^{-h} + \ldots + 2^{-h'}$

then it follows from (16), the local finiteness of $\mathcal{B}$ and $U \subset U'$ that

$U = S(U'_a, \mathcal{B}) = \bigcup \{V'_a(U'_a) | a \in \mathcal{A}, S(V_a, \mathcal{B}) \neq 0\}$

for some $U'_a \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$ with $U_a \neq U'_a$. Hence, by (13), we obtain $U \subset U'$. If $U, U'$ belong to the same covering $\mathcal{S}_n$, then $U$ and $U'$ also belong to the same covering which is impossible since $U \subset U'$. Hence we suppose that $U$ and $U'$ belong to the distinct coverings $S_{m,m}$ and $S_{m,m}$, respectively. If

$2^{-m} + \ldots + 2^{-m'} > 2^{-h} + \ldots + 2^{-h'}$

then it follows from (16), the local finiteness of $\mathcal{B}$ and $U \subset U'$ that

$U = S(U'_a, \mathcal{B}) = \bigcup \{V'_a(U'_a) | a \in \mathcal{A}, S(V_a, \mathcal{B}) \neq 0\}$

for some $U'_a \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{m_1}$ with $U_a \neq U'_a$. Hence, by (13), we obtain $U \subset U'$. If $U, U'$ belong to the same covering $\mathcal{S}_n$, then $U$ and $U'$ also belong to the same covering which is impossible since $U \subset U'$. Hence we suppose that $U$ and $U'$ belong to the distinct coverings $S_{m,m}$ and $S_{m,m}$, respectively. If

$U \subset S(U'_a, \mathcal{B}) \subset U'$.
but this contradicts the definition (18) of $S_{a_0...a_{m+1}}$. Thus in any case we can conclude $U \subset U'$ proving (7) for $\Sigma'$.

Now, to prove (8) for $\Sigma'$ we suppose

$$U_1, \ldots, U_t \in S_1 \cup \ldots \cup S_{a_{m+1}}, \quad U_1 \cap \ldots \cap U_t \neq \emptyset,$$

$$U_i \not\subset U_j \quad \text{if} \quad i \neq j.$$

If we assume

$$U_1, \ldots, U_k \in S_1 \cup \ldots \cup S_{a_{m+1}},$$

$$U_{k+1}, \ldots, U_{l} \in S_{a_{m+1}} \cup \ldots \cup S_{a_{m+1}},$$

then

$$U_a = \mathcal{S}_a(U_{a_0}, \mathbb{B}), \quad s = k + 1, \ldots, l$$

for some $U_{a_0} \in S_1 \cup \ldots \cup S_{a_{m+1}}$. Since $(U_1, \ldots, U_k)$ is independent, it follows from (7) for $\Sigma$ and (12) that $U_1, \ldots, U_k, U_{k+1}, \ldots, U_{a_0}$ are also independent. If we assume that

$$U_1 \cap \ldots \cap U_k \cap U_{a_0+1} \cap \ldots \cap U_{a_0} = \emptyset,$$

then we obtain from (10)

$$U_1 \cap \ldots \cap U_k \cap U_{a_0+1} \cap \ldots \cap U_{a_0} \cap \mathcal{S}_a(U_{a_0+1}, \mathbb{B}) \cap \mathcal{S}_a(U_{a_0}, \mathbb{B}) = \emptyset,$$

which is a contradiction. We can, therefore, conclude

$$U_1 \cap \ldots \cap U_k \cap U_{a_0+1} \cap \ldots \cap U_{a_0} \neq \emptyset,$$

and hence by (8) for $\Sigma$, we obtain $l \leq n + 1$, i.e.

$$\text{Rank } S_1 \cup \ldots \cup S_{a_{m+1}} \leq n + 1$$

proving (8) for $\Sigma'$.

Finally, to prove (9) for $\Sigma'$, we suppose

$$p \in A_1,$$

$$p \in B(U_1) \cap \ldots \cap B(U_k),$$

$$U_{k+1}, \ldots, U_l \in S_1 \cup \ldots \cup S_{a_{m+1}},$$

$$U_1 \cap \ldots \cap U_k \cap \mathcal{S}_a(U_{a_0}, \mathbb{B}) = \emptyset,$$

$$U_a \in S_1 \cup \ldots \cup S_{a_{m+1}}.$$

Since $(V_a(U_a)) \cap A) \text{ is locally finite, we obtain}$

$$B(U_a) \cap (BV_a(U_a)) \cap A \cap S(V_a, \mathbb{B}) \cap U_a = \emptyset.$$}

Hence

$$p \in B(U_1) \cap \ldots \cap B(U_k) \cap BV_{a_{m+1}} \cap BV_{a_{m+1}} \cap BV_{a_{m+1}} \cap BV_{a_{m+1}}.$$}

for some distinct members $V_{a_{k+1}}(U_{a_{k+1}}), \ldots, V_{a_{m+1}}(U_{a_{m+1}})$ of $\mathcal{S}$ defined in (17). Therefore from (17) we obtain $l \leq n - 1$, i.e.

$$\text{ord } B(S_1 \cup \ldots \cup S_{a_{m+1}}) \leq n - 1.$$

Thus $\Sigma'$ satisfies all of (6)-(9).

Finally, we shall define $S_{a_{k+1}} = S_{a_{k+1}} \cup S_{a_{m+1}}$. For the preceding covering $\mathcal{S}$ defined just before (11) we construct a locally finite open covering $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{S}$ and

$$\text{rank } S_1 \cup \ldots \cup S_{a_{m+1}} \cup \mathcal{B} \leq n + 1.$$

Since $\Sigma'$ satisfies (8) and (9) such a covering $\mathcal{B}$ can be constructed by a slight modification of the process used in [6], proof of Theorem 2.

In general, let $\mathcal{B}'$ be an open covering and $\mathcal{S}$ a locally finite open covering with $\text{Rank } S \leq n + 1$ and $\text{ord } B(S) \leq n - 1$ for $p \in A_1$; then we can assert that there exists a locally finite open covering $\mathcal{B}$ satisfying

$$\mathcal{B} \subset \mathcal{B}', \quad \text{rank } S \cup \mathcal{B} \leq n + 1.$$

Let $B_k = \{p \mid \text{ord } B(S) \geq k\}, \quad k = 0, \ldots, n$.

$$B_k \subset A_1, \ldots, \ldots, A_k+1$$

and hence we obtain

$$\dim B_k \leq n - k$$

by the decomposition theorem. Since $B(S)$ is locally finite, each $B_k$ is a closed set and satisfies $B_{k+1} \subset B_k$. For every point $p$ of $B_k - B_{k+1}$, we can choose an open neighborhood $U(p)$ of $p$ such that

$$U(p) \cap B \neq \emptyset, \quad \text{dim } B_k \leq n - k_1$$

for exactly $k$ members $S$ of $B(S)$. Then

$$U_k = \{p \mid \text{ord } B_k = B_{k+1}\}$$

is a collection which covers $B_k - B_{k+1}$ and consists of open sets which do not intersect $B_{k+1}$.

Now we shall define $a + 1$ locally finite collections $\Psi_k, \quad k = 0, \ldots, n$ of open sets such that

$$\Psi_k \subset \Psi_{k+1} \subset \mathcal{B}', \quad \text{ord } \Psi_k \leq n - k,$$

$$\Psi_k - \Psi_{k+1} \subset \mathcal{B}_{n-k}, \quad \Psi_k \text{ covers } B_{n-k}.$$

where we put

$$\Psi_k = \{p \mid \text{ord } B_k \neq \emptyset, \quad \Psi_{k+1} = \emptyset \}.$$

In general, if two subsets $U$ and $S$ satisfy this condition, then we call $U$ over-
To this end we shall show by induction on the number $m$ that for every $m$ with $0 \leq m \leq n$ we can define $m+1$ locally finite open collections $\psi_1^m, \ldots, \psi_m^m$ such that

\[
\psi_{k-1}^m \subset \psi_k^m \subset \psi_{k+1}^m, \quad \text{ord} \psi_k^m = k+1,
\]

\[
\psi_k^m - \psi_{k-1}^m \subset U_{k-1}, \quad \psi_k^m \text{ covers } B_{k-1}, \quad k = 0, \ldots, m.
\]

For $m = 0$ we choose, by use of $\dim B_0 \leq 0$, an open covering $\Omega$ of $B_0$ with

\[
\text{ord } \Omega \leq 1, \quad \Omega \subset U_0 \wedge W.
\]

Since $\Omega$ is a locally finite closed collection of order $\leq 0$ in $R$, we can easily see that there exists a locally finite open collection $\psi_0^0$ in $R$ such that

\[
\text{ord } \psi_0^0 \leq 1, \quad \Omega \subset \psi_0^0 \setminus U_0 \wedge W.
\]

Then $\psi_0$ is the desired open collection for $m = 0$.

Now, let us suppose we have defined $\psi_1^m, \ldots, \psi_m^m$ at desire. Let

\[
\psi_{k-1}^m = \{x| \alpha < \alpha_{k+1}\}, \quad k = 0, 1, \ldots, m.
\]

Since $\dim B_{m+1} \leq m+1$, we can choose a locally finite open covering $\mathcal{R}$ of $B_{m+1}$ satisfying

\[
\text{ord } \mathcal{R} \leq m+2, \quad \mathcal{R} \subset \psi_m^m \cup U_{m+1}, \quad \mathcal{R} \subset W.
\]

Since $B_{m+1}$ is closed, $\mathcal{R}$ is a locally finite closed collection in $R$ of order $\leq m+2$. Hence we can easily see that there exists a locally finite open collection $\mathcal{R}$ in $R$ such that

\[
\text{ord } \mathcal{R} \leq m+2, \quad \mathcal{R} \subset \psi_m^m \cup U_{m+1}, \quad \mathcal{R} \subset W.
\]

Consequently, $\mathcal{R}$ covers $B_{m+1}$. Then, putting

\[
P_r = \bigcup \{M| \quad \mathcal{M} \in \mathcal{R}, \quad M \subset P_r, \quad M \not\subset P_k \text{ for every } \beta < \alpha\},
\]

\[
\psi_{k+1}^{m+1} = \{x| \alpha < \alpha_{k+1}\}, \quad k = 0, 1, \ldots, m,
\]

\[
\psi_m^{m+1} = \psi_m^m + \bigcup \{M| \quad \mathcal{M} \in \mathcal{R}, \quad M \subset P_k \text{ for every } \alpha < \alpha_{m+1}\},
\]

we assert that $\psi_{k+1}^{m+1}, \ldots, \psi_m^{m+1}$ are the desired locally finite open collections for $m = m+1$. The only problem is to show that $\psi_m^{m+1}$ covers $B_{m+1}$. Since $P_k^{m+1}$ clearly covers $B_{m+1}$, we may assume $k \leq m$. To see this we note that each element of $\psi_m^m - \psi_k^m$ does not meet $B_{k-1}$, because

\[
\psi_k^m - \psi_{k-1}^m \subset U_{k-1} \cup \cdots \cup U_{m-1},
\]

and each member of $U_{m-1} \cup \cdots \cup U_{m}$ does not meet $B_{k-1}$. Furthermore note that each member of $U_{m-1} \cup \cdots \cup U_{m}$ does not meet $B_{k-1}$ either. Let

$p$ be a given point of $B_{k-1}$; then $p \in M$ for some $M \in \mathcal{M}$. From the above note and $\mathcal{M} \subset \psi_m^m \cup U_{m+1}$, it follows that

\[
p \in M \subset P
\]

for some $P \in \psi_m^m$ which means by the definition of $\psi_m^{m+1}$

\[
P \in \psi_m^{m+1} \subset P
\]

for some $P \in \psi_m^{m+1}$. Hence $\psi_m^{m+1}$ covers $B_{m+1}$. Thus we get desired collections $\psi_1^m, \ldots, \psi_m^m$ for every integer $m$ with $0 \leq m \leq n$.

Putting

\[
\mathcal{G} = \psi_k^m, \quad k = 0, 1, \ldots, n,
\]

we get the initially desired $n+1$ collections. We put

\[
\mathcal{G} = \{P_x| x \in \mathcal{F}_k\}, \quad k = 0, 1, \ldots, n.
\]

We note that $\mathcal{G}$ is a locally finite open covering of $R$. Hence there exists an open covering $\mathcal{G} = \{W_x| x \in \mathcal{F}_k\}$

of $R$ such that $W_x \subset P_x$.

If we put

\[
\mathcal{G} = \{W_x| x \in \mathcal{F}_k\}, \quad k = 0, 1, \ldots, n,
\]

then we can easily see that

\[
\mathcal{G} \subset \mathcal{G}, \quad \text{Rank } \mathcal{G} \leq \text{ord } \mathcal{G} \leq n+1,
\]

\[
\text{ord } \mathcal{G} \leq k+1, \quad \mathcal{G} - \mathcal{G} \subset U_{n-k}.
\]

Put $\mathcal{G} = \mathcal{G}$; then let us show that

\[
\text{Rank } \mathcal{G} \wedge \mathcal{G} \leq n+1.
\]

Suppose

\[
\exists x_1 \cap \cdots \cap x_n \cap U_{k+1} \cap \cdots \cap U_{n+1},
\]

\[
U_{k} \subset U_{j} \text{ if } i \neq j,
\]

\[
U_{k} \in \mathcal{G}, \quad U_{n+1} \in \mathcal{G}.
\]

Since

\[
\text{Rank } \mathcal{G} \leq n+1, \quad \text{Rank } \mathcal{G} \leq n+1,
\]

it must be

\[
1 \leq k \leq n+1.
\]

Since

\[
\text{ord } \mathcal{G} \leq n+1,
\]

at least one of $U_{k+1}, \ldots, U_{n+1}$ does not belong to $\mathcal{G}$. For example, let

\[
U_{k+1} \in \mathcal{G} = \mathcal{G}.
\]
for some \( l \geq n-k \). Then, since

\[ B_{n-l+1-l} < U_{n-l+1} \]

and each member of \( U_{n-l+1} \) overlaps exactly \( n-l-1 \) members of \( S \),

\( U_{n-l+1} \) overlaps at most \( n-l-1 \) members of \( S \). Since \( n-l-1 \leq k-1 \),

\( U_{n-l+1} \) overlaps at most \( k-1 \) members of \( S \). On the other hand, since

\[ U_1, \ldots, U_k, U_{k+1} \]

satisfies

\[ U_i \cap U_j \quad \text{if} \quad i \neq j, \]

and

\[ p \in U_1 \cap \cdots \cap U_k \cap U_{k+1}, \]

\( U_{k+1} \) overlaps \( k \) members \( U_1, \ldots, U_k \) of \( S \), which is a contradiction.

Therefore we can conclude that

\[ \text{Rank} S \cup \mathbb{B} \leq n+1. \]

Thus \( \mathbb{B} \) is the desired covering.

Let \( \mathbb{W} \) be a given member of \( \mathbb{B} \). For every member \( U \) of

\[ S_1 \cup \cdots \cup S_{n+1} \]

we assign a point

\[ g(W, U) \in W - U. \]

Then

\[ F(W) = \bigcup \{ g(W, U) : U \in \mathbb{W}, U \cap W \neq \emptyset, U \in S_1 \cup \cdots \cup S_{n+1} \}. \]

is a closed set contained in \( W \), because \( \mathbb{W} \) meets only finitely many members of \( S_1 \cup \cdots \cup S_{n+1} \) since \( \mathbb{B} \subset \mathbb{B} \subset \mathbb{B} \). Hence, by use of (9) for \( \Sigma' \), we can construct an open set \( V(W) \) for every \( W \in \mathbb{W} \) such that

\[ F(W) \subset \bigcup \{ g(W, U) : U \in \mathbb{W}, U \cap W \neq \emptyset, U \in S_1 \cup \cdots \cup S_{n+1} \}. \]

is a closed set contained in \( W \), because \( \mathbb{W} \) meets only finitely many members of \( S_1 \cup \cdots \cup S_{n+1} \) since \( \mathbb{B} \subset \mathbb{B} \subset \mathbb{B} \). Hence, by use of (9) for \( \Sigma' \), we can construct an open set \( V(W) \) for every \( W \in \mathbb{W} \) such that

\[ F(W) \subset \bigcup \{ g(W, U) : U \in \mathbb{W}, U \cap W \neq \emptyset, U \in S_1 \cup \cdots \cup S_{n+1} \}. \]

Then it is easy to see from (6), (7) for \( \Sigma' \) and (19) that

\[ \Sigma'' = (S_1, \ldots, S_{n+1}) \]

also satisfies (6), (7), (8), (9).

Since (6) and (9) are clearly satisfied by \( \Sigma'' \), we shall only concern

\[ \Sigma' \]

and (8). Let

\[ U, U' \in S_1 \cup \cdots \cup S_{n+1} \quad \text{and} \quad U \subset U'. \]

(*) See (4), Lemma 2.1.
satisfies the special condition desired in the theorem. Let \( x, y_1, \ldots, y_{n+1} \) be given \( n+3 \) points of \( R \). For every \( \varepsilon > 0 \) we obtain
\[
m_1, \ldots, m_{n+1}, \quad j = 1, \ldots, n + 2
\]
such that
\[
q(x, y_j) \leq 2^{-m_1} + \ldots + 2^{-m_{n+1}} < q(x, y_j) + \varepsilon
\]
and
\[
U_j \subseteq \mathbb{R}^{1-m_{n+1}}
\]
such that \( x, y_j \in U_j \). It follows from (5) that there exist \( U_i \) and \( U_j \) (\( i \neq j \)) such that \( U_i \cap U_j \). Therefore
\[
q(y_i, y_j) < 2^{-m_1} + \ldots + 2^{-m_{n+1}} < q(x, y_j) + \varepsilon.
\]
We take a pair \( i, j \) satisfying
\[
q(y_i, y_j) < q(x, y_j) + \varepsilon
\]
for a sequence \( (\varepsilon_m) \) of positive numbers converging to 0. Then
\[
q(y_i, y_j) < q(x, y_j)
\]
proving the necessity. Thus among the conditions (6), (7), (8), (9) for \( \mathbb{R}^{1-m_{n+1}} \), the condition (8) is essential. The other conditions are needed only to continue the inductive argument.

References


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