A connection form on $E$, we define the connection form $\omega'$ on $E'$ by setting it to be equal $f^{*}\omega$ on $f(E)$ and extending it by means of formulæ (23) of section 5.1. Then the values of $\omega'$ on $f(\pi(E))$ are contained in $h(g)$, and $\pi_{g}$ is simply identical with $h^{-1}$ (it does not depend on $x$ in this case). The induced connection coincides with the given connection $\omega$ on $E$.

References


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An arc is tame in 3-space if and only if it is strongly cellular*

by

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A set $Z$ in Euclidean $n$-space $E^n$ is tame if there exists such a homeomorphism $f$ of $E^n$ onto itself that $f(Z)$ is a polyhedron. There are known some necessary and sufficient conditions of tameness of an arc in $E^n$, e.g. [4] and [5]. We shall give here another one based on the reinforced notion of a cellular set [2]. A set $Z$ in $E^n$ is strongly cellular if there is an $n$-cell $C$ in $E^n$ and a homotopy $h: C \times I \to E^n$ such that, if $h(t) = h(x, t)$ and $S = Bd C$ is the boundary of $C$, then

(1) $h_t$ is identity mapping and $h_t|Z = id$ identity for all $t$,
(2) $h_t|S$ is homeomorphism for $t < 1$,
(3) $h_t(S) \cap h_{t'}(S) = 0$ for $t \neq t'$,
(4) $h_1(C) = Z$.

The set $Z$ will be said to be a strong deformation retract of the cell $C$.

By M. Brown's generalization of the Schoenflies Theorem [2] there will be no loss of generality if the cell $C$ is assumed to be a ball.

Theorem 1. An arc is tame in $E^3$ if and only if it is strongly cellular.

Corollary. An arc $A$ in $E^3$ is tame if and only if there are two concentric balls $B_0$ and $B$, $B \subset \text{Int} B$, and a mapping $g$ of $A$ into $E^3$ such that

(1) $f(B_0) = A$ is homeomorphism,
(2) $f(B) = A$.

Proof of the Corollary. It is obvious that the conditions of the Corollary are necessary. In order to prove that they are also sufficient let us observe first that $f(B)$ is a 3-cell by M. Brown's Theorem 1 of [2]. Then assume $C = f(B)$ and next consider the homotopy $r: B \times I \to B$ retracting $B$ to $B_0$. Now define the homotopy $h: C \times I \to C$ in the following way:

$$h(x, t) = tr^{-1}(x, t) \quad \text{for} \quad x \in C \setminus A,$$
$$h(x, t) = x \quad \text{for} \quad x \in A.$$

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It is easy to check that $A$ is a strong deformation retract of $C$ and so the proof has been reduced to the Theorem 1.

**Question.** Is the Corollary still valid if the set $B_k$ is assumed to be an acyclic continuum?

Theorem 1 is an easy corollary of the following one:

**Theorem 2.** Let $Q$ be the ball $x^2 + y^2 + z^2 \leq 4$, $I_t$ be the segment $-1 \leq x \leq 1$, $y = z = 0$ and let $A$ be an arc which is a strong deformation retract of $Q$. Then any homeomorphism of $I_t$ onto $A$ can be extended to a homeomorphism of $Q$ onto itself.

Proof of the Theorem 2 will be preceded by some lemmas. Let the symbols $Z$, $S$, $C$, $S'$ and $h$ keep the meaning they got in the definition of strong cellularity.

**Remark.** If $X$ and $Y$ are disjoint subsets of $S$ and $h_{1}(X) \cap h_{1}(Y) = 0$ then $h_{1}(X \times I) \cap h_{1}(Y \times I) = 0$.

Let henceforth $Z = A$ and $f = h_{1}/S$.

**Lemma 1.** If $B$ is a continuum in $A$ then $f^{-1}(B)$ is a continuum in $S$.

**Proof.** Suppose $f^{-1}(B) = B' \times Y$ and $X \times Y = 0$. Since the 0-dimensional homology group $H_0(B') = 0$, then by a standard duality theorem $H_{-1}(S-B') \neq 0$ and there exist two cycles $\gamma^1, \gamma^2$ in $S$ such that $\gamma^1 \cap \gamma^2 = 0$. These can be chosen so that the carrier of $\gamma^2$ is the set $[x, y]$ and the carrier of $\gamma^1$ is a continuum $M$. Then $F = f(M)$ is an arc or a point and does not intersect $B$.

Denote by $f(x)$ the arc $B \times \{x\}$ with the end-points $f(x)$ and $f(y)$. Then $J_f$ be the arc $h([x, y] \times I) \cup f(x) \cup f(y)$ and $J_f$ is an arc with the end points $x$ and $y$ which lies in $(E^3-C) \cup (x, y)$. The simple closed curve $J = J_f \cup J_f$ carries a cycle $\gamma_f$ which links $\gamma_f^1 \cap \gamma_f^2$ in $E^3$. On the other hand $h_{1}(M \times I)$ is $0$ and $f$ is continuous, so $\gamma_f^1 \cap \gamma_f^2$ is homotopic in $E^3 - J$ to a cycle lying in $Z_f$ but every cycle bounds in $F$ and so $\gamma_f^1 \cap \gamma_f^2$ in $E^3 - J$. This contradiction was a result of our false assumption that $f^{-1}(B)$ is not connected.

Further considerations will be restricted to the 3-space $E^3$.

**Lemma 2.** If $B$ is a connected subset of $A$ containing one end-point of $A$, then $f^{-1}(B)$ is a simply connected subset of $S$.

**Proof.** Let $B = A$ for otherwise the Lemma 3 is obvious. If $B$ is open in $A$ then its complement $B'$ is an arc or a point. Therefore $f^{-1}(B') = S - f^{-1}(B)$ is a connected set by Lemma 1. Hence $f^{-1}(B)$ is simply connected. If $B$ is closed in $A$ then $B'$ is open and $f^{-1}(B')$ is connected; hence $f^{-1}(B)$ is simply connected.

Let $S_{t} = h_{0}(S)$ and $C_{t}$ be the 3-cell bounded by $S_{t}$ in $E^3$.

**Lemma 3.** For any interior point $x$ of the arc $A$ there exists a topological disk $D_a$ in $C$ such that the following conditions are satisfied:

1. $D_a \cap I_t$ is a simple closed curve for $t < t$, and the point $x$ for $t = 1$.
2. $D_a$ cuts the cell $C$ between the end-points of $A$.
3. $D_a$ is locally tame at every point except perhaps at $x$.
4. For any two disks $D_a$ and $D_d$, $x \neq y$, there is a number $t_0$ such that $0 < t_0 < 1$ and $D_a \cap D_d \cap D_a = 0$.
5. If $0 < t_0 < t_1 < 1$ and $W$ is the closure of $C_{t_{0}} - C_{t_{1}}$, $Q'$ is the closure of any component of $W - D_a$ and $P$ is the closure of the component of $C_{t_{1}} - (D_a \cup D_d)$ which contains no end-points of $A$, then $Q'$ is a tame 3-cell and $P = W \cap P$ is a tame solid torus.

**Proof.** Construction of $D_a$. We adopt the following notation:

Let $a, b$ be the end-points of $A$, $B_a = \{a, b\}$ be a closed-open subarc of $A$; $G_a = f^{-1}(B_a)$ is a simply connected domain in $S$ with the boundary $B_d G_a$ contained in $f^{-1}(P)$; $F = f^{-1}(a)$ is a closed, simply connected set in $G_a$; $R_a = G_a - F$ is an open annulus.

Define a sequence $(J_{a})$ of simple closed curves in $B_a$ as follows:

- Let $J_{1} = J_{a}$ be a simple closed curve which cuts $G_a$ between $P$ and $B_d G_a$ and such that the distance $g(J_{1}, B_d G_a) = 0$. Suppose that $J_{m}$ has been already defined for some $m \geq 1$. Let $J_{m+1}$ be a simple closed curve which cuts $G_a$ between $J_{m}$ and $B_d G_a$ such that $g(J_{m+1}, B_d G_a) < 1/(m+1)$.

Denote the annulus in $B_{m+1}$ by $G_{m+1}$ and $\partial G_{m+1}$ by $R_{m+1}$ and construct the following deformations:

a. A sequence of isotopies $g^{m}_{a} : J_{m+1} \times ([m-1]/m, m/(m+1)) \rightarrow R_{m}$, $m = 1, 2, 3, ..., \ldots$, simply denoted by $g^{m}_{a}$ and such that if $g^{m}_{a}(p) = g^{m}_{a}(p, 0)$, then $g^{m}_{a}(p, m) = p$, $g^{m}_{a}(p, m+1) = J_{m+1}$.

b. An isotopic deformation $g^{m}_{a} : J_{m+1} \times (0, 1) \rightarrow R_{m+1}$ such that $g^{m}_{a}(p) = g^{m+1}_{a}(p, 0) \cdots g^{m}_{a}(p)$ for $m-1 < t < m$. $m = 1, 2, 3, ...$

c. A pseudoisotopy $\psi_{a} : J_{m} \times I \rightarrow C$ such that $\psi_{a}(p, 0) = h_{1}(f^{m}_{a}(p))$ for $0 < t < 1$.

Then $D_{a} = \psi_{a}(J_{m} \times I)$ is the desired disk. It is evident that $D_{a}$ satisfies condition (1). Now we check the other ones.

**Condition (2).** Let $J$ be that component of $C - D_{a}$ which contains the point $a$. It follows from the construction that $f(J \cap \{a\} \times h^{-1}(B_d G_a))$, but $b \in (a \times h^{-1}(B_d G_a))$ and so there is no continuum in $C - D_{a}$ which could contain both $a$ and $b$. 


Condition (3). Let \( p \in D_S \). There is a number \( s \) such that
\[ 0 < s < 1 \]
and \( p \in D_s \). Let \( D \) be the closure of a component of \( S \), \( D' \) be the closure of a component of \( S \), and \( R = D_1 \). Then
\[ p \in R \subset D \subset R \subset S' \]
and \( S' \) is a sphere. It can be easily shown that \( S' \) is bicollared and hence
\[ p \bigtriangleup S' = \triangle \]

and \( \triangle \) is a sphere. It can be easily shown that \( \triangle \) is bicollared and hence
\[ p \bigtriangleup \triangle = \triangle \]

by the theorem of Bing and Moise [1]. \( \triangle \) is a sphere.

Condition (4). By construction \( D_1 = \varphi(D_z \times I) \) and similarly
\[ D_2 = \varphi(D_z \times I) \]

Notice that \( \varphi(D_z \times I) = \varphi \). Given disjoint neighbourhoods \( U_z \) and \( U_y \) respectively we can choose a number \( t_x \) such that \( 0 < t_x < 1 \) and \( \varphi(D_z \times [t_x, 1]) \subset U_z \) for \( z = z, y \). Then \( t_x \) is the right number.

Condition (5). It is easy to see that \( q' \) is a tame sphere because its boundary is a tame sphere by much the same reason that \( S' \) is one.

In order to prove that \( T \) is a tame solid torus it will be sufficient to show that its boundary is a tame torus and that there are two tame disjoint disks in \( T \) which span \( T \).

a. \( \text{Bd} T \) is a tame torus: It is clear that \( \text{Bd} T \) is a torus for
\[ \text{Bd} T = T \cap (D_x \times D_y \cup S_x \cup S_y) \]

Also it is quite obvious that \( \text{Bd} T \) is locally tame at the interior points of the sets \( T \cap D_x, T \cap S_x \cap D_y \), and \( T \cap D_y \cap S_x \), \( i = 0, 1 \). And at the boundary points of the sets \( \text{Bd} T \) is locally tame by Doyle's theorem concerning the unions of cell pairs in \( E^3 \). Hence, by the theorem of Bing and Moise [1], \( \text{Bd} T \) is tame.

b. There are two disjoint, tame disks \( D_1, D_2 \) in \( T \) such that
\[ D_1 \cap \text{Bd} T = \text{Bd} D_1, \quad i = 1, 2 \]

By condition (4) the simple closed curves
\[ \gamma \subset \text{Bd} \]

are disjoint subsets of \( S \). In the annulus bounded by these curves choose two disjoint arcs \( A_1, A_2 \) having with each curve only one end-points in common. The mappings \( g: \gamma \to \gamma \) are an isotopy of the set \( \gamma \). Let be the desired disks.

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Proof of the Theorem 2. Let \( y: I \to A \) be a homeomorphism of the segment \( I \) onto the arc \( A \). In order to prove an extension of this homeomorphism to a homeomorphism \( \approx \) of \( O \) onto itself there will be given two isomorphic decompositions \( P_1 \) and \( P_2 \) of \( O \) into tame 3-cells, tame solid tori and points of \( I \) or \( A \) respectively.

\[ \text{Standard decomposition} \]

Let \( K_m = 0, 1, 2, \ldots \), be the set
\[ \left( \frac{m + 1}{2} \right)^2 + \frac{m + 1}{2} \leq 1 \]

Notice that \( K_m = \varphi \), \( K_m+1 \subset \text{Int} K_m \) and \( \bigcap_{m=0}^{\infty} K_m = I_0 \).

For each fraction of the form \( p/m \) in lowest terms and with \( 1 < p/m \leq 1 \)
\[ E_{p/m} = \left[ \left( x, y, z \right) \mid K_{p/m} \right] = \varphi(p/m) \]

Note that \( E_{p/m} \) is a disk which spans \( K_m \) and \( E_{p/m} \cup E_{p/m+1} \cup E_{p/m+2} \cup E_{p/m+3} \cup \cdots \cup E_{p/m+1} \) separates \( K_m \) into \( 2^m \) pieces. The closure of each component of \( K_m \) is a tame solid torus. The elements of \( K_m \) are the points of \( I_0 \) these 3-cells, and these solid tori.

Curvilinear decomposition \( P_2 \). Adopt the following notation.

The number \( s \) will stand for both the point \( (s, 0, 0) \) of \( I_0 \) and for its image in \( A \) under the homeomorphism \( \gamma \). This should cause no confusion. With the aid of Lemma 3 we construct disks \( D_{p/m} \) which meet the arc \( A \) at the points \( p/m \) and are the counterparts of the disks \( E_{p/m} \).

Let \( D_m \) be the disk \( D_m \) promised by Lemma 3. Let \( t_i \) be the number suggested by Condition 4 of that lemma such that no pair of \( \{ p_1 \} \cap D_0 = 0 \), \( D_0 \cap D_1 \cap D_2 \cap C_0 \) have a point in common. Then let \( D_{p/m} = D_{p/m} \cap D_t \) and \( D_{p/m} = D_{p/m} \cap C_t \).

Let \( t_i \) be a number such that \( t_i < t_i < 1, 1/2 < t_i \), and no pair of \( D_{p/m}, D_{p/m-1}, D_{p/m+1}, D_{p/m+2}, D_{p/m+3} \) of Lemma 3 intersect in \( C_t \). Then \( D_{p/m} \cap D_{p/m+1} \cap D_{p/m+2} \cap D_{p/m+3} \) are the intersections of the appropriate \( D_{p/m} \) and \( C_t \).

Continuing in this fashion we describe a monotone increasing sequence of numbers \( s_0 = 0, s_1, s_2, \ldots \), converging to \( 1 \) such that \( D_{p/m} \cap D_{p/m+1} \cap C_0 = 0 \) if \( s < s_0 < s_0 \), \( s < s_0 < s' \), \( s < s_0 < s' \), \( s < s_0 < s' \).

The elements of \( K_m \) are the points of \( A \) and the closure of the component of \( (C_0 \cup C_0) \) and the \( D_{p/m} \). Note that these elements are either points, tame 3-cells, or tame solid tori. The decomposition \( P_3 \) is isomorphic to the decomposition \( P_2 \).

Extending the homeomorphism \( \gamma \). Let \( \gamma: I_0 \to \text{Bd} Q \to A \to \text{Bd} Q \) be a homeomorphism such that \( \gamma_0 \) is an extension of the homeomorphism mentioned in the statement of Theorem 2. \( \gamma_0: \text{Bd} Q \to A \to \text{Bd} Q \) and \( \gamma_0 \) takes the right half of \( \text{Bd} Q \) into the closure of the component of \( Q \). \( D_0 \) containing the point \( (1, 0, 0) \) of \( A \). Extend \( \gamma_0 \) to a homeomorphism \( \gamma: I_0 \to \text{Bd} Q \to A \to \text{Bd} Q \).

Now extend \( \gamma_0 \) to a homeomorphism \( \gamma: I_0 \to \text{Bd} Q \to A \to \text{Bd} Q \). We continue extending \( \gamma \) so
that in general $\chi_i$ takes $(E_{\chi_i} - K_{\chi_i})$ onto $(Q - C_{\chi_i})$, $L_i$ onto $A_i$, and $F_{\chi_i}$ onto $D_{\chi_i}$ if $a_i \leq a_i'$. The limit of the $\chi_i$'s is the required extension taking $Q$ onto itself.

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Brou par la Réduction le 12-6-1963

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On a special metric and dimension*

by

J. Nagata (Warszawa)

Once we have characterized [3] a metric space of covering dimension $\leq n$ by means of a special metric as follows:

A metric space $R$ has dim $\leq n$ if and only if we can introduce a metric $g$ in $R$ which satisfies the following condition:

For every $\varepsilon > 0$ and for every $n + 3$ points $x, y_1, \ldots, y_{n+2}$ of $R$ satisfying

$$g(S_{\varepsilon}(x), y_i) < \varepsilon, \quad i = 1, \ldots, n + 2$$

there is a pair of indices $i, j$ such that

$$g(y_i, y_j) < \varepsilon \quad (i \neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows:

A separable metric space $R$ has dim $\leq n$ if and only if we can introduce a totally bounded metric $g$ in $R$ which satisfies the following condition:

For every $n + 3$ points $x, y_1, \ldots, y_{n+2}$ in $R$, there is a triplet of indices $i, j, k$ such that

$$g(y_i, y_j) \leq g(x, y_k) \quad (i \neq j).$$

The former theorem is not so smart though it is valid for every metric space. The problem of generalizing the latter theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

**Theorem.** A metric space $R$ has dim $\leq n$ if and only if we can introduce a metric $g$ in $R$ which satisfies the following condition:

For every $n + 3$ points $x, y_1, \ldots, y_{n+2}$ in $R$, there is a pair of indices $i, j$ such that

$$g(y_i, y_j) \leq g(x, y_i) \quad (i \neq j).$$

* The content of this paper is a development in detail of our brief note On a special metric characterizing a metric space of dim $\leq n$, Proc. of Japan Acad. 39 (1963).

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(1) $S_{\varepsilon}(x) = \{y : g(x, y) < \varepsilon/2\}.$

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13