a connection form on E, we define the connection form ω' on E' by setting it to be equal $f^{-1*}\omega$ on $f(E)$ and extending it by means of formulae (23) of section 5.1. Then the values of ω' on $f'(T(E))$ are contained in $h(\mathfrak{g})$, and π_x is simply identical with h^{-1} (it does not depend on x in this case). The induced connection coincides with the given connection ω on E.

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An arc is tame in 3-space if and only if it is strongly cellular *

by

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A set Z in Euclidean n-space E^n is *tame* if there exists such a homeomorphism f of E^n onto itself that $f(Z)$ is a polyhedron. There are known some necessary and sufficient conditions of tameness of an arc in E^3, e.g. [4] and [5]. We shall give here another one based on the reinforced notion of a cellular set [2]. A set Z in E^n is *strongly cellular* if there is an n-cell C in E^n and a homotopy h: $C\times I\to C$ such that, if $h_t(x) = h(x, t)$ and $S = \operatorname{Bd} C$ is the boundary of C, then

(1) $h_0 =$ identity mapping and $h_t|Z =$ identity for all t,

(2) $h_t|S =$ homeomorphism for $t<1$,

(3) $h_t(S)\cap h_{t'}(S) = 0$ for $t\neq t'$,

(4) $h_1(C) = Z$.

The set Z will be said to be a *strong deformation retract* of the cell C. By M. Brown's generalization of the Schoenflies Theorem [2] there will be no loss of generality if the cell C is assumed to be a ball.

THEOREM 1. *An arc is tame in E^3 if and only if it is strongly cellular.*

COROLLARY. *An arc A in E^3 is tame if and only if there are two concentric balls B_0 and B, $B_0\subset \operatorname{Int} B$, and a mapping f of B into E^3 such that*

(1) $f|B-B_0 =$ homeomorphism,

(2) $f(B_0) = A$.

Proof of the Corollary. It is obvious that the conditions of the Corollary are necessary. In order to prove that they are also sufficient let us observe first that $f(B)$ is a 3-cell by M. Brown's Theorem 1 of [2]. Then assume $C = f(B)$ and next consider the homotopy r: $B\times I\to B$ retracting B to B_0. Now define the homotopy h: $C\times I\to C$ in the following way:

$$h(x,t) = \operatorname{fr}[f^{-1}(x), t] \quad \text{for} \quad x\in C-A,$$
$$h(x,t) = x \quad \text{for} \quad x\in A.$$

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It is easy to check that A is a strong deformation retract of C and so the proof has been reduced to the Theorem 1.

QUESTION. Is the Corollary still valid if the set B_0 is assumed to be an acyclic continuum?

The Theorem 1 is an easy corollary of the following one:

THEOREM 2. *Let Q be the ball $x^2+y^2+z^2 \leqslant 4$, I_0 be the segment $-1 \leqslant x \leqslant 1$, $y = z = 0$ and let A be an arc which is a strong deformation retract of Q. Then any homeomorphism of I_0 onto A can be extended to a homeomorphism of Q onto itself.*

Proof of the Theorem 2 will be preceded by some lemmas. Let the symbols Z, C, S and h keep the meaning they got in the definition of strong cellularity.

Remark. If X and Y are disjoint subsets of S and $h_1(X) \cap h_1(Y) = 0$ then

$$h(X \times I) \cap h(Y \times I) = 0 .$$

Let henceforth $Z = A$ and $f = h_1|S$.

LEMMA 1. *If B is a continuum in A then $f^{-1}(B)$ is a continuum in S.*

Proof. Suppose $f^{-1}(B) = B' = X \cup Y$ and $\bar{X} \cap \bar{Y} = 0$. Since the 0-dimensional homology group $H_0(B') \neq 0$, then by a standard duality theorem $H_{n-1}(S-B') \neq 0$ and there exist two cycles γ^0 in B' and γ_2^{n-1} in $S-B'$ which are linked in S. They can be chosen so that the carrier of γ^0 is the set $\{x, y\}$ and the carrier of γ_2^{n-1} is a continuum M. Then $F = f(M)$ is an arc or a point and does not intersect B.

Denote by $f(x)f(y)$ the subarc of B with the end-points $f(x)$ and $f(y)$. Let J_1 be the arc $h[\{x, y\} \times I] \cup f(x)f(y)$ and J_2 be an arc with the end points x and y which lies in $(E^n - C) \cup \{x, y\}$. The simple closed curve $J = J_1 \cup J_2$ carries a cycle γ_1^1 which links γ_2^{n-1} in E^n. On the other hand $h(M \times I)$ and J are disjoint by our Remark and therefore γ_2^{n-1} is homotopic in $E^n - J$ to a cycle lying in F; but every cycle bounds in F and so $\gamma_2^{n-1} \sim 0$ in $E^n - J$. This contradiction was a result of our false assumption that $f^{-1}(B)$ is not connected.

Further considerations will be restricted to the 3-space E^3.

LEMMA 2. *If B is a connected subset of A containing one end-point of A, then $f^{-1}(B)$ is a simply connected subset of S.*

Proof. Let $B \neq A$ for otherwise the Lemma 2 is obvious. If B is open in A then its complement B' is an arc or a point. Therefore $f^{-1}(B') = S - f^{-1}(B)$ is a connected set by Lemma 1. Hence $f^{-1}(B)$ is simply connected. If B is closed in A then B' is open and $f^{-1}(B')$ is connected; hence $f^{-1}(B)$ is simply connected.

Let $S_t = h_t(S)$ and C_t be the 3-cell bounded by S_t in E^3.

LEMMA 3. *For any interior point x of the arc A there exists a topological disk D_x in C such that the following conditions are satisfied:*

(1) *$D_x \cap S_t$ is a simple closed curve for $t < 1$, and the point x for $t = 1$.*

(2) *D_x cuts the cell C between the end-points of A.*

(3) *D_x is locally tame at every point except perhaps at x.*

(4) *For any two disks D_x and D_y, $x \neq y$, there is a number t_0 such that $0 \leqslant t_0 < 1$ and $C_{t_0} \cap D_x \cap D_y = 0$.*

(5) *If $0 \leqslant t_0 < t_1 < 1$ and W is the closure of $C_{t_0} - C_{t_1}$, Q' is the closure of any component of $W - D_x$ and P is the closure of that component of $C_{t_0} - (D_x \cup D_y)$ which contains no end-points of A, then Q' is a tame 3-cell and $T = W \cap P$ is a tame solid torus.*

Proof. Construction of D_x. We adopt the following notation: Let a, b be the end-points of A; $B_x = [a, x)$ be a closed-open subarc of A; $G_x = f^{-1}(B_x)$ is a simply connected domain in S with the boundary $\operatorname{Bd} G_x$ contained in $f^{-1}(x)$; $F = f^{-1}(a)$ is a closed, simply connected set in G_x; $R_x = G_x - F$ is an open annulus.

Define a sequence $\{J_m\}$ of simple closed curves in R_x as follows: Let $J_1 = J_x$ be a simple closed curve which cuts G_x between F and $\operatorname{Bd} G_x$ and such that the distance $\varrho(J_1, \operatorname{Bd} G_x) < 1$. Suppose that J_m has been already defined for some $m \geqslant 1$. Let J_{m+1} be a simple closed curve which cuts G_x between J_m and $\operatorname{Bd} B_x$ and such that $\varrho(J_{m+1}, \operatorname{Bd} B_x) < 1/(m+1)$.

Denote the annulus in G_x bounded by J_m and J_{m+1} by R_m and construct the following deformations:

a. A sequence of isotopies $g^{(x)m}\colon J_m \times [(m-1)/m, m/(m+1)] \to R_m$, $m = 1, 2, 3, \ldots$, simply denoted by g^m and such that if $g_t^m(p) = g^m(p, t)$, then

$$g^m_{(m-1)/m}(p) = p , \quad g^m_{m/(m+1)}(J_m) = J_{m+1} .$$

b. An isotopic deformation $g^{(x)}\colon J_x \times [0, 1) \to R_x$ such that

$$g_t^{(x)}(p) = g_t^m g^{m-1}_{(m-1)/m} \ldots g^1_{1/2}(p) \quad \text{for} \quad \frac{m-1}{m} \leqslant t \leqslant \frac{m}{m+1} \text{ and } m = 1, 2, 3, \ldots$$

c. A pseudoisotopy $\psi_x\colon J_x \times I \to C$ such that

$$\psi_x(p, t) = h_t g_t^{(x)}(p) \quad \text{for} \quad 0 \leqslant t < 1 ,$$
$$\psi_x(p, 1) = x .$$

Then $D_x = \psi_x(J_x \times I)$ is the desired disk. It is evident that D_x satisfies condition (1). Now we check the other ones.

Condition (2). Let Γ be that component of $C - D_x$ which contains the point a. It follows from the construction that $\bar{\Gamma} \subset \{x\} \cup h[f^{-1}(B_x) \times I]$, but $b \notin \{x\} \cup h[f^{-1}(B_x) \times I]$ and so there is no continuum in $C - D_x$ which could contain both a and b.

Condition (3). Let $p \in D_x - \{x\}$. There is a number s such that $0 < s < 1$ and $p \in D_x - C_s$. Let D be the closure of a component of $S - D_x$, D' be the closure of a component of $S_s - D_x$ and $R = \overline{D_x - C_s}$. Then

$$p \in R \subset D \cup R \cup D' = S'$$

and S' is a sphere. It can be easily shown that S' is bicollared and hence tame by M. Brown's Theorem 1 [2]. Therefore D_x is locally tame at p.

Condition (4). By construction $D_x = \psi_x(J_x \times I)$ and similarly $D_y = \psi_y(J_y \times I)$. Notice that $\psi_z(J_z \times 1) = z$ for $z = x, y$. Given disjoint neighbourhoods U_x and U_y of x and y respectively we can choose a number t_0 such that $0 \leqslant t_0 < 1$ and $\psi_z(J_z \times [t_0, 1]) \subset U_z$ for $z = x, y$. Then t_0 is the right number.

Condition (5). It is easy to see that Q' is a tame 3-cell because its boundary is a tame sphere by much the same reason that S' is one.

In order to prove that T is a tame solid torus it will be sufficient to show that its boundary is a tame torus and that there are two tame disjoint disks in T which span $\operatorname{Bd} T$.

a. $\operatorname{Bd} T$ *is a tame torus*: It is clear that $\operatorname{Bd} T$ is a torus for

$$\operatorname{Bd} T = T \cap (D_x \cup D_y \cup S_{t_0} \cup S_{t_1}) .$$

Also it is quite obvious that $\operatorname{Bd} T$ is locally tame at the interior points of the sets $T \cap D_z$, $z = x, y$, and $T \cap S_{t_i}$, $i = 0, 1$. And at the boundary points of these sets $\operatorname{Bd} T$ is locally tame by Doyle's theorem concerning the unions of cell pairs in E^3 [3]. Hence, by the theorem of Bing and Moise [1], $\operatorname{Bd} T$ is tame.

b. *There are two disjoint, tame disks* D_1, D_2 *in* T *such that*

$$D_i \cap \operatorname{Bd} T = \operatorname{Bd} D_i, \quad i = 1, 2 .$$

By condition (4) the simple closed curves $J_{z,t_0} = g_{t_0}^{(z)}(J_z)$, $z = x, y$, are disjoint subsets of S. In the annulus bounded by these curves choose two disjoint arcs A_1, A_2 having with each curve only one end-point in common. The mappings $g_t^{(x)}, g_t^{(y)}$ define an isotopy of the set $J_{x,t_0} \cup J_{y,t_0}$ which can be extended to an isotopy

$$g^*\colon (J_{x,t_0} \cup J_{y,t_0} \cup A_1 \cup A_2) \times [t_0, t_1] \to S .$$

Let $hg^*(p, t)$ stand for $h[g^*(p, t), t]$. Then the sets $D_i = hg^*(A_i \times [t_0, t_1])$, $i = 1, 2$, are the desired disks.

Proof of the Theorem 2. Let $\chi\colon I_0 \to A$ be a homeomorphism of the segment I_0 onto the arc A. In order to produce an extension of this homeomorphism to a homeomorphism of Q onto itself there will be given two isomorphic decompositions P_1 and P_2 of Q into tame 3-cells, tame solid tori and points of I_0 or A respectively.

Standard decomposition P_1. Let K_m, $m = 0, 1, 2, \ldots$, be the set

$$\left(\frac{m+1}{m+2}\,x\right)^2 + \left(\frac{m+1}{2}\,y\right)^2 + \left(\frac{m+1}{2}\,z\right)^2 \leqslant 1 .$$

Notice that $K_0 = Q$, $K_{m+1} \subset \operatorname{Int} K_m$ and $\bigcap_{m=0}^{\infty} K_m = I_0$.

For each fraction of the form $p/2^q$ in lowest terms and with $-1 < p/2^q < 1$, let

$$E_{p/2^q} = \{(x, y, z) \in K_q \mid x = p/2^q\} .$$

Note that $E_{p/2^q}$ is a disk which spans K_q and $E_0 \cup E_{-1/2} \cup E_{1/2} \cup E_{-3/4} \cup \ldots \cup E_{1-2^{-q}}$ separates K_q into 2^{1+q} pieces. The closure of each component of $K_q - K_{q+1}$ minus the sum of the E's is either a tame 3-cell or a tame solid torus. The elements of P_1 are the points of I_0, these 3-cells, and these solid tori.

Curvilinear decomposition P_2. Adopt the following notation. The number s will stand for both the point $(s, 0, 0)$ of I_0 and for its image in A under the homeomorphism χ. This should cause no confusion. With the aid of Lemma 3 we construct disks $D'_{p/2^q}$ which meet the arc A at the points $p/2^q$ and are the counterparts of the disks $E_{p/2^q}$.

Let D'_0 be the disk D_0 promised by Lemma 3. Let t_1 be the number suggested by Condition 4 of that lemma such that no pair of $D_{-1/2} \cap C_{t_1}$, $D_0 \cap C_{t_1}$, $D_{1/2} \cap C_{t_1}$ have a point in common. Then let $D'_{-1/2} = D_{-1/2} \cap C_{t_1}$ and $D'_{1/2} = D_{1/2} \cap C_{t_1}$.

Let t_2 be a number such that $t_1 < t_2 < 1$, $1/2 < t_2$, and no pair of $D_{-3/4}$, $D_{-1/2}$, $D_{-1/4}$, D_0, $D_{1/4}$, $D_{1/2}$, $D_{3/4}$ of Lemma 3 intersect in C_{t_2}. Then $D'_{-3/4}$, $D'_{-1/4}$, $D'_{1/4}$, $D'_{3/4}$ are the intersections of the appropriate D's with C_{t_2}.

Continuing in this fashion we describe a monotone increasing sequence of numbers $t_0 = 0, t_1, t_2, \ldots$ converging to 1 such that $D_{a/2^b} \cap D_{c/2^d} \cap C_{t_i} = 0$ if $a/2^b \neq c/2^d$, $2^b \leqslant 2^i$, and $2^d \leqslant 2^i$. Then $D'_{p/2^q} = D_{p/2^q} \cap C_{t_q}$.

The elements of P_2 are the points of A and the closure of the components of the $(C_{t_i} - C_{t_{i+1}})$'s minus the D''s. Note that these elements are either points, tame 3-cells, or tame solid tori. The decomposition P_1 is isomorphic to the decomposition P_2.

Extending the homeomorphism χ. Let $\chi'_0\colon I_0 \cup \operatorname{Bd} Q \to A \cup \operatorname{Bd} Q$ be a homeomorphism such that χ'_0 is an extension of the homeomorphism mentioned in the statement of Theorem 2, $\chi'_0\colon \operatorname{Bd} E_0 \to \operatorname{Bd} D'_0$, and χ'_0 takes the right half of $\operatorname{Bd} Q = \operatorname{Bd} K_0$ into the closure of the component of $Q - D'_0$ containing the point $(1, 0, 0)$ of A. Extend χ'_0 to a homeomorphism $\chi_0\colon I_0 \cup \operatorname{Bd} Q \cup E_0 \to A \cup \operatorname{Bd} Q \cup D'_0$.

Now extend χ_0 to a homeomorphism $\chi_1\colon I_0 \cup \overline{(K_0 - K_1)} \cup E_0 \cup E_{-1/2} \cup E_{1/2} \to A \cup \overline{(Q - C_{t_1})} \cup D'_0 \cup D'_{-1/2} \cup D'_{1/2}$. We continue extending χ so that in general χ_i takes $(\overline{K_0-K_{i+1}})$ onto $(\overline{Q-C_{t_{i+1}}})$, I_0 onto A, and $E_{p/2^q}$ onto $D'_{p/2^q}$ if $2^q \leqslant 2^i$. The limit of the χ_i's is the required extension taking Q onto itself.

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On a special metric and dimension *

by

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Once we have characterized [3] a metric space of covering dimension $\leqslant n$ by means of a special metric as follows:

A metric space R has dim $\leqslant n$ *if and only if we can introduce a metric* ϱ *in R which satisfies the following condition:*

For every $\varepsilon > 0$ *and for every* $n+3$ *points* $x, y_1, \ldots, y_{n+2}$ *of R satisfying*

$$\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon, \quad i = 1, \ldots, n+2\ (^1)$$

there is a pair of indices i, j *such that*

$$\varrho(y_i, y_j) < \varepsilon \quad (i \neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows:

A separable metric space R has dim $\leqslant n$ *if and only if we can introduce a totally bounded metric* ϱ *in R which satisfies the following condition:*

For every $n+3$ *points* $x, y_1, \ldots, y_{n+2}$ *in R, there is a triplet of indices* i, j, k *such that*

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_k) \quad (i \neq j).$$

The former theorem is not so smart though it is valid for every metric space. The problem of generalizing the latter theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

THEOREM. *A metric space R has* dim $\leqslant n$ *if and only if we can introduce a metric* ϱ *in R which satisfies the following condition:*

For every $n+3$ *points* $x, y_1, \ldots, y_{n+2}$ *in R, there is a pair of indices* i, j *such that*

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_j) \quad (i \neq j).$$

* The content of this paper is a development in detail of our brief note *On a special metric characterizing a metric space of* dim $\leqslant n$, Proc. of Japan Acad. 39 (1963).

(1) $S_{\varepsilon/2}(x) = \{y \mid \varrho(x, y) < \varepsilon/2\}$.