On induced connections

by

A. Goetz (Wrocław)

The purpose of this paper [1] is to give a detailed exposition of the method of inducing connections in immerced principal fibre bundles announced in the author's paper [3].

The first chaper contains a very brief summary of the main properties of general notions, and the intention is to make clear the use of notations [2].

The second chapter deals with the main subject of the paper. Section 6 contains the definition of the immersion of principal fibre bundles and some geometrical examples, in section 7 the notion of the invariant projection of Lie algebras is introduced and its relation to the notion of weak reductivity of homogeneous spaces is given. Examples of invariant projections are the subject of section 9. Section 10 deals with the change of the immersion. In section 11 some generalizations are given, which extend the field of applicability of our scheme.

I

§ 1. Differentiable manifolds.

1.1. The differentiable manifolds as well as all mappings and functions considered in this paper are supposed to be of class C^∞. The linear space of vectors tangent to the manifold M at $x \in M$ is denoted by $T_x(M)$, the space of all tangent vectors by $T(M)$. Tangent vectors and vector fields will be denoted by the Greek letter τ with subscripts; if necessary the "contact point" x will also be indicated as a subscript.

A *tangent vector* at $x_0 \in M$ is interpreted either as a class of differentiable paths in M tangent to each other at x_0 or as a linear functional defined on the class of differentiable functions in M and satisfying the condition

$$\tau_{x_0}(F\cdot G) = (\tau_{x_0}F)\cdot G(x_0) + F(x_0)\cdot(\tau_{x_0}G)\,, \tag{1}$$

(1) The present paper was written partly during the author's stay at the University of Notre Dame, Notre Dame, Indiana, U.S.A.

(2) For more information, see Lichnerowicz [7] or Nomizu [8]. Our notations are near, but not identical, to those of Lichnerowicz.

where F and G are differentiable real functions defined in a neighbourhood of x_0. A vector field (x_0 variable) can be interpreted as a linear operation, called an *infinitesimal transformation*, which satisfies the condition

$$\tau(F\cdot G) = (\tau F)\cdot G + F\cdot(\tau G)\,. \tag{2}$$

Given two infinitesimal transformations τ_1 and τ_2, we define their *bracket* $[\tau_1, \tau_2]$ as the infinitesimal transformation $\tau = \tau_1\tau_2 - \tau_2\tau_1$, where $\tau_1\tau_2$ is the composition of the transformations τ_2 and τ_1.

1.2. Given a mapping $\varphi\colon M\to M'$, we denote by $\varphi^{\cdot}$ the derived mapping $T(M)\to T(M')$ of the tangent spaces. This mapping maps linearly each $T_x(M)$ into $T_{\varphi(x)}(M')$. If linearly independent vectors are mapped into linearly independent vectors by $\varphi^{\cdot}$, the mapping φ is called *regular*.

The derived mapping maps brackets of vector fields into brackets of vector fields:

$$\varphi^{\cdot}([\tau_1, \tau_2]) = [\varphi^{\cdot}(\tau_1), \varphi^{\cdot}(\tau_2)]\,.$$

1.3. Let $\varphi\colon M_1\times M_2\to M$ be a mapping of the Cartesian product of M_1 and M_2 into M. Consider the mappings $\varphi_x\colon M_2\to M$ defined by the formula

$$\varphi_x(y) = \varphi(x, y)\,, \qquad x\in M_1,\ y\in M_2,$$

and $\varphi_y\colon M_1\to M$ defined by the formula

$$\varphi_y(x) = \varphi(x, y)\,, \qquad x\in M_1,\ y\in M_2,$$

The derived mappings are respectively

$$\varphi_x^{\cdot}\colon T(M_2)\to T(M) \quad\text{and}\quad \varphi_y^{\cdot}\colon T(M_1)\to T(M)\,.$$

Now we can define the mappings

$$d_1\varphi\colon T(M_1)\times M_2\to T(M) \quad\text{and}\quad d_2\varphi\colon M_1\times T(M_2)\to T(M)$$

by setting

$$d_1\varphi(\tau_1, y) = \varphi_y^{\cdot}(\tau_1) \quad\text{and}\quad d_2\varphi(x, \tau_2) = \varphi_x^{\cdot}(\tau_2)\,,$$

where $x\in M_1$, $y\in M_2$, $\tau_1\in T(M_1)$, $\tau_2\in T(M_2)$.

The projections of the Cartesian product $M_1\times M_2$ onto the axes are denoted by j_1 and j_2, i.e.

$$j_1(x, y) = x \quad\text{and}\quad j_2(x, y) = y\,.$$

§ 2. Exterior differential forms.

2.1. An *exterior differential p-form* a on an n-manifold M is a function which is defined in $\bigcup_{x\in M}(T_x(M))^p$ [3] and, for a fixed x, is linear in all variables and squew symmetric, i.e.

$$a(\tau_{i_1}, \dots, \tau_{i_p}) = \varepsilon_{i_1\dots i_p}\cdot a(\tau_1, \dots, \tau_p)\,, \qquad i_k = 1, 2, \dots, p\,,$$

[3] X^p denotes the pth Cartesian power of X.

where

$$\varepsilon_{i_1\dots j_p} = \varepsilon^{i_1\dots i_p} = \begin{cases} 1 & \text{if } i_1, \dots, i_p \text{ is an even permutation of } 1, \dots, p,\\ -1 & \text{if it is an odd permutation,}\\ 0 & \text{if } i_k = i_j \text{ for some pair } k\neq j.\end{cases} \tag{3}$$

Moreover, if $\tau_1, \dots, \tau_p$ are differentiable vector fields, $a(\tau_1, \dots, \tau_p)$ is a differentiable function.

A function in M is considered as a 0-form; a 1-form is called also a linear differential form. The latter can be regarded as a mapping $a\colon T(M)\to R$, R denoting the field of real numbers.

2.2. The *exterior product* $a\wedge\beta$ of a p-form a and a q-form β is a $p+q$-form defined by the formula [4]

$$(a\wedge\beta)(\tau_1, \dots, \tau_{p+q}) = \frac{1}{p!\,q!}\varepsilon^{i_1\dots i_{p+q}}a(\tau_{i_1}, \dots, \tau_{i_p})\,\beta(\tau_{i_{p+1}}, \dots, \tau_{i_{p+q}})\,. \tag{4}$$

The exterior product satisfies the associative and distributive laws and the following commutative law

$$a\wedge\beta = (-1)^{pq}\beta\wedge a\,.$$

Every exterior differential p-form on an n-manifold M can be represented locally (in some neighbourhood $U\subset M$) in the form

$$a = \frac{1}{p!}a_{i_1\dots i_p}\vartheta^{i_1}\wedge\dots\wedge\vartheta^{i_p}, \qquad i_k = 1, 2, \dots, n\,,$$

where $\vartheta^1, \dots, \vartheta^n$ are n linear differential forms which are linearly independent in each point $x\in M$, and $a_{i_1\dots i_p}$ are functions defined in U. In particular, in a fixed coordinate neighbourhood of M we can choice $\vartheta^i = dx^i$ (the differential of the ith coordinate of x) and obtain

$$a = \frac{1}{p!}a_{i_1\dots i_p}\,dx^{i_1}\wedge\dots\wedge dx^{i_p}\,.$$

2.3. If $\varphi\colon M\to M'$ is a mapping and a' a p-form on M', we define the induced p-form $\varphi^* a'$ on M by setting

$$(\varphi^* a')(\tau_1, \dots, \tau_p) = a'\big(\varphi^{\cdot}(\tau_1), \dots, \varphi^{\cdot}(\tau_p)\big)\,, \qquad \tau_1, \dots, \tau_p\in T_x(M)\,. \tag{5}$$

In particular, if a' is a linear form regarded as a mapping $a'\colon T(M')\to R$, we have

$$\varphi^* a' = a'\circ\varphi^{\cdot}\colon T(M)\to R\,,$$

$\circ$ denoting the composition of mappings.

[4] The summation convention is used throughover the paper.

2.4. The *exterior differential* of a p-form α is the $p+1$-form $d\alpha$ whose values for $p+1$ vector fields $\tau_0, \ldots, \tau_p$ are

$$(6)\qquad d\alpha(\tau_0, \tau_1, \ldots, \tau_p) = \sum_{i=0}^{p} (-1)^i \tau_i \alpha(\tau_0, \ldots, \hat{\tau}_i, \ldots, \tau_p) - \sum_{i<j} \alpha([\tau_i, \tau_j], \tau_0, \ldots, \hat{\tau}_i, \ldots, \hat{\tau}_j, \ldots, \tau_p),$$

where the cap ˆ over a letter indicates that the letter is to be omitted.

In particular, the exterior differential of a 0-form (function) is identical with its total differential, and the exterior differential of a linear form α is defined by the formula

$$(7)\qquad d\alpha(\tau_1, \tau_2) = \tau_1 \alpha(\tau_2) - \tau_2 \alpha(\tau_1) - \alpha([\tau_1, \tau_2]) .$$

The exterior differentiation is a linear operation with the following properties:

$$(8)\qquad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

where p is the degree of the form α,

$$(9)\qquad dd\alpha = 0 .$$

If α is given in the form

$$\alpha = \frac{1}{p!} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} ,$$

then

$$(10)\qquad d\alpha = \frac{1}{p!} da^{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} .$$

The exterior differential of an induced form is identical with the induced form of the exterior differential of the original form:

$$(11)\qquad \varphi^* d\alpha' = d\varphi^* \alpha' .$$

§ 3. Lie groups.

3.1. The groups considered in this paper are *Lie groups*, i.e. groups G whose space is an analytic manifold, and the multiplication law is analytic. The dimension of the manifold is sometimes called the number of parameters of G.

The *Lie algebra* $\mathfrak{g}$ of an r-parametric group G is an r-dimensional vector space whose elements can be interpreted either as vectors of $T_e(G)$, e being the unity element of G, or as left invariant infinitesimal transformations (vector fields) on G. These elements will be denoted by the letter λ with indices.

A bracket $[\lambda_1, \lambda_2]$ of two left invariant infinitesimal transformations is itself a left invariant infinitesimal transformation, hence the bracket operation can be regarded as a kind of multiplication in $\mathfrak{g}$. This multiplication is linear in both multiplicands and satisfies the conditions

$$[\lambda_1, \lambda_2] = -[\lambda_2, \lambda_1] ,$$

$$[[\lambda_1, \lambda_2], \lambda_3] + [[\lambda_2, \lambda_3], \lambda_1] + [[\lambda_3, \lambda_1], \lambda_2] = 0 .$$

If G is a group of linear transformations of a vector space, we represent its elements by non degenerated matrices $\boldsymbol{A}$ with matrix multiplication as group operation. The Lie algebra is then an algebra of square matrices $\boldsymbol{M}$ (they can be also degenerated) with bracket operation

$$(12)\qquad [\boldsymbol{M}, \boldsymbol{N}] = \boldsymbol{MN} - \boldsymbol{NM} .$$

3.2. Any homomorphism $h\colon G \to G'$ of Lie groups induces a homomorphism, denoted by the same letter h, of their Lie algebras. If h is a monomorphism (or isomorphism) of groups so is also the induced mapping of Lie algebras.

In particular, the inner automorphism of G sending every element $g_1 \in G$ into $g g_1 g^{-1}$ induces a linear transformation of $\mathfrak{g}$ called the *adjoint transformation* and denoted by $\operatorname{adj}(g)$. This transformation is an automorphism of the Lie algebra. The image of $\lambda \in \mathfrak{g}$ by the mentioned transformation is denoted by $\operatorname{adj}(g) \cdot \lambda$. The mapping $g \to \operatorname{adj}(g)$ is a homomorphism of the group G into the group of linear transformations of $\mathfrak{g}$, called the *adjoint representation* of G. Hence

$$(13)\qquad \operatorname{adj}(g_1 g_2) = \operatorname{adj}(g_1) \operatorname{adj}(g_2)$$

and

$$(14)\qquad \operatorname{adj}(g^{-1}) = (\operatorname{adj}(g))^{-1} .$$

3.3. Let $\gamma\colon M \to G$ be a mapping of a manifold into a Lie group, and let λ be a fixed element of the Lie algebra $\mathfrak{g}$ of the group G.

THEOREM. *If the mapping $\chi\colon M \to \mathfrak{g}$ is defined by the formula*

$$\chi(x) = \operatorname{adj}(\gamma(x)^{-1}) , \qquad x \in M ,$$

then

$$(15)\qquad \chi^{\boldsymbol{\cdot}}(\tau) = -[\gamma(x)^{-1} \gamma^{\boldsymbol{\cdot}}(\tau), \operatorname{adj}(\gamma(x)^{-1}) \cdot \lambda] , \qquad x \in M,\ \tau \in T_x(M) ,$$

where $\gamma(x)^{-1} \gamma^{\boldsymbol{\cdot}}(\tau)$ denotes the element of $\mathfrak{g} = T_e(G)$ obtained from the tangent vector $\gamma^{\boldsymbol{\cdot}}(\tau)$ to G at $\gamma(x)$ by means of a left translation $\gamma(x)^{-1}$ into e.

Proof. Let us fix $x_0 \in M$ and take, for $\tau \in T_{x_0}(M)$, the path $x(t)$ tangent to τ at $t = 0$. Now, let us express $\gamma(x(t))^{-1}$ in the form

$$\gamma(x(t))^{-1} = \{\gamma(x_0)^{-1} \gamma(x(t))\}^{-1} \gamma(x_0)^{-1} .$$

Then we have

$$\chi(x(t)) = \operatorname{adj}(\{\gamma(x_0)^{-1} \gamma(x(t))\}^{-1}) \cdot \operatorname{adj}(\gamma(x_0)^{-1}) \cdot \lambda .$$

Since $\gamma(x_0)^{-1}\gamma(x(t))$ is a path in G tangent at e to $\gamma(x_0)^{-1}\gamma^{\cdot}(\tau) \in \mathfrak{g}$, and $\mathrm{adj}(\gamma(x_0)^{-1})\cdot\lambda$ is a fixed element of $\mathfrak{g}$, the tangent vector to the path $\chi(x(t))$ can be expressed in the form (5)

$$\chi^{\cdot}(\tau) = -[\gamma(x_0)^{-1}\gamma^{\cdot}(\tau),\ \mathrm{adj}(\gamma(x_0)^{-1})\cdot\lambda], \qquad \text{Q. E. D.}$$

3.4. We have to deal with exterior differential forms with values in a Lie algebra $\mathfrak{g}$. A *bracket* of two $\mathfrak{g}$-valued exterior differential forms η_1 and η_2 is defined in the same way as an exterior product provided the multiplication of values is replaced by the bracket operation. In particular, for $\mathfrak{g}$-valued linear differential forms on M, we have

$$[\eta_1, \eta_2](\tau_1, \tau_2) = [\eta_1(\tau_1), \eta_2(\tau_2)] - [\eta_1(\tau_2), \eta_2(\tau_1)] \tag{16}$$

for $\tau_1, \tau_2 \in T_x(M)$, whence it follows immediately that formula

$$[\eta_1, \eta_2] = [\eta_2, \eta_1]. \tag{17}$$

holds for $\mathfrak{g}$-valued linear differential forms.

If $\lambda_1, \ldots, \lambda_r$ form a basis of $\mathfrak{g}$, then every $\mathfrak{g}$-valued form can be expressed as

$$\eta = \eta^a\lambda_a, \qquad a = 1, 2, \ldots, r,$$

where $\eta^1, \ldots, \eta^r$ are number valued forms on M. In this case the bracket operation is given by the formula

$$[\eta_1, \eta_2] = \eta_1^a \wedge \eta_2^\beta[\lambda_a, \lambda_\beta].$$

3.5. If η is a $\mathfrak{g}$-valued linear differential form on M, and γ: $M \to G$, then

$$\vartheta = \mathrm{adj}(\gamma^{-1})\eta$$

is also a $\mathfrak{g}$-valued linear differential form, and, as it is not difficult to calculate with help of formulas (15), (7) and (16),

$$d\vartheta = \mathrm{adj}(\gamma^{-1})\cdot d\eta - [\mathrm{adj}(\gamma^{-1})\cdot\eta, \gamma^{-1}d\gamma], \tag{18}$$

where $\gamma^{-1}d\gamma$ is the linear form with values

$$(\gamma^{-1}d\gamma)(\tau) = \gamma^{-1}(x)\gamma^{\cdot}(\tau), \qquad x \in M,\ \tau \in T_x(M). \tag{19}$$

3.6. The $\mathfrak{g}$-valued linear differential form μ on G, whose value for the vector $\tau \in T_g(G)$ is $g^{-1}\tau \in T_e(G) = \mathfrak{g}$ or, in another interpretation of Lie algebras, the left invariant vector field containing τ, is clearly left invariant. We shall call it the $\mathfrak{g}$-*valued Maurer-Cartan form* of G.

Given a basis $\lambda_1, \ldots, \lambda_r$ of the Lie algebra, μ can be represented in the form

$$\mu = \mu^a\lambda_a, \qquad a = 1, 2, \ldots, r,$$

(5) See Pontrjagin [11], chap. X, § 54 F) or, in the English translation of the first edition, [12], chap. IX, § 52 B).

where $\mu^1, \ldots, \mu^r$ are left-invariant number valued differential forms on G which form a dual basis of Maurer-Cartan forms on G. Consequently, the following generalized Maurer-Cartan equation is satisfied (6)

$$d\mu = -\tfrac{1}{2}[\mu, \mu]. \tag{20}$$

The form $\gamma^{-1}d\gamma$ is nothing else than the $\mathfrak{g}$-valued form $\gamma^*\mu$ induced by the mapping γ from the $\mathfrak{g}$-valued Maurer-Cartan form μ on G. Therefore,

$$d(\gamma^{-1}d\gamma) = -\tfrac{1}{2}[\gamma^{-1}d\gamma, \gamma^{-1}d\gamma]. \tag{21}$$

§ 4. Principal fibre bundles.

4.1. A differentiable *principal fibre bundle* $E(M, G, p)$ is an $n+r$-manifold in which an r-parameter group G acts as a transformation group by right multiplication. For this multiplication we use the notations

$$zg = D_g(z) = \Psi(z, g), \qquad z \in E,\ g \in G;$$

Ψ is a mapping $E \times G \to E$. This multiplication satisfies the following conditions:

i. $(zg_1)g_2 = z(g_1g_2)$,

ii. The space $M = E/G$ of orbits, called the *basic space* of the bundle, is an n-manifold.

The natural mapping of E onto M is called the *projection* of the bundle and denoted by p. The set $p^{-1}(x)$ for $x \in M$ is called the *fibre* over x.

iii. There exist a covering $\{U_\varkappa\}$ of M and a system of mappings $\zeta_\varkappa$: $U_\varkappa \to E$ such that

1. $p(\zeta_\varkappa(x)) = x$, $x \in U_\varkappa$.
2. The mapping $(x, g) \to \zeta_\varkappa(x)g$ is a diffeomorphism of $U_\varkappa \times G$ onto $p^{-1}(U_\varkappa)$. (Consequently, the fibres are diffeomorphic to G.)
3. For $x \in U_\varkappa \cap U_\iota$ $\zeta_\iota(x) = \zeta_\varkappa(x)g_{\varkappa\iota}(x)$, where $g_{\varkappa\iota}$ is a differentiable mapping $U_\varkappa \cap U_\iota \to G$.

4.2. The mapping $d_2\Psi$: $E \times T(G) \to T(E)$ restricted to the subset $E \times T_e(G) = E \times \mathfrak{g}$ will be denoted by ψ; we use also the notation $z\lambda$ for $\psi(z, \lambda)$, $z \in E$, $\lambda \in \mathfrak{g}$. The vector $z\lambda$ is tangent at z to the fibre over $p(z)$.

Conversely, every vector τ tangent to the fibre at $z \in E$ is of the form $z\lambda$ with a uniquely determined $\lambda \in \mathfrak{g}$. If $g(t)$ is a path in G tangent to λ at $t = 0$, then $z\lambda$ is the tangent vector of the path $zg(t)$ in E.

(6) See Cohn [1], chap. IV, § 4.5, formula (20).

§ 5. Infinitesimal connections in principal fibre bundles.

5.1. An *infinitesimal connection in* $E(M, G, p)$ is defined by means of a linear differential form ω on E with values in the Lie algebra $\mathfrak{g}$ of G. This form satisfies the conditions

(22) $$\omega(z\lambda) = \lambda, \qquad z \in E,\ \lambda \in \mathfrak{g},$$

(23) $$\omega(\tau g) = \operatorname{adj}(g^{-1})\,\omega(\tau), \qquad \tau \in T(E),\ g \in G.$$

The fulfilling of these conditions is equivalent to the commutativity of the diagram

(A) $$\begin{array}{ccc} T(E) & \xrightarrow{\ \omega\ } & \mathfrak{g} \\ {\scriptstyle D_g}\uparrow & & \uparrow{\scriptstyle \operatorname{adj}(g^{-1})} \\ T(E) & \xrightarrow{\ \omega\ } & \mathfrak{g} \\ & {\scriptstyle \psi}\nwarrow \quad \nearrow{\scriptstyle j_2} & \\ & E \times \mathfrak{g} & \end{array}$$

5.2. Vectors tangent to the fibre are called *vertical*, those which satisfy $\omega(\tau) = 0$ are called *horizontal*. The space $\mathcal{V}_z$ of vertical vectors at $z \in E$ and the space $\mathcal{H}_z$ of the horizontal vectors at z are complementary vector spaces which together span $T_z(E)$. Every vector $\tau \in T(E)$ can be represented in one and only one way as a sum of its horizontal and vertical parts $\tau = \mathcal{H}(\tau) + \mathcal{V}(\tau)$. Since $\omega\big(\mathcal{H}(\tau)\big) = 0$, we have $\omega(\tau) = \omega\big(\mathcal{V}(\tau)\big)$.

Curves in E are called horizontal if their tangent vector is horizontal in every point. A horizontal curve $z(t)$ whose projection is $p\big(z(t)\big) = x(t) \in M$ is called the *horizontal lifting* of $x(t)$. Every curve $x(t)$ in M has a horizontal lifting $z(t)$ which is uniquely determined by the condition $z(t_0) = z_0$, z_0 being arbitrarily fixed in the fibre over $x(t_0)$.

5.3. The *curvature form* of the infinitesimal connection is a $\mathfrak{g}$-valued tensorial 2-form defined by the formula

(24) $$\Omega(\tau_1, \tau_2) = d\omega\big(\mathcal{H}(\tau_1), \mathcal{H}(\tau_2)\big).$$

It can be expressed by the formula (7)

(25) $$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega].$$

II

§ 6. Immersion of principal fibre bundles.

6.1. Let M and M' be two manifolds, φ: $M \to M'$ a regular immersion and h: $G \to G'$ a monomorphism of the Lie group G into the Lie group G'. Now, let $E(M, G, p)$ and $E'(M', G', p')$ be two principal fibre bundles

(7) See, e.g., Nomizu [8] chap. II, § 4, or Lichnerowicz [7], § 35. The difference between our formula (25) and that of Lichnerowicz is a consequence of a different definition of the bracket of two linear forms. According to our definition (16) it is $[\omega, \omega](\tau_1, \tau_2) = [\omega(\tau_1), \omega(\tau_2)] - [\omega(\tau_2), \omega(\tau_1)] = 2[\omega(\tau_1), \omega(\tau_2)]$ which is twice the value used by Lichnerowicz.

with basic spaces M and M', groups G and G' and projections p and p', respectively. A mapping f: $E \to E'$ is called an *immersion of principal fibre bundles compatible with* φ and h if the following conditions are satisfied

(26) $$f(zg) = f(z)\,h(g), \qquad z \in E,\ g \in G,$$

(27) $$p'\big(f(z)\big) = \varphi\big(p(z)\big),$$

or, in other words, if the diagram

(B) $$\begin{array}{ccccc} E \times G & \xrightarrow{\Psi} & E & \xrightarrow{p} & M \\ \downarrow{\scriptstyle f\times h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} \\ E' \times G' & \xrightarrow{\Psi'} & E' & \xrightarrow{p'} & M' \end{array}$$

is commutative.

The commutativity of the left part of the above diagram yields the commutativity of the diagram

(C) $$\begin{array}{ccccc} E \times \mathfrak{g} & \xrightarrow{\psi} & T(E) & \xrightarrow{D_g} & T(E) \\ \downarrow{\scriptstyle f\times h_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ E' \times \mathfrak{g}' & \xrightarrow{\psi'} & T(E') & \xrightarrow{D_{h(g)}} & T(E'). \end{array}$$

Really, by passing to the derived mappings in the left part of (B) we obtain the diagram

$$\begin{array}{ccc} E \times T(G) & \xrightarrow{d_2\Psi} & T(E) \\ \downarrow{\scriptstyle f\times h_*} & & \downarrow{\scriptstyle f_*} \\ E' \times T(G') & \xrightarrow{d_2\Psi'} & T(E') \end{array}$$

which yields (C) after restriction to $E \times \mathfrak{g}$ and $E' \times \mathfrak{g}'$. The commutativity of the right-hand part of (C) becomes clear if we put the left part of (B) in the equivalent form

$$\begin{array}{ccc} E & \xrightarrow{D_g} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ E' & \xrightarrow{D_{h(g)}} & E'. \end{array}$$

6.2. Suppose that the principal fibre bundle $E'(M, G', p)$ can be reduced to a subgroup G of G', i.e. it contains a subspace E which is stable under right multiplication by elements of G and thus has the structure of a principal fibre bundle $E(M, G, p)$. In this case the inclusion mapping f: $E \to E'$ is an immersion compatible with the identity mapping id: $M \to M$ and the inclusion homomorphism i: $G \to G'$.

If φ is one-to-one, the general case can be reduced to the above mentioned. To this end it suffices to identify z with $f(z)$, x with $\varphi(x)$ and g with $h(g)$, and to reduce the principal fibre bundle E' to $p'^{-1}(M)$, where M is a subspace of M' now.

Since h is a monomorphism, G can always be identified with $h(G)$. Consequently, G can be considered as a subgroup of G', and h as the inclusion map. Although this leads to a slight simplification of formulas, the general approach seems to be more natural in some of the applications.

6.3. Let M be a real k-manifold, M' a real n-manifold ($k < n$) and φ a regular immersion $M \to M'$ (a k-dimensional surface in M'). The set of all linear k-frames on M forms a principal fibre bundle $E(M, G, p)$, where $G = GL(k, R)$, and the projection p maps every frame in $T_x(M)$ into its origin x.

Similarly, the set of all n-frames on M' forms a principal fibre bundle $E'(M', G', p')$ with $G' = GL(n, R)$. Let us define now the monomorphism $h\colon G \to G'$ by the formula

$$h(A) = \begin{pmatrix} A & 0 \\ 0 & E_{n-k} \end{pmatrix},$$

where A is a nondegenerated $k \times k$-matrix, and E_{n-k} is the $n-k \times n-k$ unity matrix.

Suppose now that a normalization of the surface φ is given, i.e. $n-k$ vector fields $e'_{k+1}, \ldots, e'_n$ on $\varphi(M)$, linearly independent and complementary to the tangent linear space $T(\varphi(M))$ in every point. Then the mapping $f\colon E \to E'$ which sends the frame consisting of vectors $e_1, \ldots, e_k$ in $T_x(M)$ into the frame consisting of the vectors $\varphi^{\cdot}(e_1), \ldots, \varphi^{\cdot}(e_k), e'_{k+1}, \ldots, e'_n$ in $T_{\varphi(x)}(M')$ is an immersion compatible with φ and h.

6.4. A more general immersion can be obtained if a mapping $\chi\colon T(M) \to T(M')$ is given which, when restricted to $T_x(M)$ with fixed x, is a non-degenerated linear mapping $T_x(M) \to T_{\varphi(x)}(M')$, and the normalization is complementary to $\chi(T_x(M))$ in every point x. In this case the mapping $f\colon E \to E'$ which sends the frame consisting of vectors $e_1, \ldots, e_k$ into the frame consisting of $\chi(e_1), \ldots, \chi(e_k), e'_{k+1}, \ldots, e'_n$ is an immersion compatible with φ and h of the preceding example [8].

The previous immersion is a particular case of the last one with $\chi = \varphi^{\cdot}$.

6.5. An intermediate particular case arises when $\chi \neq \varphi^{\cdot}$ but $\chi(T_x(M)) = T_{\varphi(x)}(\varphi(M))$. In this case a mapping $\gamma\colon M \to G$ can be found, such that $\chi(v) = \varphi^{\cdot}(v)\, h(\gamma(x))$ for $x \in M$ and $v \in T_x(M)$.

6.6. The set of all affine frames in the affine spaces tangent to M forms a principal fibre bundle E whose group G is the group of all $k+1 \times k+1$ matrices of form

$$\begin{pmatrix} 1 & a \\ 0 & A \end{pmatrix},$$

(8) This immersion corresponds, in essence, to the treatment of linear connections by Galvani [2] used for the purpose of imbedding linearly connected spaces with non-zero torsion into affine spaces.

The immersion of section 6.3 corresponds to the classical treatment (see, e.g., Norden [10], § 38).

where a is an $1 \times k$-matrix and A a nondegenerated $k \times k$-matrix [9]. Analogously, the set of affine frames on M' forms a principal fibre bundle E' with a corresponding group G'.

The monomorphism h being defined by the formula

$$h\left(\begin{pmatrix} 1 & a \\ 0 & A \end{pmatrix}\right) = \begin{pmatrix} 1 & a & 0 \\ 0 & A & 0 \\ 0 & 0 & E_{n-k} \end{pmatrix},$$

and given an immersion $\varphi\colon M \to M'$ and a normalization $e'_{k+1}, \ldots, e'_n$ as in 6.3 (respectively in 6.4), the mapping $f\colon E \to E'$ defined by the formula

$$f\big((0; e_1, \ldots, e_k)\big) = \big(\varphi(0); \varphi^{\cdot}(e_1), \ldots, \varphi^{\cdot}(e_k), e'_{k+1}, \ldots, e'_n\big)$$

(respectively $\big(\varphi(0), \chi(e_1), \ldots, \chi(e_k), e'_{k+1}, \ldots, e'_n\big)$ is an immersion of principal fibre bundles, 0 is the origin of the frame.

§ 7. The invariant projection of Lie algebras.

7.1. Given two Lie groups G and G' with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ and a monomorphism $h\colon G \to G'$, a linear mapping $\pi\colon \mathfrak{g}' \to \mathfrak{g}$ is called an *invariant projection* of Lie algebras if the following conditions are satisfied

$$\pi(h(\lambda)) = \lambda \quad \text{for} \quad \lambda \in \mathfrak{g}, \tag{28}$$

$$\pi\big(\operatorname{adj}(h(g))\lambda'\big) = \operatorname{adj}(g)\pi(\lambda') \quad \text{for} \quad g \in G,\ \lambda' \in \mathfrak{g}' \tag{29}$$

or, in other words, if the diagram

$$\text{(D)} \qquad \begin{array}{ccccc} h(\mathfrak{g}) & \xrightarrow{i} & \mathfrak{g}' & \xrightarrow{\operatorname{adj}(h(g))} & \mathfrak{g}' \\ \uparrow h & & \downarrow \pi & & \uparrow \pi \\ \mathfrak{g} & \xrightarrow{\operatorname{id}} & \mathfrak{g} & \xrightarrow{\operatorname{adj}(g)} & \mathfrak{g} \end{array},$$

(i denotes the inclusion map) is commutative.

The commutativity of this diagram is obviously equivalent to the commutativity of the diagram

$$\text{(E)} \qquad \begin{array}{ccccc} E' \times h(\mathfrak{g}) & \xrightarrow{j_2} & \mathfrak{g}' & \xrightarrow{\operatorname{adj}(h(g))} & \mathfrak{g}' \\ \uparrow f \times h & & \downarrow \pi & & \uparrow \pi \\ E \times \mathfrak{g} & \xrightarrow{j_2} & \mathfrak{g} & \xrightarrow{\operatorname{adj}(g)} & \mathfrak{g} \end{array}$$

whatever the manifolds E, E' and the mapping $f\colon E \to E'$.

7.2. Given an invariant projection π, let us denote now

$$\varrho(\lambda') = \lambda' - h(\pi(\lambda')), \quad \lambda' \in \mathfrak{g}'. \tag{30}$$

(9) Cf. the notion of affine connection in Lichnerowicz [7].

Clearly, ϱ is a linear mapping $\mathfrak{g}'\to\mathfrak{g}'$ which has the following properties:

$$\pi\circ\varrho=0\,. \tag{31}$$

Really, for every $\lambda'\in\mathfrak{g}'$ we have $\pi(\varrho(\lambda'))=\pi(\lambda')-\pi\big(h(\pi(\lambda'))\big)=\pi(\lambda')-\pi(\lambda')=0$ (cf. (28)).

$$\varrho\circ h=0\,. \tag{32}$$

Really, $\varrho(h(\lambda))=h(\lambda)-h\big(\pi(h(\lambda))\big)=h(\lambda)-h(\lambda)=0$ for $\lambda\in\mathfrak{g}$.

$$\varrho\circ\operatorname{adj}(h(g))=\operatorname{adj}(h(g))\circ\varrho\quad\text{ for }\quad g\in G\,. \tag{33}$$

Really, $\varrho\big(\operatorname{adj}(h(g))\cdot\lambda'\big)=\operatorname{adj}(h(g))\cdot\lambda'-h\big(\pi(\operatorname{adj}(h(g))\cdot\lambda')\big)=\operatorname{adj}(h(g))\cdot\lambda'-h(\operatorname{adj}(g)\cdot\pi(\lambda'))=\operatorname{adj}(h(g))\cdot\lambda'-\operatorname{adj}(h(g))\cdot h(\pi(\lambda'))=\operatorname{adj}(h(g))\cdot\{\lambda'-h(\pi(\lambda'))\}=\operatorname{adj}(h(g))\cdot\varrho(\lambda')$, $\lambda'\in\mathfrak{g}'$, $g\in G$.

From (30) the identity

$$\lambda'=h(\pi(\lambda'))+\varrho(\lambda')$$

follows which gives a decomposition of every vector $\lambda'\in\mathfrak{g}'$ into two summands belonging to the linear spaces $h(\mathfrak{g})$ and $\varrho(\mathfrak{g}')=\mathfrak{m}$, respectively. Since these subspaces of $\mathfrak{g}'$ have only 0 in common (this follows from (31) and (32)), $\mathfrak{g}'$, as a linear space, is a direct sum

$$\mathfrak{g}'=h(\mathfrak{g})+\mathfrak{m}\,. \tag{34}$$

As a consequence of (33) the subspace $\mathfrak{m}$ is invariant under transformations of the group $\operatorname{adj}(h(G))$, i.e.

$$\operatorname{adj}(h(g))\cdot\mathfrak{m}\subset\mathfrak{m}\quad\text{ for }\quad g\in G\,. \tag{35}$$

7.3. Let us recall the notion of a *weakly reductive* homogeneous space [10]. If G' is a Lie group and H its subgroup, then the homogeneous space G'/H is called weakly reductive if the Lie algebra $\mathfrak{g}'$ of G', as a linear space, is a direct sum

$$\mathfrak{g}'=\mathfrak{h}+\mathfrak{m}\,,$$

where $\mathfrak{h}$ is the Lie algebra of H, and $\mathfrak{m}$ is a linear subspace invariant under the transformations of $\operatorname{adj}(H)$.

THEOREM. *An invariant projection π: $\mathfrak{g}'\to\mathfrak{g}$ exists if and only if $G'/h(G)$ is weakly reductive.*

Moreover, there is a one-to-one correspondence between the projections π and the decompositions of $\mathfrak{g}'$ into a direct sum, namely

$$\mathfrak{m}=\varrho(\mathfrak{g}')\,.$$

[10] See Nomizu [9]. The notion of weak reductivity was introduced by Koszul and independently by Rashevsky [13].

Proof. It is clear, in view of (34) and (35), that the weak reductivity of $G'/h(G)$ follows from the existence of an invariant projection π. Suppose now that $G'/h(G)$ is weakly reductive, and decomposition (34) is given. Then every element $\lambda'\in\mathfrak{g}'$ has a uniquely determined representation in the form

$$\lambda'=\alpha(\lambda')+\beta(\lambda')\,,$$

where $\alpha(\lambda')\in h(\mathfrak{g})$ and $\beta(\lambda')\in\mathfrak{m}$. The mapping π: $g'\to g$ defined as

$$\pi(\lambda')=h^{-1}(\alpha(\lambda'))$$

is the required invariant projection, and $\varrho=\beta$.

The decomposition (34) for a given weakly reductive $G'/h(G)$ is not necessarily unique and so is the invariant projection π: $\mathfrak{g}'\to\mathfrak{g}$.

7.4. Now consider the bracket operation in $\mathfrak{g}$ and $\mathfrak{g}'$.

First of all, since h is a monomorphism of Lie algebras, the identity

$$h([\lambda_1,\lambda_2])=[h(\lambda_1),h(\lambda_2)] \tag{36}$$

holds.

Further, if $\lambda\in\mathfrak{g}$, and $g(t)$ is the path in G tangent to λ at e, then we have, for any element $\lambda'\in\mathfrak{g}'$,

$$\operatorname{adj}\big(h(g(t))\big)\cdot\varrho(\lambda')=\varrho\big(\operatorname{adj}\big(h(g(t))\big)\cdot\lambda'\big)\,.$$

Both sides of the above identity are paths in $\mathfrak{g}'$. Passing to the tangent vectors to these paths at $t=0$ (remember that $\mathfrak{g}'$ is a linear space and tangent vectors of $\mathfrak{g}'$ can be identified, in a natural way, with elements of $\mathfrak{g}'$), we obtain the identity

$$[h(\lambda),\varrho(\lambda')]=\varrho([h(\lambda),\lambda'])\,. \tag{37}$$

Hence, in view of (31),

$$\pi([h(\lambda),\varrho(\lambda')])=0\quad\text{ for }\quad\lambda\in\mathfrak{g},\ \lambda'\in\mathfrak{g}'$$

and, in particular,

$$\pi\big([h(\pi(\lambda_1')),\varrho(\lambda_2')]\big)=0,\qquad\lambda_1',\lambda_2'\in\mathfrak{g}'\,. \tag{38}$$

Now let us calculate $\pi([\lambda_1',\lambda_1'])$. We have

$$\begin{aligned}\pi([\lambda_1',\lambda_2'])&=\pi\big([h(\pi(\lambda_1'))+\varrho(\lambda_1'),h(\pi(\lambda_2'))+\varrho(\lambda_2')]\big)\\&=\pi\big([h(\pi(\lambda_1')),h(\pi(\lambda_2'))]\big)+\pi\big([\varrho(\lambda_1'),h(\pi(\lambda_2'))]\big)+\\&\quad+\pi\big([h(\pi(\lambda_1')),\varrho(\lambda_2')]\big)+\pi([\varrho(\lambda_1'),\varrho(\lambda_2')])\,.\end{aligned}$$

The second and the third terms of the right-hand side of this equation are 0 in view of (38), and hence

$$\pi([\lambda_1',\lambda_2'])=[\pi(\lambda_1'),\pi(\lambda_2')]+\pi([\varrho(\lambda_1'),\varrho(\lambda_2')])$$

which yields

$$[\pi(\lambda_1'), \pi(\lambda_2')] = \pi\big([\lambda_1', \lambda_2'] - [\varrho(\lambda_1'), \varrho(\lambda_2')]\big). \tag{39}$$

7.5. Let a' be a $\mathfrak{g}'$-valued exterior differential p-form on M'. We define $\pi a'$ by the formula

$$(\pi a')(\tau_1, \dots, \tau_p) = \pi\big(a'(\tau_1, \dots, \tau_p)\big), \qquad \tau_1, \dots, \tau_p \in T_x(M),$$

and $\varrho a'$ by the formula

$$(\varrho a')(\tau_1, \dots, \tau_p) = \varrho\big(a'(\tau_1, \dots, \tau_p)\big).$$

Then $\pi a'$ is an exterior p-form on M with values in $\mathfrak{g}$, $\varrho a'$ is an exterior p-form with values in $\mathfrak{g}'$, and

$$d(\pi a') = \pi d a', \tag{40}$$

$$[\pi a', \pi a'] = \pi([a', a'] - [\varrho a', \varrho a']). \tag{41}$$

To prove this let us introduce a basis $\lambda_1', \dots, \lambda_{r'}'$ of the Lie algebra $\mathfrak{g}'$. Then

$$a'(\tau_1, \dots, \tau_p) = a^i(\tau_1, \dots, \tau_p)\lambda_i', \qquad i = 1, \dots, r',$$

where $a^1, \dots, a^{r'}$ are number valued p-forms.

Further,

$$(\pi a')(\tau_1, \dots, \tau_p) = a^i(\tau_1, \dots, \tau_p)\pi(\lambda_i'), \qquad i = 1, \dots, r',$$

$$d(\pi a')(\tau_0, \dots, \tau_p) = da^i(\tau_0, \dots, \tau_p)\pi(\lambda_i') = (\pi d a')(\tau_0, \dots, \tau_p),$$

which shows (40). Formula (41) follows immediately from (39).

§ 8. Examples of invariant projections.

8.1. $G' = GL(n)$, $G = GL(k)$, $k < n$. The elements are represented by non-degenerated matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad A,$$

respectively, where A is a non-degenerated square matrix of rank k, D a non-degenerated matrix of rank $n-k$, B is a matrix with k lines and $n-k$ rows, and C a matrix with $n-k$ lines and k rows. The monomorphism h: $G \to G'$ is defined by the formula

$$h(A) = \begin{pmatrix} A & 0 \\ 0 & E_{n-k} \end{pmatrix}.$$

The elements of the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are also represented by matrices (not necessarily non-degenerated), and the corresponding monomorphism of Lie algebras is

$$h(M) = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}.$$

The mapping π: $\mathfrak{g}' \to \mathfrak{g}$ defined by formula

$$\pi\left(\begin{bmatrix} M & N \\ P & Q \end{bmatrix}\right) = M \tag{42}$$

is an invariant projection, since

$$\pi\left(\operatorname{adj}(h(A)) \cdot \begin{bmatrix} M & N \\ P & Q \end{bmatrix}\right) = \pi\left(\begin{bmatrix} A & 0 \\ 0 & E_{n-k} \end{bmatrix}\begin{bmatrix} M & N \\ P & Q \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & E_{n-k} \end{bmatrix}^{-1}\right)$$

$$= \pi\left(\begin{bmatrix} AMA^{-1} & AN \\ PA^{-1} & Q \end{bmatrix}\right) = AMA^{-1} = \operatorname{adj}(A)\pi\left(\begin{bmatrix} M & N \\ P & Q \end{bmatrix}\right).$$

The mapping ϱ is given by the formula

$$\varrho\left(\begin{bmatrix} M & N \\ P & Q \end{bmatrix}\right) = \begin{bmatrix} 0 & N \\ P & Q \end{bmatrix}. \tag{43}$$

8.2. $G' = O(n)$, $G = O(k)$, the elements of G and G' represented by orthogonal matrices, the elements of the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ by squew-symmetric matrices. If h and π are defined by the same formulas as above, π is an invariant projection.

8.3. Now, let G' be the group of affine transformations of the affine n-space $\mathcal{A}^n$, and G the group of affine transformations of $\mathcal{A}^k$, $k < n$. The elements of the group G can be represented by matrices of the form

$$\begin{bmatrix} 1 & a \\ 0 & A \end{bmatrix},$$

where a is a $1 \times k$-matrix and A is a non-degenerated $k \times k$-matrix. The elements of G' are represented in an analogous way.

The corresponding Lie algebra $\mathfrak{g}$ consists of matrices of the form

$$\begin{bmatrix} 0 & m \\ 0 & M \end{bmatrix},$$

and the elements of $\mathfrak{g}'$ are represented analogously.

The monomorphism h is defined by the formula

$$h\left(\begin{bmatrix} 0 & m \\ 0 & M \end{bmatrix}\right) = \begin{bmatrix} 0 & m & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

Then the linear mapping

$$\pi\left(\begin{bmatrix} 0 & m & p \\ 0 & M & N \\ 0 & P & R \end{bmatrix}\right) = \begin{pmatrix} 0 & m \\ 0 & M \end{pmatrix} \tag{46}$$

is an invariant projection, because

$$\pi\left(\operatorname{adj}\left(h\begin{bmatrix}1 & a\\ 0 & A\end{bmatrix}\right)\cdot\begin{bmatrix}0 & m & p\\ 0 & M & N\\ 0 & P & R\end{bmatrix}\right) = \pi\left(\begin{bmatrix}1 & a & 0\\ 0 & A & 0\\ 0 & 0 & E_{n-k}\end{bmatrix}\begin{bmatrix}0 & m & p\\ 0 & M & N\\ 0 & P & R\end{bmatrix}\begin{bmatrix}1 & a & 0\\ 0 & A & 0\\ 0 & 0 & E_{n-k}\end{bmatrix}^{-1}\right)$$

$$= \pi\left(\begin{bmatrix}0 & mA^{-1}+aMA & p+aN\\ 0 & AMA^{-1} & AN\\ 0 & PA^{-1} & R\end{bmatrix}\right) = \begin{pmatrix}0 & mA^{-1}+amA^{-1}\\ 0 & AMA^{-1}\end{pmatrix}$$

$$= \begin{bmatrix}1 & a\\ 0 & A\end{bmatrix}\begin{bmatrix}0 & m\\ 0 & M\end{bmatrix}\begin{bmatrix}1 & a\\ 0 & A\end{bmatrix}^{-1} = \operatorname{adj}\left(\begin{bmatrix}1 & a\\ 0 & A\end{bmatrix}\right)\cdot\pi\left(\begin{bmatrix}0 & m & p\\ 0 & M & N\\ 0 & P & Q\end{bmatrix}\right).$$

8.4. Let G be the k-dimensional real projective group represented by equivalence classes of non-degenerated $k+1\times k+1$-matrices under the equivalence relation $A\sim tA$ (t real), and let G' be the n-dimensional real projective group ($k<n$). Consider, for even k, the monomorphism $h\colon GL(k+1, R)\to GL(n+1, R)$ defined by the formula

(47) $$h(A) = \begin{bmatrix}A & 0\\ 0 & (\det A)^{1/k+1}E_{n-k}\end{bmatrix}.$$

This monomorphism maps equivalent matrices into equivalent matrices, hence it defines a monomorphism of G into G'.

The corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ consist of equivalence classes of $k+1\times k+1$-matrices, respectively $n+1\times n+1$-matrices, under the equivalence relation $M\sim M+tE$, where t is a real number, E the unity matrix, and the induced monomorphism of Lie algebras is

(48) $$h(M) = \begin{bmatrix}M & 0\\ 0 & \frac{1}{k+1}\operatorname{tr}(M)E_{n-k}\end{bmatrix},$$

where $\operatorname{tr}(M)$ denotes the trace of the matrix M.

The linear mapping defined by the formula

(49) $$\pi\left(\begin{bmatrix}M & N\\ P & Q\end{bmatrix}\right) = M$$

maps equivalent matrices into equivalent matrices, and therefore it can be regarded as a mapping $\pi\colon \mathfrak{g}'\to\mathfrak{g}$. This mapping is an invariant projection since

$$\pi\big(h(M)\big) = \pi\left(\begin{bmatrix}M & 0\\ 0 & \frac{1}{k+1}\operatorname{tr}(M)E_{n-k}\end{bmatrix}\right) = M$$

and

$$\pi\left(\operatorname{adj}\big(h(A)\big)\begin{bmatrix}M & N\\ P & Q\end{bmatrix}\right)$$

$$= \pi\left(\begin{bmatrix}A & Q\\ 0 & (\det A)^{1/k+1}E_{n-k}\end{bmatrix}\begin{bmatrix}M & N\\ P & Q\end{bmatrix}\begin{bmatrix}A & 0\\ 0 & (\det A)^{1/k+1}E_{n-k}\end{bmatrix}^{-1}\right)$$

$$= \pi\left(\begin{bmatrix}AMA^{-1} & (\det A)^{-1/k+1}AN\\ (\det A)^{1/k+1}PA^{-1} & Q\end{bmatrix}\right) = AMA^{-1} = \operatorname{adj}(A)\pi\left[\begin{pmatrix}M & N\\ P & Q\end{pmatrix}\right].$$

The mapping ϱ is defined by formula

$$\varrho\left(\begin{bmatrix}M & N\\ P & Q\end{bmatrix}\right) = \begin{bmatrix}0 & N\\ P & Q-\frac{1}{k+1}\operatorname{tr}(M)E_{n-k}\end{bmatrix}.$$

8.5. In the case of odd k there is no monomorphism of the k-dimensional projective group into the n-dimensional projective group of a similar form. Really, suppose that a mapping of the form

$$h(A) = \begin{bmatrix}A & 0\\ 0 & \mu(A)E_{n-k}\end{bmatrix}$$

(μ a real function of the matrix A) defines a monomorphism of the projective groups. Then μ should satisfy the conditions

(51) $$\mu(AB) = \mu(A)\mu(B),$$

(52) $$\mu(tA) = t\mu(A).$$

But, according to Kucharzewski [6], μ must be of form $\mu(A) = \varphi(\det A)$, where φ is a multiplicative function, whence

$$\mu(tA) = \varphi(t^{k+1}\det A) = \varphi(t^{k+1})\mu(A).$$

On the other hand, for odd k, $\varphi(t^{k+1}) = \varphi\big((-t)^{k+1}\big)$ and, consequently, it cannot be $\varphi(t^{k+1}) = t$, and condition (52) cannot be satisfied.

8.6. For geometrical applications ([11]), the following construction may be important.

Suppose that $\alpha\colon G'\to G'$ is an involutive automorphism of G', i.e. the conditions

$$\alpha(g_1'g_2') = \alpha(g_1')\alpha(g_2') \quad\text{and}\quad \alpha\big(\alpha(g')\big) = g'$$

are satisfied.

The set $G = \{g\colon \alpha(g) = g,\ g\in G'\}$ is evidently a subgroup of G'. It is easy to prove the following theorem.

([11]) Some applications of the invariant projections of sections 8.7, 8.8 and 8.9, which are based on the construction of this section, are given in the author's paper [4].

THEOREM. *The homogeneous space G'/G is weakly reductive with*

(53) $$\mathfrak{m} = \{\lambda' \colon \alpha(\lambda') + \lambda' = 0,\ \lambda' \in \mathfrak{g}'\}.$$

If we take the inclusion monomorphism as h, then the invariant projection is

(54) $$\pi(\lambda') = \tfrac{1}{2}\big(\lambda' + \alpha(\lambda')\big),$$

and, consequently,

(55) $$\varrho(\lambda') = \tfrac{1}{2}\big(\lambda' - \alpha(\lambda')\big).$$

Proof. The subalgebra $\mathfrak{g}$ of $\mathfrak{g}'$ consists of all the elements $\lambda \in \mathfrak{g}'$ for which $\alpha(\lambda) = \lambda$. For every element $\lambda' \in \mathfrak{g}'$, it is $\frac{1}{2}\big(\lambda' + \alpha(\lambda')\big) \in \mathfrak{g}$, because $\alpha\Big(\frac{1}{2}\big(\alpha(\lambda') + \lambda'\big)\Big) = \frac{1}{2}\big(\lambda' + \alpha(\lambda')\big)$. Hence $\pi(\lambda') \in \mathfrak{g}$ and π is a linear mapping $\mathfrak{g}' \to \mathfrak{g}$. If $\lambda \in \mathfrak{g}$ then $\pi(\lambda) = \frac{1}{2}\big(\lambda + \alpha(\lambda)\big) = \frac{1}{2}(\lambda + \lambda) = \lambda$, which proves the condition (28).

Further,

$$\begin{aligned} 2\pi\big(\operatorname{adj}(g)\cdot\lambda'\big) &= \alpha\big(\operatorname{adj}(g)\cdot\lambda'\big) + \operatorname{adj}(g)\cdot\lambda' = \operatorname{adj}(g)\cdot\big(\alpha(\lambda') + \lambda'\big) \\ &= 2\operatorname{adj}(g)\pi(\lambda') \end{aligned}$$

since the automorphism α commutes with the adjoint automorphism $\operatorname{adj}(g)$. Hence the condition (29) is also satisfied and π is an invariant projection.

This theorem gives rise to the following examples.

8.7. $G' = GL(n, R)$ represented by matrices, the automorphism α given by

(56) $$\alpha(\boldsymbol{A}) = (\det \boldsymbol{A})^{-2/n}\boldsymbol{A}.$$

It is clear that this is an automorphism; to show it is involutive let us compute $\alpha\big(\alpha(\boldsymbol{A})\big)$:

$$\begin{aligned} \alpha\big(\alpha(\boldsymbol{A})\big) &= \big(\det \alpha(\boldsymbol{A})\big)^{-2/n}\alpha(\boldsymbol{A}) = \big[\det\big((\det \boldsymbol{A})^{-2/n}\boldsymbol{A}\big)\big]^{-2/n}\alpha(\boldsymbol{A}) \\ &= \big((\det \boldsymbol{A})^{-2}\det \boldsymbol{A}\big)^{-2/n}(\det \boldsymbol{A})^{-2/n}\boldsymbol{A} = \boldsymbol{A}. \end{aligned}$$

The subgroup G consists of matrices of determinant 1 or -1. Its Lie algebra $\mathfrak{g}$ consists of matrices $\boldsymbol{M}$ with trace $\operatorname{tr}(\boldsymbol{M}) = 0$. The induced automorphism of the Lie algebra is given by

(57) $$\alpha(\boldsymbol{M}) = \boldsymbol{M} - \frac{2}{n}\operatorname{tr}(\boldsymbol{M})\cdot\boldsymbol{E}_n,$$

which yields the following invariant projection π and mapping ϱ

(58) $$\pi(\boldsymbol{M}) = \boldsymbol{M} - \frac{1}{n}\operatorname{tr}(\boldsymbol{M})\boldsymbol{E}_n, \qquad \varrho(\boldsymbol{M}) = \frac{1}{n}\operatorname{tr}(\boldsymbol{M})\boldsymbol{E}_n.$$

8.8. Let us consider the matrix

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{E}_m & 0 \\ 0 & -\boldsymbol{E}_{n-m} \end{bmatrix},$$

where the fixed m is one of the numbers $0, 1, \ldots, n$.

Let now $G' = GL(n, R)$; then the mapping α defined by the formula

(59) $$\alpha(\boldsymbol{A}) = \boldsymbol{I}(\boldsymbol{A}^{\tau})^{-1}\boldsymbol{I},$$

$\boldsymbol{A}^{\tau}$ being the transposed matrix of $\boldsymbol{A}$, is an involutive automorphism.

Really, $\beta(A) = (A^{\tau})^{-1}$ is clearly an automorphism of G', whence α is the composition of the automorphism β and an inner automorphism of G'. Moreover, α is involutive, because

$$\alpha\big(\alpha(\boldsymbol{A})\big) = \boldsymbol{I}\Big(\big(\boldsymbol{I}(\boldsymbol{A}^{\tau})^{-1}\boldsymbol{I}\big)\Big)^{-1}\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{I}\boldsymbol{A}^{-1}\boldsymbol{I})^{-1}\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{I}\boldsymbol{A}\boldsymbol{I})\boldsymbol{I} = \boldsymbol{A}.$$

The subgroup G consists of matrices of rotations of the n-dimensional pseudo-Euclidean space of signature $2m-n$, i.e. of linear transformations preserving the quadratic form

$$(x^1)^2 + \ldots + (x^m)^2 - (x^{m+1})^2 - \ldots - (x^n)^2;$$

in particular, for $m = n$, the subgroup consists of matrices of rotations of the Euclidean n-space (orthogonal matrices).

The Lie subalgebra $\mathfrak{g}$ consists of matrices $\boldsymbol{M}$ which satisfy the condition

$$\boldsymbol{M} + \boldsymbol{I}\boldsymbol{M}^{\tau}\boldsymbol{I} = 0,$$

and the invariant projection $\pi\colon \mathfrak{g}' \to \mathfrak{g}$ is given by the formula

(60) $$\pi(\boldsymbol{M}) = \tfrac{1}{2}(\boldsymbol{M} - \boldsymbol{I}\boldsymbol{M}^{\tau}\boldsymbol{I}).$$

Consequently,

(61) $$\varrho(\boldsymbol{M}) = \tfrac{1}{2}(\boldsymbol{M} + \boldsymbol{I}\boldsymbol{M}^{\tau}\boldsymbol{I}).$$

In the Euclidean case ($m = n$) the subalgebra $\mathfrak{g}$ consists of all squew symmetric matrices, and the matrix $\boldsymbol{I}$ is the unity matrix.

8.9. For $n = 2m$, consider the matrix

$$\boldsymbol{J} = \begin{bmatrix} 0 & \boldsymbol{E}_m \\ -\boldsymbol{E}_m & 0 \end{bmatrix}.$$

An easy calculation, with use of the identity $\boldsymbol{J}\boldsymbol{J} = -\boldsymbol{E}$, shows that the mapping α defined by the formula

(62) $$\alpha(\boldsymbol{A}) = -\boldsymbol{J}(\boldsymbol{A}^{\tau})^{-1}\boldsymbol{J}, \qquad \boldsymbol{A} \in G',$$

is an involutive automorphism of $GL(2m, R) = G'$. The subgroup G consists of the matrices of the transformations of R^{2m}, which preserve the exterior quadratic form

$$x^1 \wedge x^{m+1} + \ldots + x^m \wedge x^n.$$

The corresponding automorphism of Lie algebras is given by formula

$$(63)\qquad \alpha(M) = JM^{\tau}J\,,$$

and the invariant projection is

$$(64)\qquad \pi(M) = \tfrac{1}{2}(M+JM^{\tau}J)\,.$$

Further,

$$(65)\qquad \varrho(M) = \tfrac{1}{2}(M-JM^{\tau}J)\,.$$

§ 9. Induced connections.

9.1. Consider two principal fibre bundles $E(M, G, p)$ and $E'(M', G', p')$ and an immersion f: $E \to E'$ compatible with φ: $M \to M'$ and the monomorphism h: $G \to G'$. Suppose, moreover, that there is an invariant projection π: $\mathfrak{g}' \to \mathfrak{g}$.

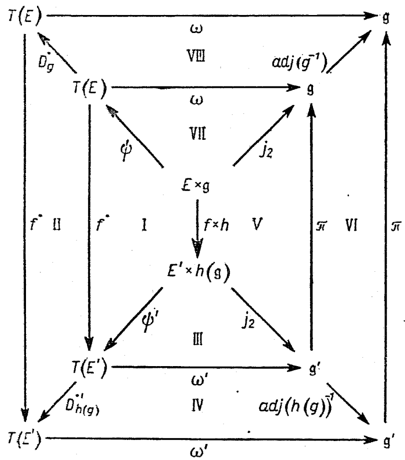
Let the $\mathfrak{g}'$-valued linear differential form ω' on E' be a connection form, in other words, let it satisfy the conditions (22) and (23) of section 5.1. Then the following theorem holds.

THEOREM. *The* $\mathfrak{g}$-*valued differential linear form*

$$(66)\qquad \omega = \pi f^{*}\omega'$$

is an infinitesimal connection form on E (¹²).

Proof. Look at the following diagram.



The commutativity of I and II follows from the definition of the immersion (diagram (C), section 6.1), that of III and IV from the definition of the infinitesimal connection ω' (diagram (A), section 5.1), and the commutativity of V and VI is a consequence of the definition of the invariant projection π (diagram (E), section 7.1). The commutativity of the two identical rectangular diagrams is nothing else than the definition of ω. Thus the diagrams VII and VIII commute, whence ω is an infinitesimal connection form on E.

The infinitesimal connection on E defined by the form $\omega = \pi f^{*}\omega'$ is called the *induced connection* (¹³).

(¹²) The notations are explained in sections 2.3 and 7.5.

9.2. The curvature form

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega]$$

of the induced connection ω can be expressed in the form

$$(67)\qquad \Omega = \pi f^{*}\Omega' - \tfrac{1}{2}\pi[\varrho f^{*}\omega', \varrho f^{*}\omega']\,,$$

where Ω' is the curvature form of the connection ω'.

Really, it is

$$\Omega = d\pi f^{*}\omega' + \tfrac{1}{2}[\pi f^{*}\omega', \pi f^{*}\omega']\,.$$

As a consequence of (39) and (40) we have

$$\begin{aligned}\Omega &= \pi f^{*}d\omega' + \tfrac{1}{2}\pi([f^{*}\omega', f^{*}\omega'] - [\varrho f^{*}\omega', \varrho f^{*}\omega'])\\ &= \pi f^{*}d\omega' + \tfrac{1}{2}\pi f^{*}[\omega', \omega'] - \tfrac{1}{2}\pi[\varrho f^{*}\omega', \varrho f^{*}\omega']\\ &= \pi f^{*}(d\omega' + \tfrac{1}{2}[\omega', \omega']) - \tfrac{1}{2}\pi[\varrho f^{*}\omega', \varrho f^{*}\omega']\\ &= \pi f^{*}\Omega' - \tfrac{1}{2}\pi[\varrho f^{*}\omega', \varrho f^{*}\omega']\,,\end{aligned}$$

Q.E.D.

The curvature form of the induced connection is split into two parts. We call the first part,

$$(68)\qquad \Omega_C = \pi f^{*}\Omega'\,,$$

the *coerced curvature*, and the second part,

$$(69)\qquad \Omega_R = -\tfrac{1}{2}\pi[\varrho f^{*}\omega', \varrho f^{*}\omega']\,,$$

the *relative curvature* of the induced connection. The curvature form of the induced connection is a sum of the coerced and the relative curvatures.

Formula (69) can be also written in the equivalent form

$$(70)\qquad \Omega_R = \tfrac{1}{2}([\omega, \omega] - \pi[f^{*}\omega', f^{*}\omega'])\,.$$

(¹³) The difference between this notion and the notion of induced connection, used by Kobayashi in [5] and other papers, is that in our case the induced connection is defined in a bundle with another group than the original bundle, whereas other authors consider only the case when both bundles have the same group.

§ 10. Change of the immersion.

10.1. The following theorem is quite obvious.

THEOREM. *If $f\colon E\to E'$ is an immersion of principal fibre bundles compatible with $\varphi\colon M\to M'$ and $h\colon G\to G'$, and $\gamma'\colon E\to G'$ satisfies the conditions*

(71) $$\gamma'(zg) = h(g^{-1})\gamma'(z)h(g), \quad g\in G,\ z\in E,$$

then the mapping $F\colon E\to E'$ defined by the formula

(72) $$F(z) = f(z)\gamma'(z)$$

is also an immersion compatible with φ and h.

Therefore we call the mapping $\gamma'\colon E\to G'$ the change of the immersion, and F the immersion changed by γ'.

10.2. Let us compute now the derived mapping of the changed immersion, i.e. $F^{\cdot}\colon T(E)\to T(E')$. We have

$$\begin{aligned} F^{\cdot}(\tau) &= f^{\cdot}(\tau)\gamma'(z) + f(z)\gamma'^{\cdot}(\tau)\\ &= f^{\cdot}(\tau)\gamma'(z) + f(z)\gamma'(z)\gamma'(z)^{-1}\gamma'^{\cdot}(\tau)\\ &= f^{\cdot}(\tau)\gamma'(z) + F(z)\gamma'(z)^{-1}\gamma'^{\cdot}(\tau), \quad z\in E,\ \tau\in T_z(E). \end{aligned}$$

The vector $\gamma'(z)^{-1}\gamma'^{\cdot}(\tau)$ is tangent to G' in e', hence it is an element of $\mathfrak{g}'$.

Consequently, if ω' is a connection form in E', then

$$\begin{aligned} F^*\omega'(\tau) &= \omega'\big(f^{\cdot}(\tau)\gamma'(z)\big) + \omega'\big(F(z)\gamma'(z)^{-1}\gamma'^{\cdot}(\tau)\big)\\ &= \operatorname{adj}\big(\gamma'(z)^{-1}\big)\cdot f^*\omega'(\tau) + \gamma'(z)^{-1}\gamma'^{\cdot}(\tau), \quad z\in E,\ \tau\in T_z(E), \end{aligned}$$

or

(73) $$F^*\omega' = \operatorname{adj}(\gamma'^{-1})f^*\omega' + \gamma'^{-1}d\gamma'.$$

If we denote the induced connection form for the changed immersion F by $\overline{\omega}$, then we have

(74) $$\overline{\omega} = \pi\operatorname{adj}(\gamma'^{-1})f^*\omega' + \pi\gamma'^{-1}d\gamma'.$$

10.3. The change of the immersion is called *inessential* if $\overline{\omega}=\omega$ for any connection ω' on E'. It is not difficult to find some practical sufficient conditions under which the change of an immersion is inessential.

Denote by K the smallest subgroup of G' generated by $\gamma'(E)$. If its Lie algebra $\mathfrak{k}$ is contained in $\mathfrak{m}=\varrho(\mathfrak{g}')$, then $\pi\gamma'^{-1}d\gamma'=0$, and

(75) $$\overline{\omega} = \pi\operatorname{adj}(\gamma'^{-1})f^*\omega'.$$

If, moreover,

(76) $$\pi\operatorname{adj}(k) = \pi \quad \text{for every} \quad k\in K,$$

then the change of the immersion is inessential, since

$$\overline{\omega} = \pi\operatorname{adj}(\gamma'^{-1})f^*\omega' = \pi f^*\omega' = \omega.$$

Particularly, the following theorem holds.

THEOREM. *If all elements of K commute with all elements of $h(G)$, and*

(77) $$\operatorname{adj}(k)\mathfrak{m}\subset\mathfrak{m} \quad \text{for} \quad k\in K,$$

then the condition (76) holds. If, moreover, $\mathfrak{k}\subset\mathfrak{m}$, then the change of the immersion is inessential.

Proof. It suffices to prove the first part of the theorem. It follows from the commutativity of $k\in K$ and $h(g)$, $g\in G$, that $\operatorname{adj}(k)\cdot\lambda'=\lambda'$ whenever $k\in K$ and $\lambda'\in h(\mathfrak{g})$. Consequently, for an arbitrary $\lambda'\in\mathfrak{g}'$, we have

$$\operatorname{adj}(k)\cdot\lambda' = \operatorname{adj}(k)\cdot\Big(h\big(\pi(\lambda')+\varrho(\lambda')\big)\Big) = h\big(\pi(\lambda')\big) + \operatorname{adj}(k)\cdot\varrho(\lambda'),$$

since $\pi(\lambda')\in\mathfrak{g}$.

On the other hand, $\operatorname{adj}(k)\cdot\varrho(\lambda')\in\mathfrak{m}$, hence $\pi\big(\operatorname{adj}(k)\cdot\varrho(\lambda')\big)=0$, and

$$\pi\big(\operatorname{adj}(k)\cdot\lambda'\big) = \pi(\lambda')$$

which shows that (75) holds.

10.4. Recalling the immersion of section 6.3 and the invariant projection of section 8.1 we see that the group K of matrices of the form

$$\begin{bmatrix} \boldsymbol{E}_k & 0\\ 0 & \boldsymbol{B}\end{bmatrix},$$

where $\boldsymbol{B}$ is a non-degenerated $n-k\times n-k$-matrix, satisfies the suppositions of the theorem of the preceding section. Consequently, any change of the immersion with $\gamma'(E)\subset K$ is inessential and does not change the induced connection.

Geometrically, this change of connection means a change of the normalization in such a way that the subspace spanned by the normalizing vectors $\boldsymbol{e}'_{k+1},\dots,\boldsymbol{e}'_n$ does not change. Hence, given a connection ω' on E', the *induced connection depends*, in fact, *only on the $n-k$-dimensional subspaces of the normalization*, and not on the particular choice of bases in them.

10.5. Another simplification of formula (74) arises, when $K\subset h(G)$. In this case we have $\gamma'(z)=h\big(\gamma(z)\big)$ for an appropriate mapping $\gamma\colon E\to G$. Now it is

$$\pi\operatorname{adj}(\gamma')\cdot f^*\omega' = \pi\operatorname{adj}\big(h(\gamma)\big)\cdot f^*\omega' = \operatorname{adj}(\gamma)\cdot\pi f^*\omega' = \operatorname{adj}(\gamma)\cdot\omega$$

and

$$\pi\gamma'^{-1}d\gamma' = \pi h(\gamma^{-1}d\gamma) = \gamma^{-1}d\gamma.$$

Hence

(78) $$\overline{\omega} = \operatorname{adj}(\gamma^{-1})\cdot\omega + \gamma^{-1}d\gamma\,.$$

10.6. Since the curvature form Ω' is a tensorial form of type adj, we have

$$F^*\Omega' = \operatorname{adj}(\gamma'^{-1})\cdot f^*\Omega'\,.$$

This can be also easy proved by direct calculations using formulae (25), (18) with $\eta = f^*\omega'$, (17) and (21).

Consequently, for the coerced and relative curvature forms $\overline{\Omega}_C$ and $\overline{\Omega}_R$ of the changed induced connection we obtain the formulae

(79) $$\overline{\Omega}_C = \pi\operatorname{adj}(\gamma'^{-1})\cdot f^*\Omega'$$

and

(80) $$\overline{\Omega}_R = -\tfrac{1}{2}\pi[\varrho\operatorname{adj}(\gamma'^{-1})\cdot f^*\omega',\ \varrho\operatorname{adj}(\gamma'^{-1})\cdot f^*\omega']\,.$$

In the case when (76) is satisfied we have

$$\overline{\Omega}_C = \Omega_C\,.$$

If the change of the immersion is inessential, in particular when the suppositions of the theorem of section 10.3 are satisfied, we have $\overline{\Omega} = \Omega$ and hence we have also

$$\overline{\Omega}_R = \Omega_R\,.$$

10.7. In the case considered in section 10.5, when $K\subset h(G)$, we have $\gamma'(z) = h(\gamma(z))$, and $\overline{\omega} = \operatorname{adj}(\gamma^{-1})\cdot\omega+\gamma^{-1}d\gamma$. By direct computations we obtain in this case

(81) $$\overline{\Omega} = \operatorname{adj}(\gamma^{-1})\Omega\,.$$

Since, on the other hand, in this case

(82) $$\overline{\Omega}_C = \operatorname{adj}(\gamma^{-1})\cdot\Omega_C\,,$$

we have also

(83) $$\overline{\Omega}_R = \operatorname{adj}(\gamma^{-1})\cdot\Omega_R\,.$$

§ 11. Two generalization.

11.1. To apply our procedure of inducing connections two conditions are required; first, we need an immersion $f\colon E\to E'$; secondly, an invariant projection $\pi\colon G'\to G$ is necessary, which exists if and only if $G'/h(G)$ is weakly reductive. However, our procedure can be modified in such a way that it covers also some cases when those requirements are not satisfied.

11.2. It may happen that there is no immersion $f\colon E'\to E'$ compatible with $\varphi\colon M\to M'$ and $h\colon G\to G'$ but there exists a covering $\{U_\varkappa\}$ of the basic space M of E and a system of immersions $f_\varkappa\colon p^{-1}(U_\varkappa)\to E'$ compatible with $\varphi|U_\varkappa\colon U_\varkappa\to M'$ and $h\colon G\to G'$ and satisfying the following condition: If $U_\varkappa\cap U_\lambda\neq\emptyset$ then, for $z\in p^{-1}(U_\varkappa\cap U_\lambda)$, the relation

$$f_\varkappa(z) = f_\lambda(z)\gamma'_{\lambda\varkappa}(z)$$

holds, $\gamma'_{\lambda\varkappa}(z)$ being an inessential change of the immersion $f_\lambda|p^{-1}(U_\varkappa\cap U_\lambda)$.

Provided we have an invariant projection $\pi\colon \mathfrak{g}'\to\mathfrak{g}$, we can define the induced connection form $\omega_\varkappa$ on $p^{-1}(U_\varkappa)$. Since the change of immersion by $\gamma'_{\lambda\varkappa}$ is inessential, the connection forms $\omega_\varkappa$ and ω_λ coincide on the intersection of $p^{-1}(U_\varkappa)$ and $p^{-1}(U_\lambda)$. Hence the induced connection is well defined on the whole bundle E.

11.3. This is the case, for instance, with E being the space of orthonormal frames on the Möbius band M, and E' the bundle of frames of the Euclidean 3-space M'. Here, G and G' are the orthogonal groups of 2 and 3 dimensions, respectively, and φ is the imbedding of the Möbius band into the 3-space.

There is, of course, no continuous normalization of the Möbius band as a whole, therefore no immersion of the kind described in section 6.3 can be found. Nevertheless, we can apply the generalized procedure of the previous section in this case.

11.4. Suppose now that there is given an immersion $f\colon E\to E'$ but $G'/h(G)$ is not weakly reductive, and hence there is no invariant projection $\pi\colon \mathfrak{g}'\to\mathfrak{g}$.

Fix $x\in M$ and consider the fibre $p^{-1}(x)$, its image $f(p^{-1}(x))$ and the set $f(p^{-1}(T_x(M)))$ of tangent vectors to E'. It may happen that the values of ω', when restricted to the last set, lie in a certain subalgebra $\mathfrak{g}''_x$ of $\mathfrak{g}'$ which corresponds to a subgroup G''_x of G' containing $h(G)$, and that $G''_x/h(G)$ is weakly reductive. Consequently, there is an invariant mapping $\pi_x\colon \mathfrak{g}''_x\to\mathfrak{g}$ which can be applied to define the induced connection form $\omega = \pi_x f^*\omega'$ on $p^{-1}(x)$. If π_x can be chosen for every x in such a way that the above form is differentiable on E, an induced connection is obtained in the whole bundle E.

This procedure is more general than the former one but, at the same time, it is more complicated, since the family of invariant projections π_x can depend essentially on the connection ω' on E'.

Mutatis mutandis this generalization can be combined with the piecewise immersion considered in section 11.2.

11.5. A trivial but characteristic example of the situation dealt with in the previous section is yielded by the homomorphic connection [14].

Let $M = M'$, φ be the identity map and $f\colon E\to E'$ an immersion compatible with φ and $h\colon G\to G'$ (a homomorphism of principal fibre bundles); f is then a one-to-one mapping $E\to f(E)$. The form ω being

[14] Cf., for example, Nomizu [8] or Kobayashi [5].

a connection form on E, we define the connection form ω' on E' by setting it to be equal $f^{-1*}\omega$ on $f(E)$ and extending it by means of formulae (23) of section 5.1. Then the values of ω' on $f'(T(E))$ are contained in $h(\mathfrak{g})$, and π_x is simply identical with h^{-1} (it does not depend on x in this case). The induced connection coincides with the given connection ω on E.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 23. 3. 1963

An arc is tame in 3-space if and only if it is strongly cellular *

by

R. H. Bing (Wisconsin, USA) and **A. Kirkor** (Warszawa)

A set Z in Euclidean n-space E^n is *tame* if there exists such a homeomorphism f of E^n onto itself that $f(Z)$ is a polyhedron. There are known some necessary and sufficient conditions of tameness of an arc in E^3, e.g. [4] and [5]. We shall give here another one based on the reinforced notion of a cellular set [2]. A set Z in E^n is *strongly cellular* if there is an n-cell C in E^n and a homotopy h: $C\times I\to C$ such that, if $h_t(x) = h(x, t)$ and $S = \operatorname{Bd} C$ is the boundary of C, then

(1) $h_0 =$ identity mapping and $h_t|Z =$ identity for all t,

(2) $h_t|S =$ homeomorphism for $t<1$,

(3) $h_t(S)\cap h_{t'}(S) = 0$ for $t\neq t'$,

(4) $h_1(C) = Z$.

The set Z will be said to be a *strong deformation retract* of the cell C. By M. Brown's generalization of the Schoenflies Theorem [2] there will be no loss of generality if the cell C is assumed to be a ball.

THEOREM 1. *An arc is tame in E^3 if and only if it is strongly cellular.*

COROLLARY. *An arc A in E^3 is tame if and only if there are two concentric balls B_0 and B, $B_0\subset \operatorname{Int} B$, and a mapping f of B into E^3 such that*

(1) $f|B-B_0 =$ homeomorphism,

(2) $f(B_0) = A$.

Proof of the Corollary. It is obvious that the conditions of the Corollary are necessary. In order to prove that they are also sufficient let us observe first that $f(B)$ is a 3-cell by M. Brown's Theorem 1 of [2]. Then assume $C = f(B)$ and next consider the homotopy r: $B\times I\to B$ retracting B to B_0. Now define the homotopy h: $C\times I\to C$ in the following way:

$$h(x, t) = \operatorname{fr}[f^{-1}(x), t] \quad \text{for} \quad x\in C-A,$$
$$h(x, t) = x \quad \text{for} \quad x\in A.$$

* Work on this paper is supported by contract NSF-611665.