COROLLARY. *If f is any epimorphism from G to a totally ordered group, or any realization of G then the decomposition $f = f_1 \circ f_0$ where f_0 is irreducible and f_1 an epimorphism is essentially unique.*

Remark. Proposition 5 is closely related to the fact that in certain integral domains, e.g. the ring of entire functions on the complex plane, each prime ideal is contained in exactly one maximal ideal [1].

6. Concluding remarks. In view of Proposition 1, the lattice-ordered group G has *no* realization iff the union W of all normal prime filters in P is smaller than $P-\{0\}$; $K_0 = \{x|\ x \in G,\ |x| \notin W\}$ is then the l-ideal of G consisting of all those elements of G which vanish under every homomorphism $G \to T$, and G/K_0 is the largest quotient group of G which does have realizations. This does not, however, describe W and K_0 internally in terms of the elements of G, and it might be of interest to have a characterization of this latter kind. It is clear that W consists of elements $a \in G$ such that no $(x_1+a-x_1) \wedge \ldots \wedge (x_k+a-x_k)$ $(x_i \in G)$ can be 0, but whether, say, W *is* the set of these elements remains an open question.

Another problem which arises naturally here is that of the existence of realizations $G \to \prod T_a$ where all T_a are archimedean. It is easy to see, for an epimorphism f: $G \to T$, that T is archimedean iff $Q(f)$ is a *minimal* normal prime filter, and hence G has realizations of the said type iff $P-\{0\}$ is the union of all minimal normal prime filters in P. Again, it seems desirable to have an alternative condition in terms of the elements of G, such as Proposition 3 provides for the existence of realizations in general.

Finally, we remark that the present approach to realizations of lattice-ordered groups might also be useful for the study of (analogously defined) realizations of lattice-ordered rings.

References

[1] B. Banaschewski, *Zur Idealtheorie der ganzen Funktionen*, Math. Nachr. 19 (1958), pp. 136-160.

[2] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ. XXV (1948).

[3] L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. 77 (1954), pp. 340-362.

[4] P. Jaffard, *Contribution à l'Etude des Groupes Ordonnées*, J. Math. Pures et Appl. 32 (1953), pp. 203-280.

[5] P. Lorenzen, *Über halbgeordnete Gruppen*, Math. Z. 52 (1949 50), pp. 483-526.

[6] P. Ribenboim, *Un théorème de realisation de groupes réticulés*, Pac. J. of Math. 10 (1960), pp. 305-308.

Reçu par la Rédaction le 7. 12. 1962

Some properties of algebraically independent sets in algebras with infinitary operations

by

J. Schmidt (Bonn)

The purpose of this paper is to continue the study of independence in algebras with infinitary operations as begun in [10]; in particular, to generalize some results of Marczewski [5] on independent subsets of finitary algebras. In Section 1, we discuss some notions of "neutral" or "singular" elements which are actually different as is shown by examples; the resulting necessary distinction represents the reason for some small complication occurring in the following sections. In Section 2, Marczewski's results on the relations between algebraic, lattice, and closure-independence are generalized (the proofs using the technique of algebraic operations here instead of Marczewski's technique of transformations of variables). In Section 3, the fundamental notion of element basis is introduced in general, only two special cases having been considered hitherto: one in Steinitz-Van Der Waerden exchange structures (MacLane [2], J. Schmidt [9]), the other in absolutely free algebras (Löwig [1], Słomiński [11]); the interrelations are studied between the existence of the element basis for element x and the representability of x by algebraic operations depending on all variables. In Section 4, the existence of the element basis for all elements x in the algebraically independent generating set M is secured in the special cases of finitary algebras and (reproducing a result of Löwig [1]) of absolutely free algebras, whereas in Section 5, an example is given for an element x in an algebra A (necessarily infinitary and not absolutely free) without element basis in the algebraically independent generating set M. The paper is reasonably self-containing; in particular, it can be read without knowledge of [10].

1. Neutral elements of different types. In [10] § 1, we have considered the natural one-one correspondence between elements of set A and operations of type Ø (empty set), i.e. nullary operations, on A. This natural correspondence is an isomorphism from algebra A onto algebra $\boldsymbol{O}^{\emptyset}(A)$ of all nullary operations on set A, the converse of this isomorphism being the natural projection $\mathrm{pr}_{\mathfrak{o}}$ ($\mathfrak{o}$ denoting the empty sequence, i.e. the empty mapping, which is the only element of $A^{\emptyset}$) of direct power $\boldsymbol{O}^{\emptyset}(A) = A^{A^{\emptyset}}$ onto factor A, i.e. the mapping $h \to h(\mathfrak{o})$ ($h \in \boldsymbol{O}^{\emptyset}(A)$). Passing to subalgebra $\boldsymbol{H}^{\emptyset}(A)$, the subalgebra of algebra $\boldsymbol{O}^{\emptyset}(A)$ generated by the empty set which consists of the nullary algebraic operations, we had the

THEOREM 1 (1). *There is one and only one homomorphism of algebra $\boldsymbol{H}^{\emptyset}(A)$ into algebra A. This homomorphism is an isomorphism onto algebra $\mathrm{C}\emptyset$, the subalgebra of A generated by the empty set; it is the restriction to subalgebra $\boldsymbol{H}^{\emptyset}(A)$ of the natural isomorphism of algebra $\boldsymbol{O}^{\emptyset}(A)$ onto algebra A.*

COROLLARY (2). *$\mathrm{C}\emptyset$ precisely consists of the values $h(\mathfrak{o})$ of nullary algebraic operations h.*

Now, all nullary operations are constant in the general sense that, for any two possible argument sequences, the value always is the same. Passing from type $\emptyset$ to an arbitrary type L, we may consider the set $\boldsymbol{C}^L(A)$ of all constant operations (algebraic or not) of type L on set A. One has the obvious

THEOREM 2. *$\boldsymbol{C}^L(A)$ is a subalgebra of algebra $\boldsymbol{O}^L(A)$.*

(In case $L = \emptyset$, one has $\boldsymbol{C}^L(A) = \boldsymbol{O}^L(A)$, the same for arbitrary L, if algebra A is of cardinal number $|A| \leqslant 1$.)

COROLLARY. *The set of values of all constant algebraic operations h of a certain type L is a subalgebra of algebra A.*

For this is trivial, if A is empty. If A is non-empty, there is a sequence $\mathfrak{a}$ of type L on A, and the natural projection $h \to \mathrm{pr}_{\mathfrak{a}}(h) = h(\mathfrak{a})$ maps set $\boldsymbol{H}^L(A) \cap \boldsymbol{C}^L(A)$ of all constant algebraic operations of type L on A onto the set of values of these operations: this set being the homomorphic image of subalgebra $\boldsymbol{H}^L(A) \cap \boldsymbol{C}^L(A) \subseteq \boldsymbol{O}^L(A)$, it is a subalgebra itself.

THEOREM 3. *Let L, M be two arbitrary non-empty sets. Then the sets of values of constant algebraic operations of types L and M respectively are the same.*

The proof is by means of "transformation of variables" as dealt with in [10], § 5. Both L and M being non-empty, there is a mapping σ of L into M. Let c be the value of a constant algebraic operation h of type L, then the operation $h_{\sigma,M}$ of type M defined by

$$h_{\sigma,M}(a_\mu |\ \mu \in M) = h(a_{\sigma(\lambda)} |\ \lambda \in L) = c$$

(1) [10], corollary 4 of theorem 5.

(2) [10], corollary 5 of theorem 5.

has constant value c by definition, moreover it has been proven to be an algebraic operation. Because of the symmetry with respect to L and M, the theorem is proved.

Let C denote the set of values of constant algebraic operations of a certain non-empty type L. By theorem 3, C is independent of L, by the corollary of theorem 2, C is a subalgebra of algebra A.

THEOREM 4. *In an arbitrary algebra A:*

$$\mathrm{C}\emptyset \subseteq C \subseteq D, \tag{1}$$

where D denotes the intersection of all non-empty subalgebras of A.

$\mathrm{C}\emptyset \subseteq C$ is trivial since $\mathrm{C}\emptyset$ is the smallest of all subalgebras of A. To prove $C \subseteq D$, we have to show that $C \subseteq B$ for any non-empty subalgebra B of A. In fact, let c be the value of a constant algebraic operation h of non-empty type L. There is an element $a \in B$, let $\mathfrak{a}$ be the corresponding constant sequence of type L, $\mathfrak{a}(\lambda) = a$ ($\lambda \in L$). Then $c = h(\mathfrak{a})$, but $\mathfrak{a}$ being a sequence in B and B being closed with respect to all algebraic operation, $c \in B$.

According to the corollary of theorem 1, inclusion $\mathrm{C}\emptyset \subseteq C$ states that all values of algebraic operations of type $\emptyset$ are values of constant algebraic operations of a certain given type L, and C may be described as the set of values of all constant algebraic operations of all (empty or non-empty) types L (3). Yet the most important consequence of (1) follows from the remark that $B = \emptyset$ or $B = D$ for any subalgebra $B \subseteq D$:

COROLLARY. *There are four possible cases:*

I. $\emptyset \subset \mathrm{C}\emptyset = C = D$;

II. $\emptyset = \mathrm{C}\emptyset \subset C = D$;

III. $\emptyset = \mathrm{C}\emptyset = C \subset D$;

IV. $\emptyset = \mathrm{C}\emptyset = C = D$.

In fact, all four cases occur, as is shown in the following examples:

EXAMPLE I. Remember that $\mathrm{C}\emptyset \neq \emptyset$ if and only if there are nullary fundamental operations f_ι, i.e. if at least one of the types K_ι of fundamental operations f_ι is empty ([10], § 1.3), this obviously being a property not only of algebra A, but of the entire species of type $(K_\iota)_{\iota \in I}$. An algebra, a type, a species of this kind has been called *with constants* in [10], *without constants* otherwise.

EXAMPLE II. Consider a group A as an algebra with multiplication and inversion as the only fundamental operations, both non-nullary (4).

(3) Marczewski [5] p. 49; also [4] p. 612 (Marczewski also writes $C = C(A) = A^{(0)}$); also Nitka [6].

(4) This is a rather frequent interpretation of a group as a general algebra.

Then $x \to x \cdot x^{-1}$ represents a constant algebraic operation with the unit element e as constant value; and since $\{e\}$ is a subalgebra, $C = \{e\}$. Nevertheless $\mathrm{C}\emptyset = \emptyset$ since one has "forgotten" to introduce the nullary operation corresponding to the unit element e as fundamental operation (as a matter of fact, this is an algebra "without constants" in the sense of example I).

EXAMPLE III. Let A consist of two different elements a, b, let f be the only non-trivial permutation of A considered as the only fundamental operation. Then there is no constant algebraic operation at all and therefore $C = \emptyset$, whereas $D = A$, since A is the only non-empty subalgebra B.

EXAMPLE IV. Any algebra of at least two elements such that any one-element subset is a subalgebra (e.g. let A be a lattice, an affine space, etc.).

Thus, we have three definitions of singular elements of an algebra which one might call *neutral elements* or even *algebraic constants* [5], these three definitions connected with sets $\mathrm{C}\emptyset$, C, and D, and coinciding in a species with constants, yet actually different in general. The distinction between these three notion plays a certain rôle in the following sections.

2. Algebraic, lattice-, and closure-independence. According to Marczewski, a subset M of algebra A is called an *independent subset of algebra* A if and only if any A-valuation β of M can be extended to a homomorphism φ of subalgebra $\mathrm{C}M$ into A; this is a special case of the notion of independence as dealt with in [10] § 3.

THEOREM 5. *Let M be an independent subset of algebra A, let M_1, M_2 be subsets of M. Then*

$$\mathrm{C}M_1 \cap \mathrm{C}M_2 \begin{cases} \subseteq C & \text{if} \quad M_1 \cap M_2 = \emptyset\,, \\ = \mathrm{C}(M_1 \cap M_2) & \text{if} \quad M_1 \cap M_2 \neq \emptyset\,. \end{cases} \tag{2}$$

Proof. An arbitrary element $c \in \mathrm{C}M_1 \cap \mathrm{C}M_2$ may be represented in the form

$$c = g(a|\ a \in M_1) = h(b|\ b \in M_2)$$

where g and h are algebraic operations of types M_1 and M_2, respectively [6]. In case $M_1 \cap M_2 = \emptyset$, g is constant, hence $c \in C$. In fact, let us consider a sequence of arbitrary elements $x_a \in A$ $(a \in M_1)$. There is a homomorphism φ of subalgebra $\mathrm{C}M$ into A such that $\varphi(a) = x_a$ for al $a \in M_1$, $\varphi(b) = b$ for all $b \in M_2$. We obtain

$$g(x_a|\ a \in M_1) = g\big(\varphi(a)|\ a \in M_1\big) = \varphi\big(g(a|\ a \in M_1)\big) = \varphi\big(h(b|\ b \in M_2)\big)$$
$$= h\big(\varphi(b)|\ b \in M_2\big) = h(b|\ b \in M_2) = c\,.$$

In case $M_1 \cap M_2 \neq \emptyset$, there is a homomorphism φ of $\mathrm{C}M$ into A such that $\varphi(a) \in M_1 \cap M_2$ for all $a \in M_1$, $\varphi(b) = b$ for all $b \in M_2$. The same calculation as above delivers $c = g\big(\varphi(a)|\ a \in M_1\big) \in \mathrm{C}(M_1 \cap M_2)$. We obtain $\mathrm{C}M_1 \cap \mathrm{C}M_2 \subseteq \mathrm{C}(M_1 \cap M_2)$; the converse inequality being trivial, even equality holds.

Let us notice that the inequality holding in the first of the two cases of (2) cannot generally be strenghtened to an equality. In fact, if $M_1 \cap M_2 = \emptyset$, we have

$$\mathrm{C}\emptyset \subseteq \mathrm{C}M_1 \cap \mathrm{C}M_2 \subseteq C\,,$$

the left-hand equality holding true if $M_1 = \emptyset$ or $M_2 = \emptyset$, the right-hand one if $M_1 \neq \emptyset$ and $M_2 \neq \emptyset$ since $C \subseteq D \subseteq \mathrm{C}M_1 \cap \mathrm{C}M_2$ in the latter case [7]. Thus, if $\emptyset = \mathrm{C}\emptyset \subset C = D$ (corollary of theorem 4, case II), then $\mathrm{C}M_1 \cap \mathrm{C}M_2 = \emptyset$ if and only if $M_1 = \emptyset$ or $M_2 = \emptyset$, $\mathrm{C}M_1 \cap \mathrm{C}M_2 = C$ if and only if $M_1 \neq \emptyset$ and $M_2 \neq \emptyset$. But for all other cases, we obtain the

COROLLARY [8]. *Let A be an algebra such that $C = \mathrm{C}\emptyset$, let M be an independent subset of A, then*

$$\mathrm{C}M_1 \cap \mathrm{C}M_2 = \mathrm{C}(M_1 \cap M_2) \tag{3}$$

for all subsets $M_1, M_2 \subseteq M$.

The hypothesis $\mathrm{C}\emptyset = C$ means that the sets of values of constant algebraic operations of a certain type L and of type $\emptyset$ respectively are the same (strengthening the statement of theorem 3); or as $\mathrm{C}\emptyset = C$ is equivalent with $C \subseteq \mathrm{C}\emptyset$, we may simply say: any value of a constant algebraic operation of arbitrary type L is the value of a nullary algebraic operation. This is the case if and only if there is no constant algebraic operation at all $(C = \emptyset)$ or there is at least one nullary *fundamental* operation $(\mathrm{C}\emptyset \neq \emptyset$, i.e. A is an algebra "with constant"). As a matter of fact, this condition is a very weak one. For in the only case it does not hold originally (case II as quoted above), we may slightly alter the algebraic structure of A by introducing one single new fundamental

(5) In [7] and [8], the elements of D or $\mathrm{C}\emptyset$, respectively, have been called neutral (with respect to closure), whereas Marczewski [4], [5] calls the elements of C algebraic constants.

(6) Cf. [10] theorem 5, corollary 1; also Marczewski [5] § 1.3 (ii).

(7) Thus, if algebra A has an independent subset M of cardinal number $|M| \geqslant 2$, then $C = D$.

(8) In the case of an algebra with finitary fundamental operations, cf. Marczewski [5] p. 56 (vi). As Marczewski gives another definition of algebraic closure, the hypothesis $\mathrm{C}\emptyset = C$ can be omitted, cf. footnote 9.

operation, namely the nullary operation corresponding to a certain element $c \in C$, thus obtaining an algebra "with constants", the closure operator C^* of which fulfils condition $\mathrm{C}^*\emptyset = C$ (C having the same meaning in the original and in the altered algebra) [9].

There is an easy lattice-theoretic interpretation of (3). Instead of set M, we may consider the family of principal subalgebras $\mathrm{C}\{a\}$ generated by elements $a \in M$. Then (3) may be written

$$\sum_{a_1 \in M_1} \mathrm{C}\{a_1\} \cap \sum_{a_2 \in M_2} \mathrm{C}\{a_2\} = \sum_{a \in M_1 \cap M_2} \mathrm{C}\{a\}, \tag{4}$$

where Σ denotes join (*sum*, *compositum*) in the complete lattice of subalgebras; (4) represents a rather strong form of lattice-independence. This lattice-theoretic property of algebraically independent sets still can be strengthened, due to

THEOREM 6. [10] *Let A be an algebra of cardinal number $|A| \neq 1$; then an independent element a is not the value of a constant algebraic operation.*

Here, the independence of element a naturally means the (algebraic) independence of set $\{a\}$.

Proof. Let us assume a to be the value of a constant algebraic operation h of, let us say, type L. Let x be an arbitrary element of A. There is a homomorphism φ of principal subalgebra $\mathrm{C}\{a\}$ into A such that $\varphi(a) = x$. Selecting an arbitrary sequence of type L in A, e.g. $a_\lambda = a$ ($\lambda \in L$), we obtain

$$x = \varphi(a) = \varphi\big(h(a_\lambda|\ \lambda \in L)\big) = h\big(\varphi(a_\lambda)|\ \lambda \in L\big) = a,$$

hence $|A| = 1$, contradicting hypothesis.

In case $|A| = 1$, the conclusion of theorem 6 becomes false, the only element $a \in A$ being independent and at the same time the value of a constant algebraic operation (all operations on A being constant).

[9] The subalgebras in the new sense are precisely the non-empty subalgebras of the original algebra and C (in [7], the author did the same with D instead of C, calling the subalgebras in this sense "ideals"). Marczewski [5] p. 49 always uses closure C^* instead of C, i.e. this restricted notion of subalgebra. The trouble is that even then we cannot generally define a subalgebra as always being non-empty (as Marczewski wanted to do [5] p. 49): cf. cases III and IV in the corollary of theorem 4. Moreover, Marczewski [5] p. 50 has hinted upon the relative character of closure C^*, whereas closure C is of absolute character.

[10] In the case of an algebra with finitary fundamental operations, cf. Marczewski [5] p. 55 (i). The converse is not generally true: there are "self-dependent" (Nitka [6]) elements which are not algebraic constants, cf. Marczewski [5], p. 55.

COROLLARY. *Let A be an algebra of cardinal number $|A| \neq 1$; then*

$$\mathrm{C}\{a\} \supset C \tag{5}$$

for any independent element $a \in A$.

For $\mathrm{C}\{a\} \supseteq D$ by definition of D, hence $\mathrm{C}\{a\} \supseteq C$ by theorem 4. But equality would imply $a \in C$ which contradicts theorem 6 [11].

In other words: all principal subalgebras $\mathrm{C}\{a\}$ generated by independent elements, in particular by elements of an independent set $M \subseteq A$ [12], are different from subalgebra C, which is the zero of the lattice of all subalgebras in the regular case $\mathrm{C}\emptyset = C$.

From "lattice-independence" as laid down in theorem 5 and the corollary of theorem 6, we immediately derive closure-independence:

THEOREM 7. [13] *Let A be an algebra of cardinal number $|A| \neq 1$, then any algebraically independent subset M of A is closure-independent, i.e. a minimal generating subset of subalgebra $\mathrm{C}M$.*

Proof. Let us assume the existence of a proper subset $M' \subset M$ such that $\mathrm{C}M' = \mathrm{C}M$: we find an element $a \in M - M'$, and since $a \in \mathrm{C}\{a\}$ and $a \in \mathrm{C}M'$, we obtain $a \in C$ by theorem 5, contradicting theorem 6.

3. Element basis and algebraic operations depending on all variables. Considering an algebra A, an independent subset M, and an element $x \in \mathrm{C}M$, we have a unique coordinate representation of x in the form $x = h(a|\ a \in M)$, where h is an algebraic operation of type M [14]. Notwithstanding this uniqueness, there might be a proper subset $M_0 \subset M$ such that $x \in \mathrm{C}M_0$: we should obtain another unique coordinate representation $x = g(a|\ a \in M_0)$, where g is an algebraic operation of this smaller type M_0. In other words: the first unique coordinate representation $x = h(a|\ a \in M)$ might be redundant, it might contain superflous "variables".

The question arises: is there a least or at least a minimal subset $M_0 \subseteq M$ such that $x \in \mathrm{C}M_0$, and what would be the meaning of the corresponding unique coordinate representation?

[11] As a matter of fact, $a \in C$ is equivalent with $\mathrm{C}\{a\} = C$, hence the corollary states essentially the same as theorem 6.

[12] Any subset of an independent set being independent, cf. Marczewski [5] p. 53 (for finitary algebras), J. Schmidt [10] theorem 7.3.

[13] In the case of an algebra with finitary fundamental operations, cf. Marczewski [5] § 2.3 (v) and (vii), the converse not being generally true according to ibid. p. 56. In the infinitary case, a direct proof (without reference to our theorems 5 and 6) has been given in [10] theorem 23.

[14] [10] theorem 11; this is essentially the original definition of independence given by Marczewski in [3] and [5].

Let us consider an operation h (algebraic or not) of type L, let L_0 be a subset of L. Then h *depends on* L_0 if there are sequences $\mathfrak{a}$, $\mathfrak{b}$ of type L which do not differ on $L-L_0$ (i.e. $\mathfrak{a}(\lambda)=\mathfrak{b}(\lambda)$ for all $\lambda\in L-L_0$) such that $h(\mathfrak{a})\neq h(\mathfrak{b})$; h *is independent of*, i.e. *does not depend on* L_0, otherwise. These definitions are frequently used in the special case $L_0=\{\lambda_0\}$; then we also say that h depends on or is independent of the "λ_0th variable" [15]. Let us consider the system $\mathfrak{J}$ of all subsets $L_0\subseteq L$ of which h is independent. It is easy to see that $\mathfrak{J}$ is an ideal: $\emptyset\in\mathfrak{J}$; if $L_0\in\mathfrak{J}$ and $L_1\subseteq L_0$, then $L_1\in\mathfrak{J}$; if $L_1, L_2\in\mathfrak{J}$, then $L_1\cup L_2\in\mathfrak{J}$. We call $\mathfrak{J}$ the *independence ideal of operation* h. There are two extreme special cases: $\mathfrak{J}=\{\emptyset\}$ means that h depends on all (its) variables; $\mathfrak{J}=\mathfrak{P}(L)$ means that h is constant.

THEOREM 8. *Let A be an algebra such that $C=\mathrm{C}\emptyset$, let h be an algebraic operation of type L on A, $(a_\lambda)_{\lambda\in L}$ a sequence without repetitions of type L in A. Let its range M_0 be minimal for the generation of $x=h(a_\lambda|\ \lambda\in L)$, i.e. $x\notin\mathrm{C}M'$ for all proper subsets $M'\subset M_0$. Then h depends on every of its variables.*

Proof. Let us assume the existence of an index $\lambda_0\in L$ such that h does not depend on λ_0. In case $L=\{\lambda_0\}$, h would be constant, hence $x\in C$, i.e. $x\in\mathrm{C}\emptyset$ by hypothesis, contradicting the minimality of $M_0=\{a_{\lambda_0}\}\neq\emptyset$. In case $L\neq\{\lambda_0\}$, we might define a transformation σ of L onto $K=L-\{\lambda_0\}$ which carries λ_0 into $\lambda_1\neq\lambda_0$, leaving fixed all other elements of L. As the sequences $(a_\lambda)_{\lambda\in L}$ and $(a_{\sigma(\lambda)})_{\lambda\in L}$ only differ on the argument $\lambda=\lambda_0$, by assumption we obtain

$$x=h(a_\lambda|\ \lambda\in L)=h(a_{\sigma(\lambda)}|\ \lambda\in L)\,,$$

hence $x\in\mathrm{C}M'$, where $M'=\{a_{\sigma(\lambda)}|\ \lambda\in L\}$. But as sequence $(a_\lambda)_{\lambda\in L}$ is without repetitions (i.e. $\lambda\to a_\lambda$ one-one), $a_{\lambda_0}\notin M'$: M' is a proper subset of M_0, again contradicting the minimality of M_0.

In the special case of an algebra with finitary fundamental operations, we obtain the

THEOREM 9. [16] *Let A be an algebra with finitary fundamental operations such that $C=\mathrm{C}\emptyset$, let M be an arbitrary subset of A. Then any element $x\in\mathrm{C}M$ can be represented in the form $x=h(a_\lambda|\ \lambda\in L)$, where h is an algebraic operation of type L depending on every of its variables, $(a_\lambda)_{\lambda\in L}$ a sequence without repetitions of type L in M. We may choose $L=M_0$, where M_0 is a finite subset of M, and $a_\lambda=\lambda$ for all $\lambda\in L=M_0$.*

Proof. Due to the finiteness property of closure, there exists a finite subset $F\subseteq M$ such that $x\in\mathrm{C}F$, and as in its power set $\mathfrak{P}(F)$ the minimal

[15] Marczewski [4] p. 611.

[16] Cf. Marczewski [5] § 1.3 (iii).

condition holds (Tarski's finiteness criterion), there is a minimal subset $M_0\subseteq F$ such that $x\in\mathrm{C}M_0$: x can be represented in the form $x=h(a_\lambda|\ \lambda\in L)$, where h is an algebraic operation of type L, $(a_\lambda)_{\lambda\in L}$ a sequence without repetitions of type L in M_0 (e.g. we may take $L=M_0$, $a_\lambda=\lambda$) and therefore in M. By theorem 8, h depends on every of its variables.

The theorem cannot be generalized to arbitrary algebras with infinitary fundamental operations, as we shall see later.

Let us still notice that we did not need the algebraic independence of M_0 in theorem 8, as will be the case in the converse

THEOREM 10. *Let A be an algebra such that $C=\mathrm{C}\emptyset$, let h be an algebraic operation of type L on A which depends on all its variables, let $(a_\lambda)_{\lambda\in L}$ be a sequence without repetitions of type L in A. Let its range M_0 be a subset of the independent set M. Then M_0 is the minimum subset of M for the generation of $x=h(a_\lambda|\ \lambda\in L)$, i.e. $M_0\subseteq M'$ for all $M'\subseteq M$ such that $x\in\mathrm{C}M'$.*

Proof. First, let us assume the existence of a proper subset $M'\subset M_0$ such that $x\in\mathrm{C}M'$. There would be an index $\lambda_0\in L$ such that $a_{\lambda_0}\notin M'$. Therefore $x\in\mathrm{C}\{a_\varkappa|\ \varkappa\in K\}$ where $K=L-\{\lambda_0\}$, hence $x=g(a_\varkappa|\ \varkappa\in K)$ where g is an algebraic operation of type K on A. Let τ be the identical transformation of K into L, then

$$h(a_\lambda|\ \lambda\in L)=x=g(a_\varkappa|\ \varkappa\in K)=g(a_{\tau(\varkappa)}|\ k\in K)=g_{\tau,L}(a_\lambda|\ \lambda\in L)\,.$$

Hence, $g_{\tau,L}$ being an algebraic operation of type L as well as h, $(a_\lambda)_{\lambda\in L}$ being a sequence without repetitions in the independent set M, we obtain $h=g_{\tau,L}$ by Marczewski's independence criterion [17]. But then h does not depend on λ_0; for let $(x_\lambda)_{\lambda\in L}$ and $(y_\lambda)_{\lambda\in L}$ be sequences of type L in A which only differ on λ_0, then

$$\begin{aligned}h(x_\lambda|\ \lambda\in L)&=g_{\tau,L}(x_\lambda|\ \lambda\in L)=g(x_\varkappa|\ \varkappa\in K)=g(y_\varkappa|\ \varkappa\in K)\\&=g_{\tau,L}(y_\lambda|\ \lambda\in L)=h(y_\lambda|\ \lambda\in L)\,,\end{aligned}$$

contradicting hypothesis. Thus, as $x\in\mathrm{C}M_0$, M_0 is a minimal subset $M'\subseteq M$ such that $x\in\mathrm{C}M'$. But then M_0 even is the minimum of these subsets M'; for as algebra A is such that $C=\mathrm{C}\emptyset$, by (3) we have $x\in\mathrm{C}M_0\cap\mathrm{C}M'=\mathrm{C}(M_0\cap M')$, hence $M_0\cap M'=M_0$, i.e. $M_0\subseteq M'$.

We call the minimum of sets $M'\subseteq M$ such that $x\in\mathrm{C}M'$ the *element basis for x in M* [18]. By combination of theorems 8 and 10, we obtain

[17] Cf. footn. 14.

[18] For the special case of exchange structures, cf. MacLane [2] theorem 5, J. Schmidt [9] p. 243. In the case of absolutely free algebras, Słomiński [11] p. 49: support.

THEOREM 11. *Let A be an algebra such that $C = \mathrm{C}\emptyset$, let M be an independent subset of A. Then for an element $x \in \mathrm{C}M$, the following properties are equivalent*:

1. *the element basis for x in M exists*;
2. *x can be represented in the form $x = h(a|\ a \in M_0)$, where h is an algebraic operation on A depending on every of its variables, of type $M_0 \subseteq M$.*

Moreover, this representation of x is unique; in particular, M_0 is the element basis for x in M (unique irredundant coordinate representation).

Proof. $1 \to 2$: Let M_0 be the element basis for x in M; because of $x \in \mathrm{C}M_0$, x can be represented in the form $x = h(a|\ a \in M_0)$ where h is an algebraic operation of type M_0 in A. But sequence $(a)_{a \in M_0}$ being without repetitions, its range M_0 being a minimal set M' such that $x \in \mathrm{C}M'$, h depends on every of its variables according to theorem 8.

$2 \to 1$: The algebraic operation h depending on every of its variables, sequence $(a)_{a \in M}$ being without repetitions, its range M_0 being a subset of the independent set M, M_0 is the element basis for x in M according to theorem 10. This also proves the uniqueness of M_0 in this representation of x. Taking another representation $x = g(a|\ a \in M_0)$, where g is an algebraic operation (depending on all its variables), we obtain $h = g$ by Marczewski's independence criterion: the above representation of x is completely unique.

Let us note that the notion of element basis only depends on the closure operator C, theorem 11 thus establishing a connection between algebraic structure and its derived closure.

In many forthcoming cases, it will be useful to have a closely related closure-theoretic notion: C being an abstract closure operator on set A and $x \in \mathrm{C}M$, we call an element $a \in M$ *indispensable for x in M* ([19]) if $x \notin \mathrm{C}(M - \{a\})$. We have the purely closure-theoretic

THEOREM 12. *Let A be a set, C a closure operator on A, M a subset of A. Then for an element $x \in \mathrm{C}M$, the following properties are equivalent*:

1. *the element basis for x in M exists*;
2. *$x \in \mathrm{C}M(x)$, where $M(x)$ is the set of all elements indispensable for x in M.*

In this case, $M(x)$ is the element basis for x in M.

Proof. If $x \in \mathrm{C}M'$, where $M' \subseteq M$, then $M(x) \subseteq M'$; for if $a \in M - M'$, then $M' \subseteq M - \{a\}$, hence $x \in \mathrm{C}(M - \{a\})$, i.e. $a \notin M(x)$. Thus, if $x \in \mathrm{C}M(x)$, $M(x)$ is the element basis for x in M. Conversely, let M_0 be the element basis for x in M; because of $x \in \mathrm{C}M_0$, we have $M(x) \subseteq M_0$. Here equality holds, for if $a \in M - M(x)$, we obtain $x \in \mathrm{C}(M - \{a\})$, hence $M_0 \subseteq M - \{a\}$, i.e. $a \notin M_0$.

([19]) Cf. Löwig [1] p. 71. We may conclude that $x \notin \mathrm{C}(M' - \{a\}$ for all subsets $M' \subseteq M$, in particular if $x \in \mathrm{C}M'$. Cf. also J. Schmidt [8] p. 34.

4. Element basis property. As we shall see later, the element basis for x in M does not always exist: $x \in \mathrm{C}M(x)$ is not generally true. It seems natural to study those subsets M such that $x \in \mathrm{C}M(x)$ holds for all $x \in \mathrm{C}M$. For sake of shortness of reference, we may define this purely closure-theoretic property of M as the *element basis property*. Again, we have the purely closure-theoretic

THEOREM 13. *Let A be a set, C a closure operator on A, M a subset of A. Then the following properties are equivalent*:

1. *M has the element basis property*;
2. *operator C is completely meet-preserving on $\mathfrak{P}(M)$, i.e.*

$$\mathrm{C}\bigcap_{t \in T} M_t = \bigcap_{t \in T} \mathrm{C}M_t \tag{6}$$

for any non-empty ([20]) *family $(M_t)_{t \in T}$ of subsets $M_t \subseteq M$* ([21]).

Proof. $1 \to 2$: $\mathrm{C}\bigcap M_t \subseteq \bigcap \mathrm{C}M_t$ is trivial. In order to prove the converse inequality, let us consider an element $x \in \bigcap \mathrm{C}M_t$. As T is non-empty, there is a $t_0 \in T$, and we have $x \in \mathrm{C}M_{t_0}$; as $M_{t_0} \subseteq M$, we also have $x \in \mathrm{C}M$. By hypothesis, there exists the element basis M_0 for x in M, and we have $M_0 \subseteq M_t$ for all $t \in T$, hence $M_0 \subseteq \bigcap M_t$; because of $x \in \mathrm{C}M_0$, we obtain $x \in \mathrm{C}\bigcap M_t$: $\bigcap \mathrm{C}M_t \subseteq \mathrm{C}\bigcap M_t$.

$2 \to 1$: Let $\mathfrak{M}_x$ be the system of all $M' \subseteq M$ such that $x \in \mathrm{C}M'$; since $x \in \mathrm{C}M$, $\mathfrak{M}_x$ is non-empty, hence $x \in \bigcap \mathrm{C}M' = \mathrm{C}\bigcap M'$: $M_0 = \bigcap M' \in \mathfrak{M}_x$, i.e. M_0 is the element basis for x in M.

Thus, the element basis property may be considered as a very strong form of lattice-independence analogous to (4).

Now, there are some special cases in which the algebraically independent subset M of an algebra A has the element basis property. First, as an immediate consequence of theorems 9 and 11, we have the important

THEOREM 14. *Let A be an algebra with finitary fundamental operations such that $C = \mathrm{C}\emptyset$, let M be an independent subset of A. Then M has the element basis property; moreover, the element bases for x in M are finite, for all elements $x \in \mathrm{C}M$.*

As it is usual to restrict the types L of finitary operations to initial intervals of natural numbers, $L = \{1, 2, \ldots, n\}$ $(n = 0, 1, 2, \ldots)$, we may

([20]) $T \neq \emptyset$; in case $T = \emptyset$, (6) may be considered as true by adequate agreement about the (relative) intersection of the empty family.

([21]) This is a transfinite strengthening of (3) first considered in [9] p. 248 (I).

give the unique irredundant coordinate representation in the case considered in theorem 14 as follows:

THEOREM 15. [22] *Let A be an algebra with finitary fundamental operations such that $C = \mathrm{C}\emptyset$, let M be an independent subset of A. M may be totally ordered by an ordering relation $<$. Then any element $x \in \mathrm{C}M$ can be represented in one and only one way in the form $x = h(a_1, \ldots, a_n)$ where h is an algebraic operation of type $L = \{1, \ldots, n\}$ depending on every of its variables, and $a_1, \ldots, a_n \in M$, moreover $a_1 < a_2 < \ldots < a_n$ in the given total ordering of M.*

For the element basis M_0 of x in M is totally ordered by the restriction of $<$; therefore, M_0 being finite, there is one and only one order-isomorphism $\lambda \to a_\lambda$ from an interval $\{1, \ldots, n\}$ onto M_0.

It is not possible to drop the finiteness hypothesis of theorem 14 without introducing other assumptions instead. For instance, there is the strongest possible independence assumption on M: M is called an *absolutely independent subset of algebra A* if and only if, for any similar algebra B, any B-valuation β of M can be extended to a homomorphism φ of subalgebra $\mathrm{C}M \subseteq A$ into B. It can be shown that in an algebra A with the fundamental operations f_i $(i \in I)$, of types K_i, respectively, subset M is absolutely independent if and only if the following *Generalized Peano Axioms* [23] hold:

P1. for any sequence $\mathfrak{a}$ of type K_i in $\mathrm{C}M$: $f_i(\mathfrak{a}) \notin M$;

P2. for any sequences $\mathfrak{a}$ and $\mathfrak{b}$ of types K_i and K_j in $\mathrm{C}M$:

$$f_i(\mathfrak{a}) = f_j(\mathfrak{b}) \quad \text{implies} \quad i = j \text{ and } \mathfrak{a} = \mathfrak{b}\,.$$

THEOREM 16. [24] *Any absolutely independent subset M of any algebra A has the element basis property.*

Proof. According to theorem 12, we have to show that $x \in \mathrm{C}M(x)$ for all $x \in \mathrm{C}M$; this will be done by algebraic induction on x. Inductive beginning: $x \in \mathrm{C}M(x)$ for all $x \in M$. In fact, if $x \in M$, then even $x \in M(x)$, i.e. $x \notin \mathrm{C}(M - \{x\})$ [25]: else we should obtain $x \in M - \{x\}$ or $x = f_i(\mathfrak{a})$ for some fundamental operation f_i and some sequence $\mathfrak{a}$ of type K_i in $\mathrm{C}(M - \{x\})$ [26], the first being impossible, the latter contradicting P1.

[22] Cf. Marczewski [5] § 2.2 (iv); Marczewski even uses this unique coordinate representation for the characterization of independent subsets.

[23] Löwig [1] p. 62 (2.1) and (2.2), Słomiński [11] p. 21 ($1.a_2$) and ($1.a_3$).

[24] Löwig [1] theorem 3.6. Słomiński [11] p. 49 tacitly assumes this theorem for granted.

[25] In other words: we show that an absolutely independent subset M (in fact, we only need P1) is closure-independent, without the hypothesis $|A| \neq 1$ as in theorem 7.

[26] General conclusion: $x \in \mathrm{C}M$ implies $x \in M$ or $x = f_i(\mathfrak{a})$ where f_i is a fundamental operation and $\mathfrak{a}$ a sequence in $\mathrm{C}M$, of the same type as f_i.

Inductive hypothesis: $x_\varkappa \in \mathrm{C}M(x_\varkappa)$ for some sequence of elements $x_\varkappa \in \mathrm{C}M$ $(\varkappa \in K_i)$; inductive conclusion: $x \in \mathrm{C}M(x)$, where $x = f_i(x_\varkappa|\ \varkappa \in K_i)$. In fact, for any $\varkappa \in K_i$, we obtain $M(x_\varkappa) \subseteq M(x)$. For if $a \in M - M(x)$, we have $x \in \mathrm{C}(M - \{a\})$, hence $x \in M - \{a\}$ or $x = f_j(y_\varkappa|\ \varkappa \in K_j)$ for some fundamental operation f_j and elements $y_\varkappa \in \mathrm{C}(M - \{a\})$; whereas the first again contradicts P1, the second delivers $i = j$ and $x_\varkappa = y_\varkappa \in \mathrm{C}(M - \{a\})$ according to P2 [27], hence $a \notin M(x_\varkappa)$ $(\varkappa \in K_i)$. Thus we have $\mathrm{C}M(x_\varkappa) \subseteq \mathrm{C}M(x)$, hence by inductive hypothesis $x_\varkappa \in \mathrm{C}M(x)$, hence $x \in \mathrm{C}M(x)$.

There is a well-known close connection between *absolutely free algebras*, i.e. algebras A with a (necessarily unique) absolutely independent generating subset M, and formal languages: M consists of the "variables", A of all "formulas", the element basis of formula x then consisting of all variables "occurring in x".

5. A counter-example. In an algebra A with infinitary fundamental operations f_i, an independent subset M need not have the element basis property. The counter-example we are going to construct is related to the fact that there are infinitary operations h not depending on any variable but nevertheless non-constant: the independence ideal $\mathfrak{J}$ of h contains all one-element and therefore all finite subsets of the index set L, but does not coincide with the ideal $\mathfrak{P}(L)$ of all subsets of L. The most simple example of such operation h is given by an arbitrary (not too trivial) Fréchet-Urysohn limit-space A; selecting a certain element $a \in A$, we may define h of type $L = N$ = set of all natural numbers by

$$h(x_\nu|\ \nu \in N) = \begin{cases} \lim\limits_{\nu\to\infty} x_\nu & \text{if } (x_\nu)_{\nu\in N} \text{ is convergent,} \\ a & \text{if } (x_\nu)_{\nu \in N} \text{ is divergent.} \end{cases}$$

In order to prepare our counter-example, we state the following counterpart to theorem 10:

THEOREM 17. *Let h be a non-constant algebraic operation of type L on algebra A which does not depend on any of its variables, let $(a_\lambda)_{\lambda\in L}$ be a sequence without repetitions of type L in A, let its range M be an independent subset of A. Then the element $x = h(a_\lambda|\ \lambda \in L)$ (which belongs to $\mathrm{C}M$) does not possess an element basis in M.*

Proof. Let us consider an arbitrary subset $M_0 \subseteq M$ such that $x \in \mathrm{C}M_0$. Let K be the set of all $\varkappa \in L$ such that $a_\varkappa \in M_0$; then $M_0 = \{a_\varkappa|\ \varkappa \in K\}$: there is an algebraic operation g of type K on A such that $x = g(a_\varkappa|\ \varkappa \in K)$. Let τ be the identical transformation of K into L. As in the proof of theorem 10, we obtain $h = g_{\tau,L}$, i.e.

$$h(x_\lambda|\ \lambda \in L) = g(x_\varkappa|\ \varkappa \in K)$$

[27] In fact, we only need a relatively weak part of P2.

for any sequence $(x_\lambda)_{\lambda \in L}$ of type L in A. Therefore, since h is non-constant and does not depend on any of its variables, so it is with g. One concludes that the type K of g is infinite, in particular, K contains two different elements $\varkappa_0$ and $\varkappa_1$. We define a transformation σ of K onto $K-\{\varkappa_0\}$ which carries $\varkappa_0$ into $\varkappa_1$, leaving fixed all other elements of K. As in the proof of theorem 8 ([28]), we obtain

$$x = g(a_{\sigma(\varkappa)} \mid \varkappa \in K),$$

hence $x \in \mathrm{C}M'$, where $M' = \{a_{\sigma(\varkappa)} \mid \varkappa \in K\} = M_0 - \{a_{\varkappa_0}\}$ is a proper subset of M_0: M_0 is not the element basis for x in M.

We are now ready to construct our counter-example. Let us consider the class $\mathfrak{A}$ of all algebras A with precisely one fundamental operation, this fundamental operation being of type N (set of natural numbers) and not depending on any of its variables. As is easily shown, the class $\mathfrak{A}$ is a closed class in the usual sense that direct products, subalgebras, and homomorphic images of algebras belonging to $\mathfrak{A}$ again belong to $\mathfrak{A}$; moreover, $\mathfrak{A}$ is non-trivial in the sense that $\mathfrak{A}$ contains at least one algebra of more than one element. Therefore, as is well known, there is an algebra $A \in \mathfrak{A}$ which is $\mathfrak{A}$-freely generated by N: $N \subseteq A$, even $\mathrm{C}N = A$, moreover N is an $\mathfrak{A}$-*independent subset of* A, i.e. for any algebra $B \in \mathfrak{A}$, any B-valuation β of N can be extended to a homomorphism φ of $\mathrm{C}N$ $(=A)$ into B. The fundamental operation f of A, which does not depend on any of its variables, is non-constant. In fact, there is a countable algebra $B \in \mathfrak{A}$, the fundamental operation g of which is non-constant (e.g. take the algebra of rational numbers with the ordinary limit-operation extended with the help of 0 as selected element); there is a mapping β of N onto B, which may be extended to a homomorphism φ of A onto B: as g is non-constant, so is f. Let us consider the sequence without repetitions $(\nu)_{\nu \in N}$, the range N of which is an independent subset of algebra A. According to theorem 17, the element $x = f(\nu \mid \nu \in N)$ does not possess an element basis in N: N does not have the element basis property.

References

[1] H. F. J. Löwig, *On the properties of freely generated algebras*, J. reine angew. Math. 190 (1952), pp. 65-74.

[2] S. MacLane, *A lattice formulation for transcendence degrees and p-bases*, Duke Math. J. 4 (1938), pp. 455-468.

[3] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. 6 (1958), pp. 731-736.

([28]) Under the additional hypothesis $C = \mathrm{C}\emptyset$ (which we carefully avoided) we might simply apply theorem 8 to the effect that M_0 cannot be minimal.

[4] E. Marczewski, *Independence in some abstract algebras*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. 7 (1959), pp. 611-616.

[5] — *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45-61.

[6] W. Nitka, *Self-dependent elements in abstract algebras*, Coll. Math. 8 (1961), pp. 15-17.

[7] J. Schmidt, *Über die Rolle der transfiniten Schlußweisen in einer allgemeinen Idealtheorie*, Math. Nachr. 7 (1952), pp. 165-182.

[8] — *Einige grundlegende Begriffe und Sätze aus der Theorie der Hüllenoperatoren*, Ber. Math. Tag. Berlin 1953, pp. 21-48.

[9] — *Mehrstufige Austauschstrukturen*, Z. math. Logik Grundl. Math. 2 (1956), pp. 233-249.

[10] — *Algebraic operations and algebraic independence in algebras with infinitary operations*, Math. Japon. 6 (1964), pp. 77-112.

[11] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Rozprawy Mat. 18 (1959).

MATHEMATISCHES INSTITUT DER UNIVERSITÄT, BONN

Reçu par la Rédaction le 1. 1. 1963