Proof. It is completely the same as in the case of $\mu = 0$, which is classical and well known ([4], p. 113), whence omitted.

The above proposition had been given by Parovicenko in [8].

THEOREM 8. *If $\mathfrak{X}$ is an ω_μ-metric space and is compact (in the sense of* [9]), *then $\mathfrak{X}$ has a basis of power $\leqslant \aleph_\mu$, whence is bicompact (in the sense of* [9]).

Proof. By Th. 3, $\mathfrak{X}$ is a $(U)_{\aleph_\mu}$-space. Since $\mathfrak{X}$ is compact, every subset X of power $\geqslant \aleph_\mu$ has in $\mathfrak{X}$ a contact point of order $\geqslant 2$ (p_0 being a *contact point* of X of order $\geqslant 2$ means that for every neighbourhood $V(p_0)$ of p_0 the set $X \cdot V(p_0)$ contains at least two points of X, [10]), then from Theorem of [10], $\mathfrak{X}$ has a basis of power $\leqslant \aleph_\mu$. Then Th. 8 follows from Lemma 2 of [10] immediately.

Recalling Cor. 1 of Th. 6, we have the following

THEOREM 9. *For a Hausdorff ω_μ-additive compact (in the sense of* [9]) *space to be ω_μ-metrisable, it is necessary and sufficient that it have a basis of power $\leqslant \aleph_\mu$.*

Proof. Sufficiency. Follows from Th. 6 immediately.

Necessity. Follows from Th. 8 immediately.

The case $\mu = 0$ of this theorem is the well-know second metrisation theorem of P. Urysohn.

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On lattice-ordered groups

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Introduction. We shall be concerned with a lattice-ordered group G, written additively though not necessarily abelian, with the set P of its positive (i.e. $x \geqslant 0$) elements, and with homomorphisms, epimorphisms, etc. from G to other such groups (mainly totally ordered ones and their products) which are *always* understood to be non-trivial, and lattice-ordered group homomorphisms, i.e. meet and join as well as sum preserving. If $K \subset G$ is an l-ideal in G then G/K denotes the quotient group as lattice ordered group, i.e. with the partial ordering defined by the image of P under the natural mapping $G \to G/K$, and we recall that for lattice-ordered groups and their homomorphisms the First Isomorphism Theorem holds, i.e. if f: $G \to G'$ is an epimorphism and $f = g \circ h$ its factorization into the natural mapping h: $G \to G/\mathrm{Ker}(f)$ and the induced mapping g: $G/\mathrm{Ker}(f) \to G'$ then h is an epimorphism and g an isomorphism (1). Our main object is to study the epimorphisms from G to totally ordered groups T, to obtain characterizing conditions for the existence of "sufficiently many" of these and hence of embeddings of G into products of such T, and to consider particular types of such embeddings. Some of our results can be regarded as an extension of those of Ribenboim [6] who restricted himself to the abelian case. The possibility of this extension is suggested by Lorenzen's theorem on regular lattice ordered groups [5] for which a proof is given in the present setting. The methods used here differ from the approach in [5] or in [6], the latter since we are able to dispense with Jaffard's notion of filet [4] in the proof of Proposition 3.

Particular subsets of P which will be of interest in the following are:

(i) the *filters* in P: the non-void subsets $F \subseteq P$ with $x \wedge y \in F$ for any $x, y \in F$ and $x \in F$ for any $x \geqslant y$ where $y \in F$;

(ii) the *prime filters* (2) in P: the proper filters Q in P for which $x + y \in Q$, x and y in P, implies $x \in Q$ or $y \in Q$;

(1) Terminology as in [2] unless stated otherwise.

(2) We use the term "prime" with respect to the group operation here rather than the lattice operation of forming the join. However, a prime filter in this sense is also prime with respect to join since $x + y \geqslant x \vee y$.

(iii) the *normal filters* in P: the filters F in P for which $a+F-a \subseteq F$ for each $a \in G$.

It will turn out that the normal prime filters in P play a key rôle in the study of the questions we are concerned with.

1. Epimorphisms to totally ordered groups. Two epimorphisms $f_1\colon G \to G_1$ and $f_2\colon G \to G_2$ will be called *associated* iff there exists an isomorphism $g\colon G_1 \to G_2$ such that $g \circ f_1 = f_2$. The relation of being associated is an equivalence on the class of all epimorphisms $f\colon G \to G'$, and the resulting equivalence classes correspond, of course, one-to-one to the l-ideals of G. However, in the case of epimorphisms to *totally ordered* groups, there are other entities within G which are in one-to-one correspondence with these equivalence classes and which may be preferable for certain purposes, namely the normal prime filters in P. This is the content of

PROPOSITION 1. *For any homomorphism f of G into a totally ordered group T, $Q(f) = \{x \mid x \in P,\ f(x) > 0\}$ is a normal prime filter and $\mathrm{Ker}(f) = \{x \mid x \in G,\ |x| \notin Q(f)\}$. Conversely, for any normal prime filter Q in P, there exists an epimorphism f from G into a totally ordered group such that $Q = Q(f)$, namely the natural homomorphism $G \to G/K$ where K is the l-ideal $\{x \mid x \in G,\ |x| \notin Q\}$.*

Proof. That $Q(f)$ (which is non-empty by the exclusion of trivial homomorphisms) is a normal prime filter follows readily from the general observation that for any homomorphism $h\colon G \to G'$ and any such filter Q' in P', $h^{-1}(Q') \cap P$ is such a filter in P, and the fact that in a totally ordered group the strictly positive elements (i.e. $x > 0$) constitute a normal prime filter.

Next, if $x \in \mathrm{Ker}(f)$ then also $f(|x|) = 0$ and hence $|x| \notin Q(f)$; conversely, if $|x| \notin Q(f)$ for $x \in G$ then $f(|x|) = 0$ and therefore also $f(x) = 0$. Hence $\mathrm{Ker}(f) = \{x \mid x \in G,\ |x| \notin Q(f)\}$.

Now, consider any normal prime filter Q in P and the set $K = \{x \mid x \in G,\ |x| \notin Q\}$ determined by it. From $|-x| = |x|$ it is clear that $-x \in K$ whenever $x \in K$. To see that also $x+y \in K$ for any $x, y \in K$, one observes first that $x+y \leqslant |x|+|y|$ and $-y-x \leqslant |y|+|x|$, and hence $|x+y| = (x+y) \vee \vee(-y-x) \leqslant |x|+|y|+|y|+|x|$; since Q is a prime filter, the last inequality shows that $|x| \in Q$ or $|y| \in Q$ if $|x+y| \in Q$. It follows that $K+K \subseteq K$, and thus K is a subgroup of G. The normality of K is a consequence of the normality of Q and the fact that $|a+x-a| = a+|x|-a$ for any $a,\ x \in G$. Finally, if $|x| \notin Q$ and $|y| \leqslant |x|$ then, clearly, $|y| \notin Q$. In all, this establishes that K is an l-ideal of G, properly contained in G since $Q \neq \emptyset$.

It remains to be shown that $G^* = G/K$ is totally ordered. Denoting by x^* the image of $x \in G$ under the natural homomorphism $G \to G^*$, let $x^* \not\geqslant 0^*$. In particular, one then has $x \notin K$ and thus $|x| \in Q$. Now, from $|x| = (x \vee 0) + \big((-x) \vee 0\big)$ and the primeness of Q it follows that $x \vee 0 \in Q$ or $(-x) \vee 0 \in Q$. Since $(x \vee 0) \wedge \big((-x) \vee 0\big) = 0$, the former would imply $(-x) \vee 0 \in K$ and hence $x^* = \big((x \vee 0) - ((-x) \vee 0)\big)^* = (x \vee 0)^* \geqslant 0^*$ which contradicts the hypothesis. Therefore, one has $(-x) \vee 0 \in Q$, hence $x \vee 0 \in K$, and finally $-x^* = (-x)^* = \big((-x) \vee 0 - (x \vee 0)\big)^* = \big((-x) \vee 0\big)^* \geqslant 0^*$. This shows that $x^* \not\geqslant 0^*$ implies $-x^* \geqslant 0^*$, i.e. G^* is totally ordered.

The relation between epimorphisms $f\colon G \to T$ and filters in P stated in Proposition 1 permits some further remarks about such epimorphisms. Let $f\colon G \to T$ be called *composite* if there exist epimorphisms $f'\colon G \to T'$ and $g\colon T' \to T$ with $f = g \circ f'$ and $\mathrm{Ker}(g) \neq 0$, and *irreducible* otherwise. Then, one has the following characterization of irreducibility:

COROLLARY 1. *An epimorphism $f\colon G \to T$ is irreducible iff $Q(f)$ is a maximal normal prime filter in P.*

Proof. If $f\colon G \to T$ is composite and $f = g \circ f'$ as above, then $\mathrm{Ker}(f') \subset \mathrm{Ker}(f)$ since $\mathrm{Ker}(g) \neq 0$, and thus $Q(f) \subset Q(f')$, i.e. $Q(f)$ not a maximal normal prime filter in P. Conversely, if there does exist a normal prime filter $Q \supset Q(f)$ in P for an epimorphism $f\colon G \to T$ then f induces a mapping $g\colon T' \to T$ on the totally ordered group $T' = G/K$ for $K = \{x \mid x \in G,\ |x| \notin Q\}$, in view of $K \subseteq \mathrm{Ker}(f)$, which is readily seen to be an epimorphism with non-zero kernel, and $f = g \circ f'$ where $f'\colon G \to T'$ is the natural homomorphism. Hence, f is composite.

Another consequence of Proposition 1 is the following factorization statement:

COROLLARY 2. *For any epimorphism $f\colon G \to T$ there exists an irreducible epimorphism $f_0\colon G \to T_0$ and an epimorphism $f_1\colon T_0 \to T$ such that $f = f_1 \circ f_0$.*

Proof. The set of all normal prime filters in P is inductive.

Of course, the f_1 in Corollary 2 will be an isomorphism iff f itself is irreducible.

Remark. One may wonder whether the normal prime filters in P might not all be maximal, since the significance of the above corollaries hinges on this question. However, one can readily obtain examples which show this not to be the case. Let G be totally ordered abelian and *non-archimedean* and consider any $a, b \in G$ such that $0 < a \ll b$ (i.e. $na \leqslant b$ for all $n = 1, 2, \ldots$). Then, the set Q of all $x \gg a$ in G is non-void, is a normal filter in P, and furthermore prime, for if $x \leqslant na$ and $y \leqslant ma$ for x and y in P with suitable n and m then $x+y \leqslant (n+m)a$, and hence $x, y \notin Q$ implies $x+y \notin Q$. Since $Q \subset P$ and P itself is a normal prime filter, Q is not maximal. It might be added that there is a fairly obvious relation between the orders of magnitude in G and the prime filters in P in this case [1].

2. Subgroups of products of totally ordered groups. We now consider the product $\prod T_a$ of a family $(T_a)_{a\in I}$ of totally ordered groups, i.e. the group of all functions $u\colon I\to\bigcup T_a$ with $u(a)\in T_a$, under functional addition and the partial order given as usual by the condition that $u\geqslant 0$ iff $u(a)\geqslant 0$ for all $a\in I$, and assume throughout this section that G is a *full subgroup* of $\prod T_a$, i.e. a subgroup (and sublattice!) such that at each $a\in I$ the $u\in G$ take on all available values in T_a. With each $u\in G$ we associate the subset $S(u)=\{a|\ u(a)\neq 0\}$ of I. Since G is full, it follows that $I=\bigcup S(u)$ $(u\in G)$, and the fact that the T_a are totally ordered implies that $S(u\wedge v)=S(u)\cap S(v)$ and $S(u\vee v)=S(u)\cup S(v)$ for any $u,v\geqslant 0$ in G. Since $u\in G$ implies $|u|\in G$ and $S(u)=S(|u|)$, the sets $S(u)$, $u\in G$, therefore form the basis of a topology $\mathfrak{D}$ in I. If the $S(u)$, $u\in G$, are all closed in this topology, besides being open by definition, then G will be called *zero-dimensional*. We are interested here in obtaining a criterion for G to have this property.

PROPOSITION 2. *G is zero-dimensional iff, for each $a\in I$, the normal prime filter $Q_a=\{u|\ u\in P,\ u(a)>0\}$ in P is an ultrafilter.*

Proof. First, let G be zero-dimensional and take any $u\in P$ such that $u\notin Q_a$, i.e. $u(a)=0$. Then, $a\notin S(u)$ implies by hypothesis the existence of a $v\in G$ such that $a\in S(v)$ and $S(v)\cap S(u)=\emptyset$. Of course, $v\geqslant 0$ may be assumed since $S(v)=S(|v|)$, and this shows that $v\in Q_a$ with $u\wedge v=0$. It follows that Q_a is an ultrafilter.

Conversely, let Q_a be an ultrafilter for some particular $a\in I$ and take any $u\in P$ such that $a\notin S(u)$. This means $u(a)=0$ and hence $u\notin Q_a$. By hypothesis, it then follows that $u\wedge v=0$ for some $v\in Q_a$, and one therefore has that $a\in S(v)$ and $S(u)\cap S(v)=\emptyset$. Hence, for any $a\in I$ whose Q_a is an ultrafilter, if $a\notin S(u)$ then a does not belong to the closure of $S(u)$, either. Now, if *all* Q_a are ultrafilters, this latter statement holds for *all* $a\in I$, and then each $S(u)$, $u\in G$, is closed in the topology $\mathfrak{D}$.

The following example shows that *a large subgroup of a product* $\prod T_a$ *need not be zero-dimensional*. Let G be the additive group of all continuous real functions on a completely regular connected Hausdorff space E, partially ordered as usual. Then, G is a large subgroup of $\prod T_x$ $(x\in E)$ where $T_x=R$, the additive group of reals in its natural order. For each $u\in G$, $S(u)$ is open and the topology of E is generated by these sets; hence no $S(u)$ distinct from 0 and E is also closed since E is connected. Thus, G is not a zero-dimensional subgroup of $\prod T_x$. The spaces E for which G *is* a zero-dimensional subgroup of $\prod T_x$ constitute a well-known class: they are the P-spaces of [3]. Moreover, if E is chosen suitably (e.g. $E=R$) then each $x\in E$ belongs to the boundary of some $S(u)$, $u\in G$, and in this case, *none* of the normal prime filters Q_x is an ultrafilter in P. In actual fact, in this case the ultrafilters in P can be seen to correspond to the maximal filters $\mathfrak{M}$ in the lattice of all open sets; for any such $\mathfrak{M}$, $\{u|\ S(u)\in\mathfrak{M},\ u\geqslant 0\}$ is an ultrafilter in P, and all ultrafilters in P are of this kind.

3. Realizations by totally ordered groups. A monomorphism f of G into the product of a family $(T_a)_{a\in I}$ of totally ordered groups for which the image $f(G)$ is a large subgroup of $\prod T_a$ will be called a *realization of G by totally ordered groups* or, simply, a realization of G. If $f(G)$ is, moreover, a zero-dimensional subgroup of $\prod T_a$ then the realization f will be referred to as zero-dimensional. According to Ribenboim [6], G does have zero-dimensional realizations if it is abelian; to what extent this remains true for arbitrary G will be settled by [3]

PROPOSITION 3. *The following conditions are equivalent:*

(i) *G has a zero-dimensional realization.*

(ii) *G has a realization.*

(iii) *No strictly positive element of G is disjoint from any of its conjugates.*

(iv) *Each ultrafilter in P is normal.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii). By hypothesis, $\bigcap\mathrm{Ker}(f)=0$ where f ranges over all homomorphisms $f\colon G\to T$, or perhaps better over a representative set for the different classes of associated epimorphisms $G\to T$. Hence $P-\{0\}=\bigcup Q(f)$ where the $Q(f)$ are normal prime filters in P by Proposition 1. It follows from this that each non-zero $a\in P$ belongs to a normal proper filter in P and therefore cannot be disjoint from any $x+a-x$.

(iii)$\Rightarrow$(iv). Let U be any ultrafilter in P and suppose that $x+a-x\notin U$ for some $a\in U$ and $x\in G$. Since U is an ultrafilter and P a distributive lattice, there then exists a $b\in U$ such that $(x+a-x)\wedge b=0$. Now consider $c=a\wedge b$ which belongs to U and hence is not 0; $c\leqslant a$ implies $x+c-x\leqslant x+a-x$, and from $c\leqslant b$ one then obtains $(x+c-x)\wedge c\leqslant(x+a-x)\wedge b=0$, hence $(x+c-x)\wedge c=0$. This contradicts (iii).

(iv)$\Rightarrow$(i). Apart from being normal by hypothesis, any ultrafilter U in P is also prime: if $x,y\in P$ do not belong to U then $x\wedge a=y\wedge b=0$ for suitable $a,b\in U$, and for $c=a\wedge b\in U$ one has $(x+y)\wedge c=0$ [2], i.e. $x+y\notin U$. Now, let Ω be the set of all ultrafilters in P, and for each $U\in\Omega$ consider the l-ideal $K(U)=\{x|\ x\in G,\ |x|\notin U\}$ and the natural homomorphism $h_U\colon G\to T(U)=G/K(U)$. Then $\bigcap K(U)(U\in\Omega)=0$ since each non-zero $a\in P$ belongs to the proper filter $[a,\to]$ in P and thus to some ultrafilter. It follows that the mapping $h\colon G\to\prod T(U)$ $(U\in\Omega)$

[3] The equivalence (ii)$\leftrightarrow$(iii) is due to Lorenzen [5] who calls a lattice-ordered group *regular* iff it satisfies (iii).

defined by $h(x) = \big(h_U(x)\big)_{U\in\Omega}$ is a realization, and since $U = \{x | x \in P, h_U(x) > 0\}$, h is zero-dimensional by Proposition 2.

In analogy to the definitions in Section 1, let a realization $f\colon G \to \prod T_a$ $(a \in I)$ be called *composite* if there exists a realization $f'\colon G \to \prod T'_a$ $(a \in I)$ and a family of epimorphisms $g_a\colon T'_a \to T_a$ such that $g = \prod g_a$ has non-zero kernel and $f = g \circ f'$, and *irreducible* otherwise. Then, the following characterization of irreducibility is a ready consequence of Proposition 3.

Corollary 1. *A realization $f\colon G \to \prod T_a$ is irreducible iff it is zero-dimensional.*

Proof. Let $f_a\colon G \to T_a$ be the epimorphism determined by f and the projection of the product to T_a. It is then clear that f is irreducible iff each f_a is. By Corollary 1 of Proposition 1 this holds iff each $Q(f_a)$ is a maximal normal prime filter, and since G has a realization here this holds, by Proposition 3, iff each $Q(f_a)$ is an ultrafilter. The assertion now follows from Proposition 2.

Another consequence of Proposition 3 is:

Corollary 2. *Any realization of G can be factored into an irreducible realization followed by a suitable epimorphism.*

Proof. With the same notations as above, let $f_a = g_a \circ f'_a$ be a decomposition of f_a such that $f'_a\colon G \to T'_a$ is an irreducible epimorphism. Then, $f = g \circ f'$ where f' is the realization $x \to \big(f'_a(x)\big)_{a\in I}$ and g the epimorphism and $\prod g_a$, it follows from our previous results that f' is irreducible.

Finally, the proof of Proposition 3 leads to the following observation:

Corollary 3. *If G has any realizations then $h\colon G \to \prod T(U)$ $(U \in \Omega)$ is universal in the sense that for any realization f of G, $f = g \circ h$ with suitable g.*

Proof. The epimorphism g can be described as first projecting $\prod T(U)$ $(U \in \Omega)$ onto a suitable partial product and then mapping the factors of this onto the factors which occur in the product for the representation f. Incidentally, h is, of course, essentially characterized by this property of universality.

We close this section with an example of a lattice-ordered group without realizations. Let H be the group of all increasing mappings of the real unit interval I onto itself, under functional composition (written multiplicatively) with $u \in H$ taken as positive iff $u(x) \geqslant x$ for all $x \in I$, i.e. $u \geqslant e$ (e the identity function) in the usual partial order of real-valued functions [2]. Now, let $u \in H$ be given by the straight line segments in $I \times I$ from $(0,0)$ to $(\frac{1}{4}, \frac{1}{2})$ and from $(\frac{1}{4}, \frac{1}{2})$ to $(1,1)$, and take any $v \in H$ such that $X = [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$ is its set fo fixed points and $e < v < u$, e.g. v given by the straight line segments from $(0,0)$ to $(\frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{2})$ to $(1,1)$ and suitably low circular arc above the straight line segment from $(\frac{1}{4}, \frac{1}{4})$ to $(\frac{1}{2}, \frac{1}{2})$. It follows that $u(X) = [0, \frac{1}{2}] \cup [u(\frac{1}{2}), 1]$, hence $I = u(X) \cup X$, and thus also $I = X \cup u^{-1}(X)$. Now, take any $a \in I$. If $a \in X$ then $a = v(a)$ and hence $u(a) = (uv)(a)$; on the other hand, if $a \in u^{-1}(X)$ then $u(a) \in X$ and $u(a) = (vu)(a)$. The function v being positive, one always has $u(a) \leqslant (uv)(a)$, $(vu)(a)$ and what has just been shown therefore amounts to the fact that $u(a)$ is always the smaller one of $(uv)(a)$ and $(vu)(a)$, i.e. $u = (uv) \wedge (vu)$. This implies that $e = v \wedge (u^{-1}vu)$ although $v > e$.

4. Filets.

A comparison of our method of proving Proposition 3 with the technique used by Ribenboim [6] in the abelian case can be obtained by making a few remarks about filets. Since everything needed for this purpose is purely lattice theoretic, i.e. independent of the presence of the group operation, we shall consider first any arbitrary distributive lattice L with least element o (in place of P). If $D(a) = \{x | x \in L, a \wedge x = 0\}$ for each $a \in L$ then $D(a \vee b) = D(a) \cap D(b)$, and $D(a \wedge b)$ is the smallest $D(x) \supseteq D(a) \cup D(b)$. Therefore, the set of all $[a] = \{x | D(x) = D(a)\}$, i.e. of the equivalence classes with respect to the equivalence relation $D(x) = D(y)$, can by partially ordered by setting $[a] \leqslant [b]$ iff $D(a) \supseteq D(b)$ and is a lattice with respect to this ordering, *the lattice $\mathfrak{F}(L)$ of filets of L*, such that $\Phi\colon x \to [x]$ is a lattice epimorphism from L to $\mathfrak{F}(L)$. $\mathfrak{F}(L)$ is distributive, has $[0]$ as its least element, and is disjunctive, i.e. if $[a] < [b]$ then there exists a non-zero $[c] \leqslant [b]$ disjoint from $[a]$; in addition we remark that any lattice homomorphism f of L onto a disjunctive distributive lattice L' which maps only 0 to 0 induces a lattice isomorphism $g\colon \mathfrak{F}(L) \to L'$ such that $f = g \circ \Phi$.

Φ induces a one-to-one correspondence between the ultrafilters in L and those in $\mathfrak{F}(L)$. First, one observes that for any ultrafilter U in L, $a \in U$ implies $[a] \subseteq U$ since $b \in [a]$ and $b \notin U$ would lead to $b \wedge x = 0$ for some $x \in U$ and hence to $a \wedge x = 0$ by $D(b) = D(a)$. Now, $\Phi(U)$ is clearly a filter in $\mathfrak{F}(L)$; moreover, $[a] \notin \Phi(U)$ means $[a] \cap U = \emptyset$ and hence $a \notin U$, which implies $a \wedge x = 0$ for some $x \in U$ and therefore $[a] \wedge [x] = [0]$. This shows that $\Phi(U)$ is an ultrafilter, and from $U = \bigcup [a]$ $(a \in U)$ it follows that $U \to \Phi(U)$ is one-to-one. It remains to show that this correspondence is onto. If $\mathfrak{V} \subseteq \mathfrak{F}(L)$ is any ultrafilter in $\mathfrak{F}(L)$, then $\Phi^{-1}(\mathfrak{V})$ is a filter in L, proper since $\mathfrak{V}$ is proper, and for any ultrafilter $U \supseteq \Phi^{-1}(\mathfrak{V})$ in L one has $\Phi(U) \supseteq \mathfrak{V}$ and therefore $\Phi(U) = \mathfrak{V}$.

As an immediate consequence one sees that the collections of subsets $\{x | x \in L, x \notin U\}$ and $\{x | x \in L, [x] \notin \mathfrak{V}\}$ of L, U ranging over the ultrafilters of L and $\mathfrak{V}$ over those of $\mathfrak{F}(L)$, are the same. For the lattice-ordered group G, this shows that the family $\big(T(U)\big)_{U\in\Omega}$ of totally ordered groups used in the previous section is essentially identical with the family of totally ordered groups employed in [5].

In order to obtain some information about the effect of a realization of the group G on its lattice of filets (meaning $\mathfrak{F}(P)$), we again consider first the lattice L (as above). If Δ is any set of proper filters in L, we shall call the topology on Δ generated by the sets $\Delta(a) = \{F \mid a \in F \in \Delta\}$ the *natural* topology on Δ, and denote the associated closure operator by Γ. Clearly, the restriction of the natural topology on the set Ψ of all proper filters in L to any $\Delta \subseteq \Psi$ is the natural topology on Δ. A subset $\Delta \subseteq \Psi$ is dense (in Ψ) iff $L - \{0\} = \bigcup F$ $(F \in \Delta)$.

For any dense subset Δ of Ψ, the sets $\Delta(a)$ $(a \in L)$ are closed in Δ, besides being open by definition, iff each member of Δ is an ultrafilter. Let $U \in \Delta$ be an ultrafilter and $a \in L$. Then $U \notin \Delta(a)$ implies that $a \wedge x = 0$ for some $x \in U$, and hence $U \in \Delta(x)$ and $\Delta(x) \cap \Delta(a) = \emptyset$; thus if $U \notin \Delta(a)$ then also $U \notin \Gamma\Delta(a)$. Therefore, if all members of Δ are ultrafilters then all $\Delta(a)$ are closed in Δ. Conversely, assume that $F \in \Delta$ is not maximal and take $a \in L$ such that $a \notin F$ but $a \wedge x \neq 0$ for all $x \in F$. Then $\Delta(a) \cap \Delta(x) = \Delta(a \wedge x) \neq \emptyset$, the latter since Δ is dense, for all $x \in F$; thus F belongs to the closure of $\Delta(a)$ in Δ but not to $\Delta(a)$, i.e. $\Delta(a)$ is not closed.

For any dense set Δ consisting of prime filters ([4]), *the relation θ: $[x] \to \Gamma\Delta(x)$ is a one-to-one join preserving mapping from $\mathfrak{F}(L)$ to $\mathfrak{P}(\Delta)$.* Since Δ is dense, one has $CD(a) = \{x \mid a \wedge x \neq 0\} = \bigcup F(F \in \Delta(a))$. Hence $D(a) \subseteq D(b)$ implies $\bigcup F(F \in \Delta(b)) \subseteq \bigcup F(F \in \Delta(a))$, and this leads to $\Gamma\Delta(b) \subseteq \Gamma\Delta(a)$ since $\Gamma\Delta(x) = \{G \mid G \in \Delta,\ G \subseteq \bigcup F(F \in \Delta(x))\}$. All steps being reversible, one therefore has that $[b] \leqslant [a]$ iff $\Gamma\Delta(b) \subseteq \Gamma\Delta(a)$. This proves that θ is, in fact, a mapping from $\mathfrak{F}(L)$ to $\mathfrak{P}(\Delta)$, one-to-one and order preserving. Concerning joins, one has $\theta([a] \vee [b]) = \theta([a \vee b]) = \Delta(a \vee b)$, and since Δ consists only of prime filters, one has $\Delta(a \vee b) = \Delta(a) \cup \Delta(b)$ and therefore also $\theta([a] \vee [b]) = \theta([a]) \cup \theta([b])$.

Combining the last two remarks, one now obtains: *For any dense set Δ consisting of ultrafilters, the correspondence $x \to \Delta(x)$ is a lattice monomorphism from $\mathfrak{F}(L)$ into $\mathfrak{P}(\Delta)$.*

We now return to the lattice ordered group G. If f: $G \to \prod T_a$ $(a \in I)$ is a realization of G then the set I is in an obvious correspondence (not necessarily one-to-one) with a dense set Δ of prime filters in P such that the natural topology on Δ corresponds to the topology determined by $f(G)$ on I as described in Section 2. Denoting now by $S(a)$ the subset of I consisting of those a for which $f(a)$ has non-zero value and by Γ the closure operator in the given topology on I, one has, as an immediate consequence from the above lattice theoretic considerations, the following generalization of Lemma 1 of [6]:

([4]) In the absence of other operations, "prime" here refers to the join in the lattice.

PROPOSITION 4. *For any realization $G \to \prod T_a$ $(a \in I)$, the correspondence $[x] \to \Gamma S(x)$ is a one-to-one join preserving mapping from the lattice of filets of G to $\mathfrak{P}(I)$; in particular, if the realization is zero-dimensional this correspondence reduces to $x \to S(x)$ and is a lattice monomorphism.*

5. Complete lattice-ordered groups. We now assume that the lattice-ordered group G under consideration is complete. As is well known, G is then abelian [2]; hence all prime filters in P, and in particular the ultrafilters, are normal and G possesses realizations. The completeness of G has certain interesting consequences for the epimorphisms f: $G \to T$, and, analogously, for the realizations of G which we shall investigate in this section.

In the following, $a \sim b$ will stand for $D(a) = D(b)$ where $a, b \in P$.

LEMMA. *For any a, $b \in P$, if $a \leqslant b$ then b has a unique sum decomposition $b = u + v$ with u and v in P, $u \sim a$ and $u \wedge v = 0$.*

Proof. The set $\{x \mid x \leqslant b,\ x \sim a\}$ is non-void and bounded and hence has a supremum which will be denoted by u. From $u \geqslant a$ it follows that $u \wedge y = 0$ implies $a \wedge y = 0$ for any $y \in P$. Conversely, if $a \wedge y = 0$ for $y \in P$ then $u \wedge y = y \wedge \bigvee x(a \sim x \leqslant b) = \bigvee (y \wedge x)(a \sim x \leqslant b) = 0$ by the infinite distributivity laws which hold in G [2]. Therefore, one has $u \sim a$. Now, $u \leqslant b$ implies that $b = u + v$ with $v \in P$. In order to see that $u \wedge v = 0$, consider $u + (u \wedge v)$. Clearly, $u + (u \wedge v) \sim u$, and by $u + (u \wedge v) \leqslant b$ and the resulting $u + (u \wedge v) \sim a$ one has $u + (u \wedge v) \leqslant u$; hence $u \wedge v = 0$. As to the uniqueness of this decomposition, let $b = x + y$ with $x \sim a$ and $x \wedge y = 0$ for any $x, y \in P$. By definition of u one has $x \leqslant u$, and by $x \sim u$ also $u \wedge y = 0$; from $b = x + y = x \vee y$ it now follows that $u = u \wedge x \leqslant x$, and thus $u = x$ and $v = y$.

We remark that, as a consequence of this lemma, the lattice of filets of G is relatively complemented and hence a Boolean lattice (without unit, though).

PROPOSITION 5. *Each prime filter in P is contained in exactly one ultrafilter.*

Proof. Let Q be any prime filter in P, U any ultrafilter containing Q, and $Q^* = \{x \mid x \sim a$ for some $a \in Q\}$. From the previous section it is clear that $Q^* \subseteq U$. Now, for any $b \in U$, take an arbitrary $c \in Q$ and consider the decomposition $c = u + v$ where $u \sim b \wedge c$, $u \wedge v = 0$ and $u, v \in P$. Then it follows from $b \wedge c \in U$ and $(b \wedge c) \wedge v = 0$ that $v \notin Q$, and hence $u \in Q$ since Q is prime. Therefore, one also has $b \vee u \in Q$, and since $b \wedge x = 0$ implies $(b \wedge c) \wedge x = 0$, thus $u \wedge x = 0$ and finally $(b \vee u) \wedge x = 0$ one obtains $b \sim b \vee u$, i.e. $b \in Q^*$. This proves that $U = Q^*$; hence Q^* is an ultrafilter, and thus the only ultrafilter containing Q.

As an immediate consequence of this proposition one now has the following

COROLLARY. *If f is any epimorphism from G to a totally ordered group, or any realization of G then the decomposition $f = f_1 \circ f_0$ where f_0 is irreducible and f_1 an epimorphism is essentially unique.*

Remark. Proposition 5 is closely related to the fact that in certain integral domains, e.g. the ring of entire functions on the complex plane, each prime ideal is contained in exactly one maximal ideal [1].

6. Concluding remarks. In view of Proposition 1, the lattice-ordered group G has *no* realization iff the union W of all normal prime filters in P is smaller than $P-\{0\}$; $K_0 = \{x|\ x \in G,\ |x| \notin W\}$ is then the l-ideal of G consisting of all those elements of G which vanish under every homomorphism $G \to T$, and G/K_0 is the largest quotient group of G which does have realizations. This does not, however, describe W and K_0 internally in terms of the elements of G, and it might be of interest to have a characterization of this latter kind. It is clear that W consists of elements $a \in G$ such that no $(x_1+a-x_1) \wedge \ldots \wedge (x_k+a-x_k)$ $(x_i \in G)$ can be 0, but whether, say, W *is* the set of these elements remains an open question.

Another problem which arises naturally here is that of the existence of realizations $G \to \prod T_a$ where all T_a are archimedean. It is easy to see, for an epimorphism f: $G \to T$, that T is archimedean iff $Q(f)$ is a *minimal* normal prime filter, and hence G has realizations of the said type iff $P-\{0\}$ is the union of all minimal normal prime filters in P. Again, it seems desirable to have an alternative condition in terms of the elements of G, such as Proposition 3 provides for the existence of realizations in general.

Finally, we remark that the present approach to realizations of lattice-ordered groups might also be useful for the study of (analogously defined) realizations of lattice-ordered rings.

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Some properties of algebraically independent sets in algebras with infinitary operations

by

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The purpose of this paper is to continue the study of independence in algebras with infinitary operations as begun in [10]; in particular, to generalize some results of Marczewski [5] on independent subsets of finitary algebras. In Section 1, we discuss some notions of "neutral" or "singular" elements which are actually different as is shown by examples; the resulting necessary distinction represents the reason for some small complication occurring in the following sections. In Section 2, Marczewski's results on the relations between algebraic, lattice, and closure-independence are generalized (the proofs using the technique of algebraic operations here instead of Marczewski's technique of transformations of variables). In Section 3, the fundamental notion of element basis is introduced in general, only two special cases having been considered hitherto: one in Steinitz-Van Der Waerden exchange structures (MacLane [2], J. Schmidt [9]), the other in absolutely free algebras (Löwig [1], Słomiński [11]); the interrelations are studied between the existence of the element basis for element x and the representability of x by algebraic operations depending on all variables. In Section 4, the existence of the element basis for all elements x in the algebraically independent generating set M is secured in the special cases of finitary algebras and (reproducing a result of Löwig [1]) of absolutely free algebras, whereas in Section 5, an example is given for an element x in an algebra A (necessarily infinitary and not absolutely free) without element basis in the algebraically independent generating set M. The paper is reasonably self-containing; in particular, it can be read without knowledge of [10].

1. Neutral elements of different types. In [10] § 1, we have considered the natural one-one correspondence between elements of set A and operations of type Ø (empty set), i.e. nullary operations, on A. This natural correspondence is an isomorphism from algebra A onto algebra $O^{\emptyset}(A)$ of all nullary operations on set A, the converse of this isomor-