On a certain result of Leray

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The result to which we refer is Lemma 8 on page 121 of [5]. We propose to obtain a more general result by using a modern proof based on the theory of spectral sequences. We then derive as a corollary the well-known theorem on acyclic coverings due also to Leray.

The theory of covertures has been set up by Leray in [5] and [6]; good accounts of it may also be found in [1] and [7]. We recall some basic definitions from this theory and refer the reader for the basic facts concerning spectral sequences to [6], [4], or [3].

Let A be a ring and X a topological space. An A-*complex* C on X is an A-module with whose each element $c \in C$ is associated a closed subset of X, called the *carrier* of c and denoted by $S(c)$, such that the following relations hold:

(S_1) $S(c) = \emptyset$ if and only if $c = 0$,

(S_2) $S(c+c') \subset S(c) \cup S(c')$ $(c, c' \in C)$,

(S_3) $S(ac) = S(c)$ $(c \in C,\ a \in A,\ a \neq 0)$.

If C is graded and if c, c' are homogeneous of different degrees, we require that equality hold in (S_2); if C is differential, then we must have

(S_4) $S(dc) \subset S(c)$ $(c \in C)$,

where d is the differential operator.

Let Y be a subspace of X and C an A-complex on X. Let C_{X-Y} denote the elements of C whose carriers are contained in $X-Y$. We denote by YC the A-module C/C_{X-Y} and by Yc the image of $c \in C$ under the natural homomorphism $C \to C/C_{X-Y}$. It is easily seen that we may define for YC a structure of A-complex on Y by setting $S(Yc) = S(c) \cap Y$.

If M is a graded differential A-module, we denote as usual by $H^q(M)$ the q-th cohomology module of M.

By an A-*coverture* on X we mean [1] a graded differential A-complex K on X such that the degrees are $\geqslant 0$, the differential raises the degree by 1, and for each $x \in X$ the following conditions are satisfied:

[1] It should be noted that our definition is more general than the usual ones (cf. [6], [1], [7]) in that we do not require K to have the structure of an algebra or its elements to have compact carries, nor that K have a unit relatively to each compact subset of X.

(i) There exists a canonical isomorphism ψ_x of A onto $H^0(xK)$,

(ii) $H^p(xK) = 0$ for all $p > 0$.

The *intersection* $C \circ C'$ of two A-complexes on X is defined as follows: For each $g \in C \otimes C'$, define the carrier $S(g)$ to be the set of points $x \in X$ such that $(p_x \otimes p'_x) g \neq 0$, where $p_x\colon C \to xC$ and $p'_x\colon C' \to xC'$ are the natural homomorphisms.

Then $C \circ C'$ is the factor module of $C \otimes C'$ with respect to the submodule consisting of the elements with empty carrier. For any $c \in C$, $c' \in C'$, the image of $c \otimes c'$ under the natural homomorphism $C \otimes C' \to C \circ C'$ is denoted by $c \circ c'$.

Let A be a ring and X a topological space. An A-complex L is said to be *free*, if there exists a family $(l_i)_{i \in I}$ of elements of L such that each element $l \in L$ may be written uniquely as

$$l = \sum_{i \in I} a_i l_i \qquad (a \in A),$$

where all but a finite number of the a_i's are equal to zero; moreover,

$$S(l) = \bigcup S(l_i),$$

where the union is taken over all indices i for which $a_i \neq 0$ in the above representation of l. The family of elements $(l_i)_{i \in I}$ is said to be a *base* of L.

Let A be a commutative ring with a unit element and K an A-coverture on X. If Y is a subset of X, an element $u \in K$ is said to be a *unit relatively to* Y provided that for each $x \in Y$ the element xu of xK is homogeneous of degree 0, it is a cocycle and its cohomology class in $H^0(xK)$ corresponds to the unit element of A under the isomorphism ψ_x given by the definition of an A-coverture.

LEMMA. *Let A be a commutative ring with a unit element and X a Hausdorff space. Let K be an A-coverture and L a free differential A-complex on X, satisfying the following condition:*

(Γ) *For each element l_i of the base of L, there exists a unit $u_i \in K$ relatively to $S(l_i)$.*

Let $l = \sum_i a_i l_i$ be an arbitrary element of L. Then the element $\sum_i u_i \circ a_i l_i$ of $K \circ L$ does not depend on the choice of the units u_i.

The mapping defined by

$$f(l) = \sum_i u_i \circ a_i l_i$$

is a monomorphism of the differential module L into the differential module $K \circ L$.

Proof. Let u'_i be a second unit of K relatively to $S(l_i)$. Since $x(K \circ L) = xK \otimes xL$ ([7], p. 165) and since for $x \notin S(l_i)$, $xl_i = 0$ and for $x \in S(l_i)$, $xu_i = xu'_i$, we may write for any $x \in X$:

$$x(u_i \circ a_i l_i - u'_i \circ a_i l_i) = (xu_i - xu'_i) \otimes a_i \cdot xl_i = 0 .$$

Hence $S(u_i \circ a_i l_i - u'_i \circ a_i l_i) = \emptyset$ and, by (S_1), $u_i \circ a_i l_i = u'_i \circ a_i l_i$.

Assume now $f(l) = \sum_i u_i \circ a_i l_i = 0$. This means that, for each $x \in X$, we have

$$x\Big(\sum_i u_i \circ a_i l_i\Big) = \sum_i xu_i \otimes xa_i l_i = \sum_i a_i xu_i \otimes xl_i = 0 .$$

But xL is a free module having as base the elements xl_i such that $x \in S(l_i)$. It follows ([2], Exp. XI, p. 5) that $xK \otimes xL$ coincides with the group of finite formal linear combinations of elements xl_i with coefficients in xK. Hence we have for each i such that $x \in S(l_i)$, $a_i \cdot xu_i = 0$, i.e. $a_i = 0$. As the point x is arbitrary, we infer that $a_i = 0$ for all i, and therefore $l = 0$. Thus f is a monomorphism.

We now prove that f commutes with d. We have

$$df(l_i) = d(u_i \circ l_i) = du_i \circ l_i + u_i \circ dl_i .$$

Since $xdu_i = d(xu_i) = 0$ for any $x \in S(l_i)$ and $xl_i = 0$ for any $x \notin S(l_i)$, we have $x(du_i \circ l_i) = 0$ for any $x \in X$ and therefore $du_i \circ l_i = 0$. Hence

$$df(l_i) = u_i \circ dl_i .$$

On the other hand, if $dl_i = \sum_j b^i_j l_j$ we have

$$fd(l_i) = f\Big(\sum_j b^i_j l_j\Big) = \sum_j u_j \circ b^i_j l_j .$$

Let x be an arbitrary point of X. We may write

$$x\big(df(l_i) - fd(l_i)\big) = x\Big(u_i \circ \sum_j b^i_j l_j - \sum_j u_j \circ b^i_j l_j\Big)$$
$$= x\Big(\sum_j (u_i - u_j) \circ b^i_j l_j\Big) = \sum_j (xu_i - xu_j) \otimes b^i_j xl_j .$$

If $x \notin S(l_j)$, we have $xl_j = 0$; on the other hand, since

$$\bigcup_{b^i_j \neq 0} S(l_j) = S(dl_i) \subset S(l_i),$$

u_i is a unit relatively to $S(l_j)$ for each j, so that for $x \in S(l_j)$ we have $xu_i - xu_j = 0$. It follows that $x\big(df(l_i) - fd(l_i)\big) = 0$, whence $df(l_i) = fd(l_i)$. We conclude that $df = fd$, i.e. f is a homomorphism of differential modules.

We call f the *canonical homomorphism* of L into $K \circ L$.

THEOREM. *Let A be a commutative ring with a unit element and X a Hausdorff space. Let K be an A-coverture and L a free differential A-complex on X, satisfying the following condition:*

(Λ) *There exists a base $(l_i)_{i \in I}$ of L such that for each $i \in I$:*

(i) *There exists an isomorphism φ_i of A onto $H^0\big(S(l_i)K\big)$ such that for each $x \in S(l_i)$ the diagram*

$$\begin{array}{ccc} & A & \\ {}^{\varphi_i}\swarrow & & \searrow{}^{\psi_x} \\ H^0\big(S(l_i)\cdot K\big) & \xrightarrow[p^*]{} & H^0(xK) \end{array}$$

is commutative, where p^* is induced by the natural homomorphism of $S(l_i)\cdot K$ onto $x\cdot K$ and ψ_x is the canonical isomorphism of A onto $H^0(x\cdot K)$ given by the definition of a coverture.

(ii) *$H^p\big(S(l_i)\cdot K\big) = 0$ for all $p > 0$.*

Then condition (Γ) stated in the above lemma is satisfied and the canonical homomorphism f of L into $K \circ L$ induces an isomorphism f^* of $H(L)$ onto $H(K \circ L)$.

Proof. For each $i \in I$, select an element $u_i \in K$ such that

$$\varphi_i(1) = S(l_i)u_i,$$

where 1 is the unit element of the ring A.

By the commutativity of the above diagram, we have for each $x \in S(l_i)$

$$x\cdot u_i = p^*\big(S(l_i)\cdot u_i\big) = p^*\varphi_i(1) = \psi_x(1),$$

hence u_i is a unit of K relatively to $S(l_i)$ and condition (Γ) is satisfied.

We now introduce on the module $T = K \circ L$ a filtration by means of the following submodules:

$$T^{-p} = \sum_{i \leqslant p} K^i \circ L,$$

where p is an arbitrary integer.

On L we consider the null-filtration; we then have in its spectral sequence:

$$E_r = L \quad (r \leqslant 0), \qquad E_r = H(L) \quad (r > 0).$$

The canonical homomorphism f is compatible with the filtrations and with the differentials, so that it induces a homomorphism of the spectral sequence of L into that of $K \circ L$.

The following relation holds in the spectral sequence of $K \circ L$:

$$T^{-p} = C_{-1}^{-p} = K^p \circ L + T^{-p+1}.$$

Let d_1 be the partial differential of $K \circ L$ with respect to K and let Z and D be the cocycles and the coboundaries with respect to d_1. Then

$$C_0^{-p} = T^{-p} \cap d^{-1}(T^{-p}) = Z(K^p \circ L) + T^{-p+1},$$

$$D_{-1}^{-p} + C_{-1}^{-p+1} = T^{-p} \cap dT^{-p+1} + T^{-p+1} \cap d^{-1}(T^{-p}) = D(K^p \circ L) + T^{-p+1}.$$

The first of these relations is evident; to check the second one, notice that

$$T^{-p+1} \cap d^{-1}(T^{-p}) \subset T^{-p+1},$$

$$T^{-p} \cap dT^{-p+1} \subset D(K^p \circ L) + T^{-p+1},$$

hence the left-hand member is contained in the right-hand one. Conversely, if $a \in D(K^p \circ L)$, a is of the form

$$a = \sum_j d\alpha_j^{p+1} \circ \beta_j = d\Big(\sum_j \alpha_j^{p-1} \circ \beta_j\Big) + (-1)^p \Big(\sum_j \alpha_j^{p-1} \circ d\beta_j\Big)$$

$$(\alpha_j^{p-1} \in K^{p-1},\ \beta_j \in L),$$

hence $a \in T^{-p} \cap dT^{-p+1} + T^{-p+1} \cap d^{-1}(T^{-p})$. Finally, we have

$$T^{-p+1} \subset T^{-p+1} \cap d^{-1}(T^{-p}).$$

According to the two relations we have just proved, we may write in the spectral sequence of $K \circ L$:

$$\bar{E}_0^{-p} = C_0^{-p}/C_{-1}^{-p+1} + D_{-1}^{-p} = Z(K^p \circ L)/D(K^p \circ L).$$

Consider now a d_1-cocycle $z \in Z(K^p \circ L)$; it has the form

$$z = \sum_{i \in J} \alpha_i^p \circ l_i,$$

where $\alpha_i^p \in K^p$ and J is a finite subset of I. For each $x \in X$, we have $xd_1z = 0$, i.e. in $xK^p \otimes xL$:

$$\sum_{i \in J} xd\alpha_i^p \otimes xl_i = 0.$$

But xL is a free module, having as base the elements xl_i such that $x \in S(l_i)$. It follows ([2], Exp. XI, p. 5) that $xK^p \otimes xL$ coincides with the group of finite formal linear combinations of elements xl_i with coefficients in xK^p. Hence we deduce that the relation $xl_i \neq 0$, i.e. $x \in S(l_i)$, implies $xd\alpha_i^p = 0$, i.e. $x \notin S(d\alpha_i^p)$; we have therefore for each $i \in J$:

$$S(d\alpha_i^p) \cap S(l_i) = \emptyset.$$

This relation and the condition (Λ) imply that:

a) If $p > 0$, there exists an element $\alpha_i^{p-1} \in K^{p-1}$ such that

$$S(l_i)\,d\alpha_i^{p-1} = S(l_i)\,\alpha_i^p,$$

whence

$$da_i^{p-1} \circ l_i = a_i^p \circ l_i .$$

Since this relation holds for each $i \in J$, it follows that $z = d_1\big(\sum_{i \in J} a_i^{p-1} \circ l_i\big)$, hence $z \in D(K^p \circ L)$.

Finally we have

$$Z(K^p \circ L) = D(K^p \circ L) \qquad (p > 0) .$$

b) If $p = 0$, there exists an element $a_i \in A$ such that

$$\varphi_i(a_i) = S(l_i)\, a_i^0 .$$

If u_i is the unit of K relatively to $S(l_i)$ defined by

$$\varphi_i(1) = S(l_i) \cdot u_i ,$$

we have

$$S(l_i) \cdot (a_i u_i) = S(l_i) \cdot a_i^0 ,$$

whence

$$a_i u_i \circ l_i = a_i^0 \circ l_i .$$

Since this relation holds for each $i \in J$, it follows that

$$z = \sum_{i \in J} u_i \circ a_i l_i ,$$

i.e. $z \in f(L)$.

Conversely, let $z \in f(L)$, i.e. $z = \sum_{i \in J} u_i \circ a_i l_i$. Then

$$d_1 z = \sum_{i \in J} du_i \circ a_i l_i .$$

For each $x \in X$, we have

$$x d_1 z = \sum_{i \in J} x(du_i) \otimes x(a_i l_i) .$$

If $x \notin S(l_i)$, then $x(a_i l_i) = a_i(x l_i) = 0$. If $x \in S(l_i)$, then $x(du_i) = d(x u_i) = 0$, because $x u_i = \varphi_x(1)$, so that $x u_i$ is a cocycle of xK.

Consequently, for each $x \in X$, $x d_1 z = 0$. Hence $d_1 z = 0$, i.e. $z \in Z(K^0 \circ L)$.

Finally we have

$$Z(K^0 \circ L) = f(L) .$$

From the above considerations we infer that we have in the spectral sequence of $K \circ L$:

$$\bar{E}_0^0 = f(L) , \qquad \bar{E}_0^p = 0 \qquad (p \neq 0) .$$

Since $H(\bar{E}_r^p) = \bar{E}_{r+1}^p$, this implies that $\bar{E}_r^p = 0$ for $r > 0$, $p \neq 0$, hence $d_r = 0$ for $r > 0$. Therefore

$$\bar{E}_1 = \bar{E}_2 = \ldots = \bar{E}_\infty = G\big(H(K \circ L)\big) ,$$

where $G\big(H(K \circ L)\big)$ is the graded module associated to the filtration of $H(K \circ L)$.

But

$$G\big(H(K \circ L)\big) = H(K \circ L) .$$

For, in the filtration of $H(K \circ L)$ induced by the filtration of $K \circ L$ there appears only one filtrant grade; to check this it is sufficient to show that for $p > 0$ we have

$$C^{-p} = C^{-p+1} + D^{-p} .$$

It is obvious that the right-hand member is contained in the left-hand one. Conversely, let $v \in C^{-p}$. Then v is of the following form:

$$v = w + \sum_i a_i^p \circ l_i , \qquad w \in T^{-p+1}, \qquad a_i^p \in K^p .$$

But

$$dv = dw + \sum_i da_i^p \circ l_i + (-1)^p \sum_i a_i^p \circ dl_i = 0 .$$

Since $K \circ L$ is the direct sum of its submodules $K^i \circ L$, it follows that

$$\sum_i da_i^p \circ l_i = d_1\Big(\sum_i a_i^p \circ l_i\Big) = 0 .$$

However, we have proved above that for $p > 0$, $Z(K^p \circ L) = D(K^p \circ L)$, which yields a $t \in K^{p-1} \circ L$ such that

$$d_1 t = \sum_i a_i^p \circ l_i .$$

We may therefore write

$$v = w + d_1 t = w + (-1)^p d_2 t + dt ,$$

where d_2 is the partial differential of $K \circ L$ with respect to L.

We have

$$dt \in D^{-p}, \qquad w + (-1)^p d_2 t \in C^{-p+1},$$

since

$$w + (-1)^p d_2 t \in T^{-p+1}$$

and

$$d\big(w + (-1)^p d_2 t\big) = d(v - dt) = dv - d^2 t = 0 .$$

The inclusion $C^{-p} \subset C^{-p+1} + D^{-p}$ is thus proved.

The canonical homomorphism f induces a homomorphism of the spectral sequence of L into that of $K \circ L$; according to the above lemma, it induces an isomorphism of $E_0 = L$ onto $\bar{E}_0 = f(L)$; it induces therefore an isomorphism f^* of $H(E_0) = E_1 = H(L)$ onto $H(\bar{E}_0) = \bar{E}_1 = H(K \circ L)$.

This concludes the proof of the theorem.

We may derive from the theorem the following

COROLLARY (Theorem of Leray on acyclic coverings). *Let X be a locally compact Hausdorff space and let A be a principal ideal ring. Let U be a locally finite covering of X consisting of compact subsets such that for each finite non-void intersection F of members of U the relations*

$$H^0(F, A) \approx A, \quad H^p(F, A) = 0 \quad (p > 0)$$

hold, where $H^p(F, A)$ denotes the p-th Alexander-Spanier cohomology module of F with A as coefficients.

Then the simplicial cohomology module based on finite A-cochains of the nerve of the convering U is isomorphic to the Alexander-Spanier cohomology module of the space X with compact carriers and A as coefficients.

Proof. Let L be the differential complex of finite A-cochains of the nerve of U where the differential is the usual coboundary operator and the carriers are defined as follows: to each simplex s^p of the nerve we associate a compact subset $S(s^p)$ of X, namely the intersection of the members of U corresponding to the vertices of s^p. For each finite A-cochain $c^p \in L$ we define its carrier $S(c^p)$ to be the union of $S(s^p)$, where s^p runs through all simplexes on which c^p is not zero. As easily checked, L is an A-coverture and at the same time a free differential A-complex on the space X.

Now let K be a fine coverture of the space X ([7], p. 141). For each element l_i of the base of L, $S(l_i)\cdot K$ is a fine coverture of the space $S(l_i)$. According to our assumptions on $S(l_i)$ and to the uniqueness theorem of [7], p. 153, condition (Λ) in the above theorem is fulfilled. The theorem then yields an isomorphism f^* of $H(L)$ onto $H(K \circ L)$. On the other hand, according to [6], p. 54, $K \circ L$ is a fine coverture of X, so that, again by the uniqueness theorem, $H(K \circ L)$ coincides with the Alexander-Spanier cohomology module of X with compact carriers and A as coefficients.

Since $H(L)$ is nothing else than the simplicial cohomology module based on finite A-cochains of the nerve of U, the proof of the corollary is completed.

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Reçu par la Rédaction le 2. 2. 1963