On a certain result of Leray

by

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The result to which we refer is Lemma 8 on page 121 of [5]. We propose to obtain a more general result by using a modern proof based on the theory of spectral sequences. We then derive as a corollary the well-known theorem on acyclic coverings due also to Leray.

The theory of coverings has been set up by Leray in [5] and [6]; good accounts of it may also be found in [1] and [7]. We recall some basic definitions from this theory and refer the reader for the basic facts concerning spectral sequences to [6], [4], or [3].

Let $A$ be a ring and $X$ a topological space. An $A$-complex $C$ on $X$ is an $A$-module with whose each element $c \in C$ is associated a closed subset of $X$, called the carrier of $c$ and denoted by $\text{S}(c)$, such that the following relations hold:

\begin{enumerate}
\item[(S1)] $\text{S}(0) = \emptyset$ if and only if $c = 0$,
\item[(S2)] $\text{S}(c+c') \subseteq \text{S}(c) \cup \text{S}(c')$, ($c, c' \in C$),
\item[(S3)] $\text{S}(ac) = \text{S}(c) \cap S(a, a \in A, a \neq 0)$.
\end{enumerate}

If $C$ is graded and if $c, c'$ are homogeneous of different degrees, we require that equality hold in (S3); if $C$ is differential, then we must have

\begin{enumerate}
\item[(S4)] $\text{S}(dc) \subseteq \text{S}(c) \cap S(c, c \in C, d$ is the differential operator).
\end{enumerate}

Let $Y$ be a subspace of $X$ and $C$ an $A$-complex on $X$. Let $C_{X-Y}$ denote the elements of $C$ whose carriers are contained in $X-Y$. We denote by $YC$ the $A$-module $C Y_{X-Y}$ and by $Ye$ the image of $e \in C$ under the natural homomorphism $e \rightarrow Ye$. It is easily seen that we may define for $YC$ a structure of $A$-complex on $Y$ by setting $\text{S}(Ye) = \text{S}(e) \cap Y$.

If $M$ is a graded differential $A$-module, we denote as usual by $H^q(M)$ the $q$-th cohomology module of $M$.

By an $A$-cover of $X$ we mean a graded differential $A$-complex $K$ on $X$ such that the degrees are $\geq 0$, the differential raises the degree by 1, and for each $x \in X$ the following conditions are satisfied:

\begin{enumerate}
\item[(1)] It should be noted that our definition is more general than the usual ones (cf. [6], [1], [7]) in that we do not require $K$ to have the structure of an algebra or its elements to have compact carriers, nor that $K$ have a unit relatively to each compact subset of $X$.
\end{enumerate}
(i) There exists a canonical isomorphism \( \psi_p \) of \( A \) onto \( H^p(\mathbb{Z}K) \).

(ii) \( H^p(\mathbb{Z}K) = 0 \) for all \( p > 0 \).

The intersection \( C \cap C' \) of two \( A \)-complexes on \( X \) is defined as follows:

For each \( \gamma \in C \cap C' \), define the carrier \( S(\gamma) \) to be the set of points \( \gamma \in X \) such that \( \langle p\gamma, \bar{p}\gamma \rangle \neq 0 \), where \( p : C \to \mathbb{Z}^n \) and \( \bar{p} : C' \to \mathbb{Z}^n \) are the natural homomorphisms.

Then \( C \cap C' \) is the factor module of \( C \cap C' \) with respect to the sub-module consisting of the elements with empty carrier. For any \( \gamma \in C \cap C' \), the image of \( \gamma \) under the natural homomorphism \( C \cap C' \to C' \) is denoted by \( \gamma' \).

Let \( A \) be a ring and \( X \) a topological space. An \( A \)-complex \( L \) is said to be free, if there exists a family \( (l_i)_{i \in I} \) of elements of \( L \) such that each element \( l_i \in L \) may be written uniquely as

\[
l_i = \sum_{a \in A} a l_i \quad (a \in A),
\]

where all but a finite number of the \( a_i \)s are equal to zero; moreover,

\[
S(l) = \bigcup_{a \in A} S(l_i),
\]

where the union is taken over all indices \( i \) for which \( a_i \neq 0 \) in the above representation of \( l \). The family of elements \( (l_i)_{i \in I} \) is said to be a base of \( L \).

Let \( A \) be a commutative ring with a unit element and \( K \) an \( A \)-cov-erature on \( X \). If \( Y \) is a subset of \( X \), an element \( u \in K \) is said to be a unit relatively to \( X \) provided that for each \( \gamma \in Y \) the element \( ku \) of \( K \) is homogenous of degree 0, it is a cocycle and its homology class in \( H^n(\mathbb{Z}K) \) corresponds to the unit element of \( A \) under the isomorphism \( \psi_n \) given by the definition of an \( A \)-cov-erature.

**Lemma.** Let \( A \) be a commutative ring with a unit element and \( X \) a Hausdorff space. Let \( K \) be an \( A \)-cov-erature and \( L \) a free differential \( A \)-complex on \( X \), satisfying the following conditions:

(i) For each element \( l_i \) of the base of \( L \), there exists a unit \( u_i \in K \) relatively to \( S(l_i) \).

Let \( l = \sum l_i \) be an arbitrary element of \( L \). Then the element \( \sum u_i \cdot a l_i \) of \( K \cdot L \) does not depend on the choice of the units \( u_i \).

The mapping defined by

\[
f(l) = \sum f(l_i) = \sum u_i \cdot a l_i\]

is a monomorphism of the differential module \( L \) into the differential module \( K \cdot L \).

**Proof.** Let \( \psi_n^0 \) be a second unit of \( K \) relatively to \( S(l_i) \). Since \( \psi(K \cdot L) = \psi(K \cdot \mathbb{Z}X) \), and since for \( \gamma \in S(l_i) \), \( \psi_{\gamma} = 0 \) and for \( \gamma \in S(l_i) \), \( \psi_{\gamma} = \psi_{\gamma} \), we may write for any \( \gamma \in X \):

\[
x(\psi_n \cdot a l_i - u_i \cdot a l_i) = \psi_{\gamma} - \psi_{\gamma} = 0.
\]

Hence \( S(u_i \cdot a l_i - u_i \cdot a l_i) = 0 \) and, by (8), \( u_i \cdot a l_i = u_i \cdot a l_i \).

Assume now \( f(l) = \sum u_i \cdot a l_i = 0 \). This means that, for each \( \gamma \in X \), we have

\[
\sum u_i \cdot a l_i = \sum \psi_{\gamma} = 0.
\]

But \( u \cdot L \) is a free module having as base the elements \( u_i \) such that \( u \in S(l_i) \). It follows (2), Exp. XI, p. 3, that \( \psi(K \cdot \mathbb{Z}X) \) coincides with the group of finite formal linear combinations of elements \( u_i \) with coefficients in \( \mathbb{Z}K \). Hence we have for each \( \gamma \) such that \( \psi(\gamma) \in S(l_i) \), \( u_i \cdot \psi_{\gamma} = 0 \), i.e. \( u_i = 0 \).

As the point \( \gamma \) is arbitrary, we infer that \( u_i = 0 \) for all \( i \), and therefore \( f(l) = 0 \). Thus \( f \) is a monomorphism.

We now prove that \( f \) commutes with \( d \). We have

\[
d(f(l)) = d(u_i \cdot a l_i) = d(u_i \cdot a l_i + u_i \cdot d l_i).
\]

Since \( u_i \cdot \psi_{\gamma} = 0 \) for any \( \gamma \in S(l_i) \) and \( d l_i = 0 \) for any \( \gamma \in S(l_i) \), we have \( d(u_i \cdot a l_i) = 0 \) for any \( \gamma \in X \) and therefore \( d(u_i \cdot a l_i) = 0 \).

Hence

\[
d(f(l)) = u_i \cdot d l_i.
\]

On the other hand, if \( d l_i = \Psi B_i \), we have

\[
f(\psi_{\gamma}) = \sum f(B_i) = \sum u_i \cdot a l_i.
\]

Let \( \gamma \) be an arbitrary point of \( X \). We may write

\[
x(f(l) - f(\psi_{\gamma})) = x(\sum u_i \cdot a l_i - \sum u_i \cdot a l_i) = \sum (u_i \cdot a l_i - u_i \cdot a l_i).
\]

If \( \gamma \in S(l_i) \), we have \( u_i \cdot a l_i = 0 \); on the other hand, since

\[
S(l) = S(\psi_{\gamma}) C S(l_i),
\]

\( u_i \) is a unit relatively to \( S(l_i) \) for each \( i \), so that for \( \gamma \in S(l_i) \), we have \( u_i \cdot a l_i = 0 \). It follows that \( x(f(l) - f(\psi_{\gamma})) = 0 \), whence \( f(\psi_{\gamma}) = f(\psi_{\gamma}) \).

We conclude that \( df = fd \), i.e. \( f \) is a homomorphism of differential modules.
We call $f$ the canonical homomorphism of $L$ into $K \ast L$.

**Theorem.** Let $A$ be a commutative ring with a unit element and $X$ a $\mathcal{H}$-complex on $X$, satisfying the following conditions:

(A) There exists a base $(i)_{i \in I}$ of $L$ such that for each $i \in I$:

(i) There exists an isomorphism $\varphi_i$ of $A$ onto $H^p(S(i)_0K)$ such that for each $x \in S(i)_0$ the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi_i} & H^p(S(i)_0K) \\
\downarrow{\otimes} & \downarrow{\otimes} & \downarrow{\otimes} \\
\otimes H^p & \xrightarrow{\otimes} & H^p(\otimes K)
\end{array}
$$

is commutative, where $p^*$ is induced by the natural homomorphism of $S(i)_0K$ onto $x \cdot K$ and $\varphi_2$ is the canonical isomorphism of $A$ onto $H^p(x \cdot K)$ given by the definition of $\mathcal{T}$.

(ii) $H^p(S(i)_0K) = 0$ for all $p > 0$.

Then condition (i) stated in the above lemma is satisfied and the canonical homomorphism of $L$ into $K \ast L$ induces an isomorphism of $H^p(L)$ onto $H^p(K \ast L)$.

**Proof.** For each $i \in I$, select an element $u_i \in K$ such that

$$\varphi_i(u_i) = S(i)_0u_i,$$

where $1$ is the unit element of the ring $A$.

By the commutativity of the above diagram, we have for each $x \in S(i)_0$

$$x \cdot u_i = p^*[S(i)_0 \cdot u_i] = p^*[\varphi_i(u_i)] = \varphi_2(u_i),$$

hence $u_i$ is a unit of $K$ relatively to $S(i)_0$ and condition (ii) is satisfied.

We now introduce the module $T = K \ast L$ a filtration by means of the following submodules:

$$T^p = \sum_{i \in I} K \ast L,$$

where $p$ is an arbitrary integer.

On $L$ we consider the null-filtration; we then have in its spectral sequence:

$$E_r = L (r < 0), \quad E_r = H(L) (r > 0).$$

The canonical homomorphism $f$ is compatible with the filtrations and with the differentials, so that it induces a homomorphism of the spectral sequence of $L$ into that of $K \ast L$.

The following relation holds in the spectral sequence of $K \ast L$:

$$T^p = C^{\infty}_p = K \ast L + T^{p+1}.$$
whence
\[ da_i^{p-1} \cdot l_i = a_i^p \cdot l_i. \]

Since this relation holds for each \( i \in J \), it follows that
\[ z = d_1 \left( \sum_{i \in J} a_i^{p-1} \cdot l_i \right), \]

hence \( z \in D(K^p \cdot L) \).

Finally we have
\[ Z(K^p \cdot L) = D(K^p \cdot L) \quad (p \neq 0). \]

b) If \( p = 0 \), there exists an element \( a_0 \in A \) such that
\[ \varphi(a_0) = S(l_0) \cdot a_0, \]

where \( u_0 = 1 \) is the unit of \( K \) relatively to \( S(l_0) \) defined by
\[ \varphi(1) = S(l_0) \cdot u_0, \]

whence
\[ S(l_0) \cdot a_0 u_0 = S(l_0) \cdot a_0, \]

i.e. \( a_0 \in f(L) \).

Conversely, let \( z \in f(L) \), i.e. \( z = \sum_{i \in J} a_i u_i a_i l_i \). Then
\[ d_1 z = \sum_{i \in J} d_1 a_i u_i a_i l_i. \]

For each \( x \in X \), we have
\[ x d_1 z = \sum_{i \in J} x (d_1 a_i u_i a_i l_i). \]

If \( x \in S(l_i) \), then \( x(a_i l_i) = a_i d(x l_i) = 0 \). If \( x \notin S(l_i) \), then \( x \cdot d(u_i) = d(x u_i) = 0 \), because \( x u_i = y_0 u_i \), so that \( x u_i \) is a cocycle of \( x K \).

Consequently, for each \( x \in X \), \( x d_1 z = 0 \). Hence \( d_1 z = 0 \), i.e. \( z \in Z(K^p \cdot L) \).

Finally we have
\[ Z(K^p \cdot L) = f(L). \]

From the above considerations we infer that we have in the spectral sequence of \( K \cdot L \):
\[ E^{p}_0 = f(L), \quad E^{p}_{p-1} = 0 \quad (p \neq 0). \]

Since \( H(E^{p}_{p}) = E^{p+1}_{p} \), this implies that \( E^{p}_{r} = 0 \) for \( r > 0 \), \( p \neq 0 \), hence \( d_r = 0 \) for \( r > 0 \). Therefore
\[ E^{p}_1 = E^{p}_2 = \ldots = E^{p}_{m} = G(H(K \cdot L)), \]

where \( G(H(K \cdot L)) \) is the graded module associated to the filtration of \( H(K \cdot L) \).

But
\[ G(H(K \cdot L)) = H(K \cdot L). \]

Accordingly, let \( v \in C^{-p} \). Then \( v \) is of the following form:
\[ v = w + \sum_{i} a_i^{p} \cdot l_i, \quad w \in T^{-p+1}, \quad a_i^{p} \in K^p. \]

But
\[ d_1 v = d_1 w + \sum_{i} d_1 a_i^{p} \cdot l_i + (-1)^{p} \sum_{i} d_1 a_i^{p} \cdot l_i = 0. \]

Since \( K \cdot L \) is the direct sum of its submodules \( K^i \cdot L \), it follows that
\[ \sum_{i} d_1 a_i^{p} \cdot l_i = d_1 \left( \sum_{i} a_i^{p} \cdot l_i \right) = 0. \]

However, we have proved above that for \( p > 0 \), \( Z(K^p \cdot L) = D(K^p \cdot L) \), which yields a \( i \in K^{p-1} \cdot L \) such that
\[ d_i l = \sum_{i} a_i^{p} \cdot l_i. \]

We may therefore write
\[ v = w + d_i l = w + (-1)^i d_i l + dt, \]

where \( d_i \) is the partial differential of \( K \cdot L \) with respect to \( L \).

We have
\[ dt \in D^{-p}, \quad w + (-1)^i d_i l \in C^{-p+1}, \]

since
\[ w + (-1)^i d_i l \in T^{p+1}, \]

and
\[ d(w + (-1)^i d_i l) = d(w - dt) = dv - dt = 0. \]

The inclusion \( C^{-p} \subset C^{-p+1} \cdot D^{-p} \) is thus proved.

The canonical homomorphism \( f \) induces a homomorphism of the spectral sequence of \( L \) into that of \( K \cdot L \) according to the above lemma, it induces an isomorphism of \( E_n = L \) onto \( E_n = f(L) \); it induces therefore an isomorphism \( f^* \) of \( H(E_0) = E_1 = H(K \cdot L) \) onto \( H(E_1) = H(K \cdot L) \).
This concludes the proof of the theorem.
We may derive from the theorem the following

Corollary (Theorem of Leray on acyclic coverings). Let $X$ be a locally compact Hausdorff space and let $A$ be a principal ideal ring. Let $U$ be a locally finite covering of $X$ consisting of compact subsets such that for each finite non-void intersection $F$ of members of $U$ the relations

$$H^p(F, A) \approx A, \quad H^p(F, A) = 0 \quad (p > 0),$$

hold, where $H^p(F, A)$ denotes the $p$-th Alexander-Spanier cohomology module of $F$ with $A$ as coefficients.

Then the simplicial cohomology module based on finite $A$-cochains of the nerve of the covering $U$ is isomorphic to the Alexander-Spanier cohomology module of the space $X$ with compact carriers and $A$ as coefficients.

Proof. Let $L$ be the differential complex of finite $A$-cochains of the nerve of $U$ where the differential is the usual coboundary operator and the carriers are defined as follows: to each simplex $\sigma^p$ of the nerve we associate a compact subset $S(\sigma^p)$ of $X$, namely the intersection of the members of $U$ corresponding to the vertices of $\sigma^p$. For each finite $A$-cochain $\omega \in L$ we define its carrier $S(\omega)$ to be the union of $S(\sigma^p)$, where $\sigma^p$ runs through all simplexes on which $\omega$ is not zero. As easily checked, $L$ is an $A$-coverture and at the same time a free differential $A$-complex on the space $X$.

Now let $K$ be a fine covering of the space $X$ ([7], p. 141). For each element $l_k$ of the base of $L$, $S(l_k) - K$ is a fine covering of the space $S(l_k)$. According to our assumptions on $S(l_k)$ and to the uniqueness theorem of [7], p. 153, condition (A) in the above theorem is fulfilled. The theorem then yields an isomorphism $\psi$ of $H(L)$ onto $H(K \cdot L)$. On the other hand, according to [6], p. 54, $K \cdot L$ is a fine covering of $X$, so that, again by the uniqueness theorem, $H(K \cdot L)$ coincides with the Alexander-Spanier cohomology module of $X$ with compact carriers and $A$ as coefficients.

Since $H(L)$ is nothing else than the simplicial cohomology module based on finite $A$-cochains of the nerve of $U$, the proof of the corollary is completed.

References
