Then B_{r_λ, s_λ} occurs as a term in the representation (in the form (2), with condition C) of x_{k,y_k} for some $k < \varrho_0$, say $k = k_3$. But then

$$\bigcap_{k<\max(\varrho'',\varrho''',k_3)+1} x_{k,y_k} \leqslant (A_{i',j} \wedge A_{i',k'} \wedge B_{j,k'}) \leqslant A_{i,j}\,,$$

a contradiction, since $\max(\varrho'', \varrho''', k_3)+1 < \varrho_0$.

Thus our original assumption, that $F_{\gamma,a}$ is not (γ, ∞) distributive, has led to a contradiction, and Theorem 2 is proved.

THEOREM 3. *If γ is an infinite regular cardinal, then there does not exist a free complete (γ, ∞) distributive Boolean algebra on γ complete generators.*

Proof. Theorem 3 follows from the remarks preceding Theorem 2 and Theorem 2 itself.

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On the Lebesgue measurability and the axiom of determinateness

by

Jan Mycielski (Wrocław) and S. Świerczkowski (Glasgow)

It is the purpose of this paper to show that the axiom of determinateness (A) (see [2], [3]) implies that all linear sets are Lebesgue measurable. We will use (A) in the following form: *every infinite positional game with perfect information and a denumerable set of positions is determined.* (Let us recall another form (see [2]) which does not use notions of the theory of games: *for every set P of sequences of natural numbers there exists a function f defined on all finite (or empty) sequences of natural numbers, taking natural values and such that for every sequence* $n_1, n_2, \ldots$

$$\big(n_1, f(n_1), n_2, f(n_1, n_2), n_3, f(n_1, n_2, n_3), \ldots\big) \in P$$

or for every sequence $n_1, n_2, \ldots$

$$\big(f(\emptyset), n_1, f(n_1), n_2, f(n_1, n_2), n_3, \ldots\big) \notin P\,.\,)$$

(A) implies also the property of Baire of every linear set (see [2]) and the proof of the result of this paper, although more complicated, is based on an analogous idea as the proof of this fact. Let us mention that the development of the theory of measure, e.g. the denumerable additivity, is based on a weak form of the axiom of choice which is a consequence of (A) (see [2], prop. $\mathcal{C}$). Of course our result could be formulated as follows: the existence of a non-measurable set implies the existence of non-determined games of the prescribed form (and the existence of sets P without the above mentioned property). Clearly the axiom of choice is not used in this paper.

1. THEOREM. (A) *implies the Lebesgue measurability of every linear set* [1].

First we note that it is enough to show for every subset X of the closed interval $\langle 0, 1\rangle$ the following proposition:

(P) (A) *implies* $|X|_i > 0$ *or* $|cX|_i > 0$. [2]

[1] For a generalization of this result, see section 2.

[2] $|\cdot|_i$ denotes the interior measure and cX the complement of X in $\langle 0, 1\rangle$.

In fact, if there exists a nonmeasurable linear set, then it is easy to construct an $X \subseteq \langle 0, 1\rangle$ with interior measure 0 and exterior measure 1, i.e. (P) would disprove (A).

Let $r_1, r_2, \ldots$ be a sequence of positive rational numbers satisfying:

(i) $$\sum_{n=1}^{\infty} r_n < \infty \quad \text{and} \quad 1/2 > r_1 > r_2 > \ldots$$

let J_k ($k = 0, 1, 2, \ldots$) be the class of all subsets S of $\langle 0, 1\rangle$ which have the following properties;

(ii) S is a finite union of closed intervals $\langle a, b\rangle$, where a and b are rational numbers;

(iii) The diameter $\delta(S) = \sup_{x,y \in S} |x-y|$ satisfies $\delta(S) \leqslant 1/2^k$;

(iv) $|S| = r_1 \cdot r_2 \cdot \ldots \cdot r_k$, where $|\cdot|$ denotes the Lebesgue measure [3].

We take the notation $S_0 = \langle 0, 1\rangle$. An ordered $(k+1)$-sequence $S_0, S_1, \ldots, S_k$, where $S_i \subset S_{i-1}$ and $S_i \in J_i$ is denoted by $\overline{S}_k$. The set of all sequences $\overline{S}_k$ is denoted by $\overline{J}_k$. If τ is a mapping of $\overline{J}_k$ into J_{k+1}, then $\overline{\tau}(\overline{S}_k)$ denotes the ordered sequence $S_0, S_1, \ldots, S_k, \tau(\overline{S}_k)$.

We consider the following game between two players I and II, determined by the set $X \subseteq \langle 0, 1\rangle$. I chooses a set $S_1 \in J_1$, then II chooses a set $S_2 \in J_2$ with $S_2 \subset S_1$, then again I chooses $S_3 \in J_3$ with $S_3 \subset S_2$, etc. infinitely many times. If $\bigcap_{n=1}^{\infty} S_n \subseteq X$, then I wins; if $\bigcap_{n=1}^{\infty} S_n \subseteq cX$, then II wins (exactly one of the two inclusions hold since, by (ii) and (iii), $\bigcap_{n=1}^{\infty} S_n$ is a one point set).

By (ii), the set $\bigcup_{k=1}^{\infty} J_k$ is denumerable, therefore the statement (A) implies that I or II has a winning strategy [4]. Therefore, in view of (i), (P) follows from the following theorem:

(T) (a) *If I has a winning strategy, then*

$$|X|_i \geqslant r_1 \prod_{n=1}^{\infty} (1-2r_{2n}) .$$

(b) *If II has a winning strategy, then*

$$|cX|_i \geqslant \prod_{n=1}^{\infty} (1-2r_{2n-1}) .$$

For proving (T) we need some lemmas.

[3] Hence the only member of J_0 is the interval $\langle 0, 1\rangle$.

[4] In fact this requires a small reasoning or the application of Theorem 2 of [2].

(L_1) *Let $\overline{S}^0_{n-1} \in \overline{J}_{n-1}$ and let τ be a mapping of $\overline{J}_n$ into J_{n+1} such that $\tau(\overline{S}_n) \subset S_n$ for every $\overline{S}_n \in \overline{J}_n$ $\left(\overline{\tau}(\overline{S}_n) \in \overline{J}_{n+1}\right)$. Then there exists a finite sequence $S^1_n, \ldots, S^m_n \in J_n$, $S^i_n \subset S^0_{n-1}$, such that*

($\geqslant$) $$\Big|\bigcup_{i=1}^{m} \tau(\overline{S}^i_n)\Big| \geqslant |S^0_{n-1}|(1-2r_n) \text{ [5]}$$

and moreover the sets $\tau(\overline{S}^1_n), \ldots, \tau(\overline{S}^m_n)$ are disjoint.

Proof. We define the sets S^i_n by induction. Suppose that $S^1_n, \ldots, S^j_n$ ($j \geqslant 0$) have already been defined, we consider the set

$$R_j = S^0_{n-1} - \bigcup_{i=1}^{j} \tau(\overline{S}^i_n) .$$

If $|R_j| > 2|S^0_{n-1}|r_n$ then R_j contains a subset P of diameter $\delta(P) \leqslant \delta(R_j)/2 \leqslant 1/2^n$ such that $|P| > |S^0_{n-1}|r_n$; moreover P can be a finite union of rational intervals. Then there exists a set $S^{j+1}_n \subseteq P$ which belongs to J_n. In this way we define consecutively the terms of the sequence $S^1_n, S^2_n, \ldots$ till we arrive to S^m_n such that $|R_m| \leqslant 2|S^0_{n-1}|r_n$, i.e. ($\geqslant$) holds. Since for every $l = 1, 2, \ldots, m-1$ we have $\tau(\overline{S}^{j+1}_n) \subset S^{j+1}_n \subset S^0_{n-1} - \bigcup_{i=1}^{j} \tau(\overline{S}^i_n)$, it follows that the sets $\tau(\overline{S}^i_n)$ are disjoint, q.e.d.

Let τ be a strategy for player I, i.e. a mapping of $\bigcup_{k=0}^{\infty} \overline{J}_{2k}$ into $\bigcup_{k=0}^{\infty} J_{2k+1}$ such that $\overline{\tau}(\overline{J}_{2k}) \subseteq \overline{J}_{2k+1}$.

Let us denote by $I_\tau(\overline{S}^0_{n-1})$ a set $\{\overline{S}^1_n, \ldots, \overline{S}^m_n\}$ given by (L_1), it is clear that, owing to (ii), a function I_τ exists effectively (i.e., without using the axiom of choice). We put

$$I^n_\tau = \bigcup_{\overline{S}_2 \in I_\tau(\overline{\tau}(\overline{S}_0))} \ \bigcup_{\overline{S}_4 \in I_\tau(\overline{\tau}(\overline{S}_2))} \cdots \bigcup_{\overline{S}_{2(n-1)} \in I_\tau(\overline{\tau}(\overline{S}_{2(n-2)}))} I_\tau\big(\overline{\tau}(\overline{S}_{2(n-1)})\big)$$

and

$$A_n = \bigcup_{\overline{S}_{2n} \in I^n_\tau} \tau(\overline{S}_{2n}) .$$

Now we prove two other lemmas

(L_2) $|A_n| \geqslant r_1 \prod_{i=1}^{n} (1-2r_{2i})$ *and* $A_{n+1} \subseteq A_n$ *for* $n = 1, 2, \ldots$

Proof. We observe that by (L_1) all the sets $\tau(\overline{S}_{2n})$ occurring in the union A_n are disjoint (n being fixed). Now we prove the inequality of (L_2) by induction. For $n = 1$ it holds clearly by (L_1). Suppose that it holds for some n. We have by (L_1):

$$\Big|\bigcup_{\overline{S}_{2(n+1)} \in I_\tau(\overline{\tau}(\overline{S}_{2n}))} \tau(\overline{S}_{2(n+1)})\Big| \geqslant |\tau(\overline{S}_{2n})|\,(1-2r_{2(n+1)}) ;$$

[5] $\overline{S}^i_n$ denotes the sequence $S^0_0, S^0_1, \ldots, S^0_{n-1}, S^i_n$.

and thus by (L_1) (the disjointness),

$$|A_{n+1}| = \sum_{\bar{S}_{2n}\in I_\tau^n} \Big| \bigcup_{\bar{S}_{2(n+1)}\in I_\tau(\bar{\tau}(\bar{S}_{2n}))} \tau(\bar{S}_{2(n+1)})\Big|$$

$$\geqslant \sum_{\bar{S}_{2n}\in I_\tau^n} |\tau(\bar{S}_{2n})|(1-2r_{2(n+1)})$$

$$= \Big|\bigcup_{\bar{S}_{2n}\in I_\tau^n} \tau(\bar{S}_{2n})\Big|(1-2r_{2(n+1)})$$

$$= |A_n|(1-2r_{2(n+1)}) \geqslant r_1 \prod_{i=1}^{n+1}(1-2r_{2i})$$

and this concludes the proof of the inequality. The inclusion of (L_2) clearly follows from (L_1), q.e.d.

(L_3). *For every point*

$$p \in \bigcap_{n=1}^{\infty} A_n$$

there exists a strategy σ_p for player II, such that if I plays by means of τ and II by means of σ_p then $\bigcap_{n=1}^{\infty} S_n = \{p\}$ ($S_1, S_2, \ldots$ denote the consecutive choices of the players).

Proof. The sets $\tau(\bar{S}_{2n})$ $(\bar{S}_{2n}\in I_\tau^n)$ being disjoint (for fixed n), let us denote by $\bar{S}_{2n}^p$ this (unique) sequence $\bar{S}_{2n}\in I^n$ for which $p\in\tau(\bar{S}_{2n})$ and $S_0^p = \langle 0,1\rangle$. Suppose that σ_p is a strategy for player II such that

$$\sigma_p\big(\bar{\tau}(\bar{S}_{2(n-1)}^p)\big) = S_{2n}^p \quad \text{for} \quad n=1,2,\ldots$$

It is clear that $p\in\bigcap_{n=1}^{\infty} S_n$ if the choices S_n are performed by means of τ and σ_p and by (iii) we have the conclusion, q.e.d.

Proof of (T). (a). If τ is a winning strategy for player I, then by (L_3)

$$\bigcap_{n=1}^{\infty} A_n \subseteq X$$

and by (L_2)

$$\Big|\bigcap_{n=1}^{\infty} A_n\Big| \geqslant r_1 \prod_{i=1}^{\infty}(1-2r_{2i}).$$

Then (a) follows.

The proof of (b) is analogous (by means of some lemmas analogous to (L_2) and (L_3)), q.e.d.

2. Remark. Our Theorem permits getting a stronger consequence of (A):

If E is a separable metric space and μ is a denumerably additive finite measure on the field $\boldsymbol{B}(E)$ of Borel subsets of E, then every set $X\subseteq E$ is μ-measurable i.e. there are such $B_1, B_2\in\boldsymbol{B}(E)$, that

$$B_1\subseteq X\subseteq B_2 \quad \textit{and} \quad \mu(B_1)=\mu(B_2).$$

Without loss of generality we can suppose that μ is not purely atomic. Let h be a homeomorphism of E into the Hilbert cube H [6]. Let A be the set of atoms i.e. $A=\{x\in E\colon \mu(x)>0\}$. Since μ is finite A is denumerable and the set $E^* = H - h(A)$ is borelian in H. Let be $m(Y)=\mu\big(h^{-1}(Y)\big)$ for each $Y\in\boldsymbol{B}(E^*)$. It is clear that $\langle E^*, \boldsymbol{B}(E^*), m\rangle$ is a finite atom-free separable Borel measure-space. Then this space is pointwise isomorphic (by a Borel homeomorphism [6]) to the Lebesgue measure-space over a segment of the real line (see e.g. [1], § 4.1). It follows assuming the conclusion of the Theorem that all the subsets of E^* are m-measurable and all the subsets of E are μ-measurable.

[6] Its existence requires only a weak form of the axiom of choice $\mathcal{C}$ which follows from (A) (see [2]).

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