A note on the theory of propositional types

by

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Let $H$ be the theory of propositional types described by L. Henkin in paper [1]. The primitives of $H$ are: $\lambda$-operator, the function of application written as $(X_\alpha Y_\beta)$, and the denumerable number of constants $Q_{\alpha\alpha\alpha\alpha}$, denoting, for every type $\alpha$, the relation of identity in type $\alpha$. For example, $Q_{0000}$ is the familiar relation of propositional equivalence:

$\alpha_0 = \alpha_0 \equiv (Q_{0000} \alpha_0) \alpha_0$.

Using quantifiers and higher types, we may define all connectives by means of equivalence, as was first shown by A. Tarski in [4] and [5]:

$(\alpha_0 \land \alpha_0) = \forall \theta \phi \exists \alpha_0 (\alpha_0 = (\exists \alpha_0 \phi))$,

$\phi = \forall \theta \alpha_0 \phi$,

$\neg \alpha_0 = \neg \alpha_0 \equiv \phi$.

Conjunction may be defined, of course, in many ways.

Henkin formulate these definitions by using $\lambda$-operator instead of quantifiers. The aim of this note is to show that:

(1.0) Every constant $Q_{\alpha\alpha\alpha\alpha}$ is definable by means of $\lambda$-operator, application, and constants: $Q_{\alpha\alpha\alpha\alpha}$, conjunction $\land$, and $\phi$.

Proof. Let us write $Q^\phi$ instead of $Q_{\alpha\alpha\alpha\alpha}$. We shall prove (1.0) by means of three lemmas:

1. $Q^\phi$ is definable by means of $Q^\phi$ and $Q^{\phi^\phi}$:

$(\phi^\phi \land \phi^\phi) \equiv \lambda x_0 \lambda x_0 ((\phi^\phi \land \phi^\phi) \alpha_0 (\phi^\phi \land \phi^\phi) \alpha_0)$

where $T$ is the constant of truth ($T = (\phi = \phi)$).

According to Henkin’s definition of quantifier,

$\forall \theta \alpha_0 \alpha_0 = ((\phi^\phi \land \phi^\phi) \alpha_0 \alpha_0)$,

the formula (1.1) may be rewritten in the following informal way:

$\alpha_0 \alpha_0 Q^\phi \alpha_0 = \forall \theta \alpha_0 \alpha_0 \phi \alpha_0 \alpha_0 \alpha_0$

which shows that (1.1) is an application of the principle of extensionality.
Two last definitions are analogous to (0.3) and (0.2). Hence, from (1.0) we obtain (2.0).

The reduction of primitive notions does not finitize the axiom system of $H$. We need (so remarked first S. Leśniewski) the rule or the denumerable number of axioms having the form of extensionalities (e.g. $(p = q \land (\alpha \theta p)) \rightarrow (\alpha \theta q)$) or the form of generalization rules (e.g. $(\alpha \theta F) \land (\alpha \theta T) \rightarrow (\alpha \theta Q)$). The first proof of the completeness-theorem for the theory of propositional types (in Leśniewski’s formulation) was given by J. Šupecki in [3].

References


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