A note on the theory of propositional types

by

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Let H be the theory of propositional types described by L. Henkin in paper [1]. The primitives of H are: λ-operator, the function of application written as $(X_{\alpha\beta}Y_\beta)$, and the denumerable number of constants $Q_{(0\alpha)\alpha}$ denoting, for every type α, the relation of identity in type α. For example, $Q_{(00)0}$ is the familiar relation of propositional equivalence:

(0.1) $$x_0 \equiv y_0 = \big((Q_{(00)0}x_0)y_0\big).$$

Using quantifiers and higher types, we may define all connectives by means of equivalence, as was first shown by A. Tarski in [4] and [5]:

(0.2) $$x_0 \wedge y_0 = \mathbf{V}z_{00}\Big(x_0 \equiv \big((z_{00}x_0) \equiv (z_{00}y_0)\big)\Big),$$

(0.3) $$F = \mathbf{V}x_0x_0,$$

(0.4) $$\sim x_0 = x_0 \equiv F.$$

Conjunction may be defined, of course, in many ways.

Henkin formulate these definitions by using λ-operator instead of quantifiers. The aim of this note is to show that:

(1.0) *Every constant $Q_{(0\alpha)\alpha}$ is definable by means of λ-operator, application, and constants: $Q_{(00)0}$, conjunction $\wedge$, and F.*

Proof. Let us write Q^α instead of $Q_{(0\alpha)\alpha}$. We shall prove (1.0) by means of three lemmas:

L$_1$. *$Q^{\alpha\beta}$ is definable by means of Q^α and $Q^{0\beta}$:*

(1.1) $$Q^{\alpha\beta} = \lambda x_{\alpha\beta}\lambda y_{\alpha\beta}\Big(\big(Q^{0\beta}\lambda z_\beta T\big)\lambda z_\beta\Big(\big(Q^\alpha(x_{\alpha\beta}z_\beta)\big)(y_{\alpha\beta}z_\beta)\Big)\Big)$$

where T is the constant of truth $(T = (F \equiv F))$.

According to Henkin's definition of quantifier,

$$\mathbf{V}z_\beta A_0 = \big((Q^{0\beta}\lambda z_\beta T)\lambda z_\beta A_0\big),$$

the formula (1.1) may be rewritten in the following informal way:

$$x_{\alpha\beta}Q^{\alpha\beta}y_{\alpha\beta} = \mathbf{V}z_\beta(x_{\alpha\beta}z_\beta)Q^\alpha(y_{\alpha\beta}z_\beta)$$

which shows that (1.1) is an application of the principle of extensionality.

L$_2$. *$Q^{0\beta}$ is definable by means of the connectives and all individuals $B_1^\beta, \ldots, B_n^\beta$ of type β:*

$$(1.2)\quad Q^{0\beta} = \lambda x_{0\beta}\lambda y_{0\beta}\{((x_{0\beta}B_1^\beta) \equiv (y_{0\beta}B_1^\beta)) \wedge \ldots \wedge ((x_{0\beta}B_n^\beta) \equiv (y_{0\beta}B_n^\beta))\}.$$

The meaning of (1.2) is evident.

L$_3$. *All individuals $B_1^\beta, \ldots, B_n^\beta$ of type β are definable by means of connectives, by means of all individuals of all types $\alpha \precsim \beta$, and by means of all Q^α for all $\alpha \precsim \beta$.*

(I write $\alpha \precsim \beta$ if the index α is a part of the index β.)

Every type-index β may be written as

$$\beta = ((0\beta_1)\beta_2) \ldots \beta_k$$

where $\beta_1, \beta_2, \ldots, \beta_k$ are type-indexes and obviously $\beta_i \precsim \beta$ for $1 \leqslant i \leqslant k$.

Hence, for every j $(1 \leqslant j \leqslant n)$, the definition of B_j^β can have the following form:

$$B_j^\beta = \lambda x_{\beta_1} \ldots \lambda x_{\beta_k}\{[((Q^{\beta_1}x_{\beta_1})B^{\beta_1}) \wedge \ldots \wedge ((Q^{\beta_k}x_{\beta_k})B^{\beta_k}) \wedge V^1] \vee \ldots$$
$$\vee [((Q^{\beta_1}x_{\beta_1})B_{t_1^s}^{\beta_1} \wedge \ldots \wedge ((Q^{\beta_k}x_{\beta_k})B_{t_k^s}^{\beta_k}) \wedge V^s]\}.$$

This definition consists of an alternative of s conjunctions. Every conjunction contains a component V^i $(1 \leqslant i \leqslant s)$. We have $V^i = F$ or $V^i = T$ according to the truth-table of B_j^β. B_j^β is considered as a function of k arguments of types $\beta_1, \ldots, \beta_k$ and taking the values of type 0 (T or F). Every conjunction represents a k-dimensional point in the k-dimensional truth-table of B_j^β. Hence the construction of the definition is evident.

(1.0) follows from the lemmas L_1-L_3 by induction.

If we want to have only the equivalence as primitives, we can assume two:

$$Q_1 = Q_{(0(00))(00)} \quad \text{and} \quad Q_2 = Q_{(0(0(00)))(0(00))}.$$

(2.0) *Every constant $Q_{(0\alpha)\alpha}$ is definable by means of λ-operator, application and the constants Q_1 and Q_2.*

Proof.

$$p \equiv q = ((Q_1\lambda x_0 p)\lambda x_0 q),$$
$$\mathrm{as} = \lambda x_0 x_0,$$
$$T = ((Q_1 \mathrm{as})\mathrm{as}),$$
$$\mathrm{ver} = \lambda x_0 T,$$
$$F = ((Q_1 \mathrm{as})\mathrm{ver}),$$
$$p \wedge q = ((Q_2\lambda x_{00}T)\lambda x_{00}(p \equiv ((x_{00}p) \equiv (x_{00}q)))).$$

Two last definitions are analogous to (0.3) and (0.2). Hence, from (1.0) we obtain (2.0).

The reduction of primitive notions does not finitize the axiom system of H. We need (as remarked first S. Leśniewski) the rule or the denumerable number of axioms having the form of extensionalities (e.g. $(p \equiv q \wedge (g_{00}p)) \rightarrow (g_{00}q)$) or the form of generalization rules (e.g. $((g_{00}F) \wedge (g_{00}T)) \rightarrow (g_{00}q)$). The first proof of the completeness-theorem for the theory of propositional types (in Leśniewski's formulation) was given by J. Słupecki in [3].

References

[1] L. Henkin, *A theory of propositional types*, Fund. Math. 52 (1963), pp. 323-334.

[2] S. Leśniewski, *Grundzüge eines neuen Systems der Grundlagen der Mathematik*, Fund. Math. 14 (1929), pp. 1-81.

[3] J. Słupecki, *St. Leśniewski's proptothetics*, Studia Logica 1 (1953), pp. 44-111.

[4] A. Tarski, *O wyrazie pierwotnym logistyki (On the primitive term of logistic)*, Przegląd Filozoficzny 26 (1923), pp. 65-89.

[5] A. Tajtelbaum, *Sur le terme primitif de la Logistique*, Fund. Math. 4 (1924), pp. 196-200.

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