Local invertibility *

by

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This report is a continuation of the study of invertible spaces ([2], [3], [4], [5]). In particular, the material presented here stems from seminar discussions held at Michigan State University during the spring and summer of 1962. A good part of the credit for the final form of several results here is due to Mr. M. D. Guay and Mr. H. V. Kronk who also worked under NSF Grant GP-31.

Throughout this paper we use the notation $\mathcal{G}(S)$ to denote the group of all homeomorphisms of a topological space S onto itself.

DEFINITION 1. A topological space S is said to be *invertible about a point* $p \in S$ if, for each open neighborhood U of p, there exists $h \in \mathcal{G}(S)$ such that $h(S-U)$ lies in U. Such a point p is called an *invert point* of S. The set of all invert points of S is called the *invert* of S and will be denoted by $I(S)$.

As an example we point out that the n-cube I^n has non-empty invert and that, indeed, $I(I^n) = \mathrm{Bd}(I^n) = S^{n-1}$. (The equality of two topological spaces denotes topological equivalence.) Also we remark that $I(S) = S$ if and only if S is invertible.

The following characterization is an obvious generalization of Theorem 6 of [3].

THEOREM 1. *A space S is invertible about a point $p \in S$ if and only if, for each closed set C in $S-p$ and each open neighborhood U of p, there exists $h \in \mathcal{G}(S)$ such that $h(C)$ lies in U.*

Proof. If S has the assumed property, then S is evidently invertible about the point p. Suppose then that S is invertible about p and that the sets C and U are given. Since $U-C$ is not empty, then $U-C$ is itself an open neighborhood of p. Hence there is a homeomorphism $h \in \mathcal{G}(S)$ such that $h\big(S-(U-C)\big)$ lies in $U-C$. Clearly, h carries C into U.

Note: We must make the restriction that C lies in $S-p$. For if $S = I^1$, then the closed set $0 \cup 1$ cannot be carried into any proper open neighborhood of the invert point 0.

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THEOREM 2. *If S is a T_1-space, then $I(S)$ contains $0, 1, 2$ or an infinite number of points.*

Proof. Assume that $I(S)$ has a finite number $n > 2$ of points and let $p \in I(S)$. There is an open set U such that $p \in U$ and such that $U \cap [I(S) - p]$ is empty. Let h be an inverting homeomorphism for U. Since $h(S-U)$ lies in U, the $n-1$ points in $I(S)-p$ are carried into U. But this implies that U contains an invert point other than p, contradicting the choice of U.

THEOREM 3. *The invert $I(S)$ of a space S is an invertible subspace of S.*

Proof. Let $p \in I(S)$ and let V be any open neighborhood of p in the subspace topology of $I(S)$. Then there is an open set U in S such that $U \cap I(S) = V$. Since U is a neighborhood of p, there exists $h \in \mathcal{G}(S)$ such that $h(S-U)$ lies in U. Let $g = h|I(S)$ be the restriction of h to $I(S)$. Since $I(S)$ is carried onto itself by elements of $\mathcal{G}(S)$, it follows that $g \in \mathcal{G}(I(S))$. Clearly, $g[I(S)-V]$ lies in $U \cap I(S) = V$ and hence $I(S)$ is invertible.

THEOREM 4. *The invert $I(S)$ of a space S is a closed subset of S.*

Proof. Let p be a limit point of $I(S)$. Then any open neighborhood U of p contains an invert point. Hence there is an inverting homeomorphism for U and it follows that p is also an invert point.

COROLLARY. *If $I(S)$ is dense in S, then S is invertible.*

THEOREM 5. *If the invert $I(S)$ contains a non-empty open subset of the T_1-space S, then S is invertible.*

Proof. Suppose that the non-empty open set U of S is contained in $I(S)$. Then each point of S can be carried into U by some element of $\mathcal{G}(S)$, whence each point is in $I(S)$.

COROLLARY. *Either the T_1-space S is invertible or its invert $I(S)$ is a closed nowhere dense subset of S.*

The characterization of the n-sphere given in [2] can be stated in the present terminology as "The only n-manifold without boundary which has a non-empty invert is the n-sphere". It is not surprising perhaps that the following characterization should also hold.

THEOREM 6. *Let M^n be an n-manifold with non-empty boundary B. If $I(M^n)$ is not empty, then M^n is a topological n-cube.*

Proof. If p is an invert point of M^n, then p must be a point in the boundary B because an element of $\mathcal{G}(M^n)$ cannot carry a boundary point into an open n-cell neighborhood of an interior point of M^n.

Next we show that the boundary B is an $(n-1)$-sphere. Each element of $\mathcal{G}(M^n)$ carries B onto itself, of course. If V is an open neighborhood of $p \in I(M^n)$ in the subspace topology on B, then there is an open set U in M^n such that $U \cap B = V$. By assumption there is a homeomorphism $h \in \mathcal{G}(M^n)$ such that $h(M^n - U)$ lies in U and clearly $h|B$ carries $B-V$ into V. Thus B is an $(n-1)$-manifold which is invertible about the point p. By the characterization theorem stated above, B is an $(n-1)$-sphere.

Now let $S^{n-1} \times [0, 1)$ be mapped be a homeomorphism g onto a collar on the boundary B, i.e., g is an imbedding of $S^{n-1} \times [0, 1)$ into M^n such that $g(S^{n-1} \times 0) = B$. (See [1].) Let U be an open neighborhood of the point p such that $\overline{U}$ is a closed n-cell. Then there is an element $h \in \mathcal{G}(M^n)$ which carries the $(n-1)$-sphere $g(S^{n-1} \times 1/2)$ into U. Clearly $hg(S^{n-1} \times 1/2)$ lies in $U \cap \operatorname{Int} M^n$ (the interior of M^n). The bi-collared sphere $hg(S^{n-1} \times 1/2)$ separates $\overline{U}$ into an open n-cell A and a half-open annulus $\overline{U} - \overline{A}$. In any case, $h^{-1}(\overline{A})$ is a closed n-cell bounded by $g(S^{n-1} \times 1/2)$ and so M^n is the union of $h^{-1}(\overline{A})$ and the obviously compatible annulus $g(S^{n-1} \times [0, 1/2])$.

To indicate a direction in which possibly fruitful results may be obtained we include the following definition and theorem.

DEFINITION 2. A space S is *invertible about a subset* A of S if, for each open neighborhood U of A, there exists $h \in \mathcal{G}(S)$ such that $h(S-U)$ lies in U. The subset A may be called an *inversion subset*. The inversion subset A of S is *minimal* if S is not invertible about a proper subset of A.

THEOREM 7. *Let T be a metric space which has a bi-collared simple closed curve J as a minimal inversion subset. Then T is a torus or the Klein bottle.*

Proof. By definition there is an imbedding f of $S^1 \times (-1, 1)$ into T such that (a) $f(S^1 \times 0) = J$ and (b) $f[S^1 \times (-1, 1)] = U$ is an open neighborhood of J. Obviously T is locally euclidean in two dimensions at each point of U and by the assumption of invertibility about J, any point of T can be carried into U by a homeomorphism $h \in \mathcal{G}(T)$. It follows that T is a 2-manifold but T is not a 2-sphere because each minimal inversion subset in a 2-sphere is a single point.

Now in the bi-collar U on J we select an open parameter annulus A bounded by parameter 1-spheres $J_1 = f(S^1 \times -1/2)$ and $J_2 = f(S^1 \times 1/2)$. Since J lies in A there exists $h \in \mathcal{G}(T)$ carrying $T-A$ into A and thus $T-A$ is a compact 2-manifold with boundary $J_1 \cup J_2$. Since $h(T-A)$ lies in the annulus A, it follows that $T-A$ is itself a closed annulus. Hence T is the union of two closed annuli joined along their common boundary $J_1 \cup J_2$.

Our interest in suspensions, which one sees in several currently unsolved problems, led us to the following results.

THEOREM 8. *The suspension uSv of a space S has at least two invert points.*

Proof. We show that the cone points u and v are in $I(uSv)$. Let U be any open neighborhood of the cone point u. Then U contains an open set of the form $g(S\times(t_0,1])$ where g is a homeomorphism on $S\times(t_0,1)$ and $g(S\times 1)=u$. Let r denote the reflection map which carries the point (x,t), $x\in S$, $-1<t<1$, onto the point $(x,-t)$ and which interchanges the cone points u and v. Let h be a homeomorphism of the interval $[-1,1]$ onto itself which leaves the endpoints fixed and carries the point $-t_0$ onto the point t_0. Define the map $k(x,t)=\big(x,h(t)\big)$. The composite map kr is an inverting homeomorphism for the open set U. Since $r(u)=v$, the point v is also in $I(uSv)$.

Note that a space may have exactly two invert points and not be a suspension. Figure 1 pictures a plane Peano continuum having this property. If this continuum is denoted by C, then clearly $I(C)=p\cup q$.

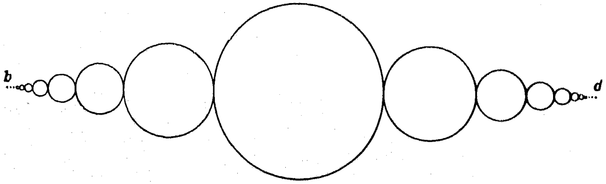

Fig. 1

THEOREM 9. *Let S be a near-homogeneous space* (see [3]) *and let uSv be the suspension of S. If uSv contains more than two invert points, then uSv is invertible.*

Proof. Suppose that (p,t_0), $-1<t_0<1$, is an invert point of uSv other than u and v. Homeomorphisms such as k in the proof above will carry the point (p,t_0) onto any point (p,t) and homeomorphisms of the form $h(x,t)=\big(g(x),t\big)$, $g\in\mathcal{G}(S)$, will carry the point (p,t_0) onto any point (x,t_0) in a dense set in $S\times t_0$. Thus $I(uSv)$ is dense in uSv and the Corollary to Theorem 4 applies.

Another direction in which we found interesting questions involves the quotient space $Q=S/I(S)$ of a space S modulo its invert.

THEOREM 10. *Let the space S have non-empty invert $I(S)$. Then the quotient space $Q=S/I(S)$ has non-empty invert.*

Proof. Let q be the quotient map of S onto Q and let $q\big(I(S)\big)=w$. Then w is an invert point of Q. For if U is any open neighborhood of w, then $q^{-1}(U)$ is an open neighborhood of $I(S)$ in the space S. Hence there is an element $h\in\mathcal{G}(S)$ carrying $S-q^{-1}(U)$ into $q^{-1}(U)$. Since $I(S)$ is invariant under elements of $\mathcal{G}(S)$, the composition qhq^{-1} is a one-to-one transformation of Q onto itself. This composition is a homeomorphism because both qhq^{-1} and $(qhq^{-1})^{-1}=qh^{-1}q^{-1}$ are closed. Moreover, qhq^{-1} carries $Q-U$ into U. Hence $w\in I(Q)$.

The example which gave rise of Theorem 10 is the n-cube I^n. Clearly, $I^n/I(I^n)=S^n$. For another example (not connected), let C be the Cantor set on the x-axis in E^2. At the points $(0,0)$ and $(1,0)$ erect vertical line segments (i.e. parallel to the y-axis) having length 1 and having the points $(0,0)$ and $(1,0)$ as midpoints. At the points $(1/3,0)$ and $(2/3,0)$ erect vertical line segments of length $1/2$ having these points as midpoints. In general, at the nth stage in the usual construction of C, 2^{n-1} open intervals are removed. At the endpoints of these intervals erect vertical line segments of length $1/n$ bisected by these endpoints. The set X is the union of C and the line segments so constructed. It is easy to see that $I(X)=C$.

If Theorem 10 is applied to the set X just described, then the quotient space $X/I(X)$ is homeomorphic to the plane Peano continuum having exactly one invert point which we picture in Figure 2.

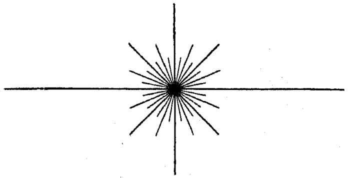

Fig. 2

The situation described above leads to an interesting question which we have not been able to answer. Suppose that we let $X/I^1(X)$ denote $X/I(X)$ and define inductively, for $n>1$, the space $X/I^n(X)$ $=[X/I^{n-1}(X)]/I[X/I^{n-1}(X)]$. Does there always exist an integer N, depending upon the space X, such that for $n>N$, the spaces $X/I^n(X)$ are all homeomorphic? And in the opposite direction, is there a continuum X for each integer N such that, for $n<N$, the spaces $X/I^n(X)$ are all different? As an example in which $N=3$, consider the continuum X shown in Figure 3. It is quite obvious that $I(X)=S^1$ and that $X/I(X)$ is a topological disk. It follows that $X/I^2(X)=S^2$ and that $X/I^3(X)$ is a single point. This example can also be modified to yield an example for which $N=4$ but we have no example for $N=5$.

In view of Theorems 8 and 10 the following result arose quite naturally.

THEOREM 11. *Let X be the suspension of a continuum C and let Y be the space obtained by identifying the cone points of X. If Y is near-homogenous, then C is a single point (and hence Y is a simple closed curve).*

Proof. Let q be the quotient map of X onto Y and let w be the image of the cone points. Any neighborhood of w contains a neighborhood consisting of two open cones with common vertex w. Thus w is a local cutpoint of Y. Since Y is near-homogeneous, there exists $g \in \mathcal{G}(Y)$ such that $g(w) \in Y-w$. The point $g(w)$ is also a local cutpoint of Y and hence $q^{-1}g(w)$ is a local cutpoint of X. We select an open product neighborhood $h(C\times(a,b))$ in X of the point $q^{-1}g(w) = h(x_0, t_0)$, where $x_0 \in C$, $a < t_0 < b$, and h is a homeomorphism. If the continuum C is non-degenerate, then the set

$$\{(x,t)|\ x\in C,\ t\neq t_0\}\cup\{(x,t)|\ x\neq x_0,\ a<t<b\}$$

is connected and its image under h is $h(C\times(a,b))-q^{-1}g(w)$. This contradicts the statement that $q^{-1}g(w)$ is a local cutpoint.

Corollary. *Let X be the suspension of a continuum C and suppose that $I(X) = S^0$. Then $X/I(X)$ is invertible if and only if C is a single point.*

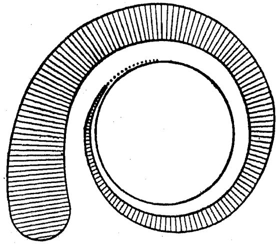

Fig. 3

We had remarked earlier [3] that invertibility is a very strong form of near-homogeneity. Thus local invertibility led us to discuss the much weaker property contained in the following definition which, while quite natural, seems to be new.

Definition 3. A space S is *near-homogeneous at a point* $p \in S$ if, for each open neighborhood U of p and each point $x \in S$, there exists $h \in \mathcal{G}(S)$ such that $h(x) \in U$. The set of all points of S at which S is near-homogeneous will be denoted by $N(S)$.

We note that the space S is near-homogeneous if and only if $S = N(S)$. The following facts may also be noted. They are established just as were the corresponding theorems above.

1. *$N(S)$ is carried onto itself by each $h \in \mathcal{G}(S)$.*
2. *$N(S)$ is a closed subset of S.*
3. *$N(S)$ is a near-homogeneous subspace of S.*
4. *If $N(S)$ contains a non-empty open subset of S, then S is near-homogeneous.*
5. *If $N(S)$ is non-empty, then $N[S/N(S)]$ is also non-empty.*

Our chief interest here lies in the relation between the subsets $N(S)$ and $I(S)$ of a space S. The investigation of local near-homogeneity will be carried out in some detail in another paper.

Theorem 12. *If $I(S)$ is not empty, then $I(S) = N(S)$.*

Proof. It is obvious that $I(S)$ is a subset of $N(S)$. So let p be a point of $N(S)$ and q be a point of $I(S)$. If U is any open neighborhood of p, then by the near-homogeneity of S at p, there exists $h \in \mathcal{G}(S)$ such that $h(q) \in U$. But $h(g)$ is also a point in $I(S)$. Hence p is a point of $\overline{I(S)}$ and, since $I(S)$ is closed, p is a point of $I(S)$. Thus $N(S)$ is also a subset of $I(S)$.

Theorem 13. *Let S and T be spaces. If $p \in N(S)$ and $q \in N(T)$, then the point $(p,q) \in S\times T$ is in $N(S\times T)$.*

Proof. Let (x,y) be any point in $S\times T$ and let $U\times V$ be a product neighborhood of (p,q). By the near-homogeneity of S at p and of T at q, there exist homeomorphisms $g_1 \in \mathcal{G}(S)$ and $g_2 \in \mathcal{G}(T)$ such that $g_1(x) \in U$ and $g_2(y) \in V$. Clearly the homeomorphism $h(s,t) = (g_1(s), g_2(t))$ carries (x,y) into $U\times V$.

Corollary 1. *If $I(S\times T)$ is not empty, then $I(S)\times I(T)$ is a subset of $I(S\times T)$.*

Corollary 2. *The product of two invertible spaces is either invertible or has empty invert.*

This latter corollary provides some further information concerning a problem which we have not solved and which can be stated as follows: Can the product of two finite dimensional continua be invertible? Another result along this same line can be given to indicate the sort of conditions one must have in order to give an affirmative answer to this question.

Remark. If $(p,q) \in I(S\times S)$, then $(q,p) \in I(S\times S)$.

Theorem 14. *Let S be a space with the following properties:*

(1) *$I(S\times S)$ is not empty,*

(2) *$I(S)\times I(S) \neq I(S\times S)$, and*

(3) *$\mathcal{G}(S)$ acts almost transitively on $S-I(S)$.*

Then $[S\times I(S)]\cup[I(S)\times S]$ is a subset of $I(S\times S)$ and if these two sets are not equal, then $S\times S$ is invertible.

Note. A group of homeomorphisms acts almost transitively on a set X if, for each point p in X and each relatively open set U in X, there is a homeomorphism in the group which carries p into U.

Proof of Theorem 14. Since $I(S\times S)$ is not empty, it contains the set $I(S)\times I(S)$. Let $(p,q) \in I(S\times S)-I(S)\times I(S)$. Without loss of generality we may assume that $p \notin I(S)$. Then by (3), the orbit O_p of the point p is dense in S. If $q \in I(S)$, then the fact that $I(S) = N(S)$ implies that $S\times I(S)$ lies in $I(S\times S)$. By the remark above, $I(S)\times S$

also lies in $I(S\times S)$. If $q\notin I(S)$, then $(p,q)\in[S-I(S)]\times[S-I(S)]$ and again by (3), we have the product $O_p\times O_q$ of orbits as a dense subset of $S\times S$. In this case, $I(S\times S)=S\times S$ and $S\times S$ is invertible.

For the case of an infinite product, the situation is quite different. The following results are included here to give a comparison between the two cases.

THEOREM 15. *Let $S=S_i$, $i=1,2,3,\ldots$, be a space with the property that for some point $p\in S$, the set $T_n=p\times \mathop{X}\limits_{i=2}^{n} S_i$ lies in the invert $I(\mathop{X}\limits_{i=1}^{n} S_i)$ for $n=2,3,\ldots$ Then the infinite product $\mathop{X}\limits_{i=1}^{\infty} S_i$ is invertible.*

Proof. The images of the points in $\bigcup\limits_{n=2}^{\infty} T_n$ under space homeomorphisms are evidently dense in the infinite product. It suffices to show then that a point of $\bigcup\limits_{n=2}^{\infty} T_n$ is an invert point of $\mathop{X}\limits_{i=1}^{\infty} S_i$. But this is obvious because T_n lies in T_{n+1} and is contained in the invert of the finite product $\mathop{X}\limits_{i=2}^{\infty} S_i$.

COROLLARY. *The Hilbert cube is invertible.*

As another interesting example, consider the subspace Γ of the Hilbert cube consisting of all points $(a_1,a_2,a_3,\ldots)$ with only a finite number of non-zero coordinates. This space Γ has the following properties:

(1) *Every simplex can be imbedded in Γ,*

(2) *Γ is a dense subset of the Hilbert cube,*

(3) *Γ is a monotone union of closed n-cells,*

(4) *Γ is connected,*

(5) *Γ is invertible.*

We verify property (5). First we observe that for each n, the point $(0,a_2,\ldots,a_n)$ is an invert point of I^n, for any choice of the coordinates a_i in I^1. Let $U=(B_1\times B_2\times\ldots\times B_n\times I^\omega)\cap\Gamma$ be an open set in Γ containing the point $(0,a_2,\ldots,a_n,0,0,\ldots)$. (We are considering Γ as a subspace of the Hilbert cube, of course.) Since $(0,a_2,\ldots,a_n)$ is an invert point of I^n, there is a homeomorphism h of Γ onto itself which is pointwise fixed outside $(B_1\times B_2\times\ldots\times B_n)\cap\Gamma$ and which carries $\Gamma-U$ into U. Therefore the point $(0,a_2,\ldots,a_n,0,0,\ldots)$ is an invert point of the space Γ. We consider now the space homeomorphism g_n which simply interchanges the first and the $(n+1)$th coordinates of a point. Clearly the image $g_n(a_1,a_2,\ldots,a_n,0,0,\ldots)$ of a point in Γ becomes an invert point $(0,a_2,\ldots,a_n,a_1,0,\ldots)$. Thus each point of Γ is an invert point and Γ is invertible.

THEOREM 16. *Hilbert space is invertible.*

Proof. According to Klee [6], Hilbert space is homeomorphis to the unit sphere S^ω in Hilbert space. Since S^ω is certainly near-homogeneous, we need only show the existence of one invert point. Consider the point $p=(1,0,0,\ldots)\in S$ and let U be any open neighborhood of p. There exists $\varepsilon>0$ such that every point $q=(a_1,a_2,\ldots)$ of S satisfying $1-\varepsilon<a_1\leqslant 1$ lies in U. Represent each ponit x of S by $x=x_1+x_2$, where x_1 is a point of a one-dimensional linear subspace L_1 of Hilbert space and where x_2 is a point in the complementary linear subspace L_2 orthogonal to L_1. Consider the map h of L_1 onto itself given by

$$\begin{aligned} h(x_1) &= -1+\frac{1}{2-\varepsilon}(x_1+1), & -1\leqslant x_1\leqslant 1-\varepsilon,\\ &= 1-(1-x_1)^2/\varepsilon^2, & 1-\varepsilon<x_1\leqslant 1,\\ &= x_1, & |x_1|>1. \end{aligned}$$

it is a matter of elementary calculation to show that the map

$$H(x_1+x_2)=h(x_1)+\left(\frac{1-h^2(x_1)}{1-x_1^2}\right)^{1/2}x_2,$$

when restricted to S^ω, is a homeomorphism of S^ω onto itself carrying the open set $\{x_1+x_2|\ 1-\varepsilon<x_1\leqslant 1\}$ onto the open set $\{x_1+x_2|\ 0<x_1\leqslant 1\}$. If R is the reflection of S^ω in L_2, then the map $H^{-1}RH$ provides a homeomorphism of S^ω onto itself such that $H^{-1}RH(S^\omega-U)$ lies in U. Thus the point p is an invert point of S^ω.

The comparison between the Corollary to Theorem 15 above and Theorem 16 is perhaps surprising. Another such result follows the next lemma.

LEMMA. *Let J denote the open unit interval $(0,1)$ and let $U=U_1\times\times U_2\times\ldots\times U_n$ be an open set in the product J^n such that for some i, $1\leqslant i\leqslant n$, the open set U_i is of the form $(0,t)$. Then there exists a homeomorphism h of J^n onto itself such that $h(J^n-U)$ lies in U.*

The proof of this lemma is trivial.

THEOREM 17. *The countably infinite product J^ω of open intervals is invertible.*

Proof. Let $U=U_1\times U_2\times\ldots\times U_n\times J\times\ldots$ be an open set in the Tychonoff basis for J^ω. Restricting attention to the first $n+1$ coordinates, and thus to a product J^{n+1}, the lemma above provides a homeomorphism h of J^{n+1} onto itself such that $h(J^{n+1}-U_1\times\ldots\times U_n\times J)$ lies in $U_1\times\ldots\times\times U_n\times J$. Extending h to be the identity map on each coordinate x_k, $k>n+1$, we obtain a homeomorphism $\tilde{h}$ of J^ω onto itself which carries $J^\omega-U$ into the open set U.

We conclude this report with a brief outline of a theory analogous to that above. The purpose here is to localize the concept of continuous invertibility (see [4]) and this led us to the definition of continuous near-homogeneity given below.

The symbol $\mathcal{H}(S)$ will denote the group of all homeomorphisms of the space S onto itself which are isotopic to the identity map on S. That is, for each element $h \in \mathcal{H}(S)$, there is a continuous map $f\colon S \times I^1 \to S$ with the properties (1) $f(x, 0) = x$, (2) $f(x, 1) = h(x)$ for all $x \in S$ and (3), for each $t \in I^1$, $f|S \times t$ is a homeomorphism.

In the light of the foregoing development, the following definitions are quite natural.

DEFINITION 4. A space S is *continuously invertible at a point* $p \in S$ if, for each open neighborhood U of p, there exists $h \in \mathcal{H}(S)$ such that $h(S-U)$ lies in U. The set of all points at which S is continuously invertible will be denoted by $CI(S)$.

DEFINITION 5. A space S is *continuously near-homogeneous at a point* $p \in S$ if, for each open neighborhood U of p and each point $x \in S$, there exists $h \in \mathcal{H}(S)$ such that $h(x) \in U$. The set of all points at which S is continuously near-homogeneous will be denote by $CN(S)$. The space S is *continuously near-homogeneous* if $S = CN(S)$.

The first six theorems below are proved in exactly the same way as were their counterparts above. We use the convention of square brackets to enable us to write two theorems as one.

THEOREM 18. *The set* $CI(S)$ $[CN(S)]$ *is carried onto itself by each* $h \in \mathcal{H}(S)$.

THEOREM 19. *The set* $CI(S)$ $[CN(S)]$ *is closed in* S.

THEOREM 20. *If* $CI(S)$ $[CN(S)]$ *contains a non-empty open subset of* S, *then* S *is continuously invertible [continuously near-homogeneous].*

THEOREM 21. *The set* $CI(S)$ $[CN(S)]$ *is a continuously invertible [continuously near-homogeneous] subspace of* S.

THEOREM 22. *If* $CI(S)$ $[CN(S)]$ *is non-empty then* $CI(S/CI(S))$ $[CN(S/CN(S))]$ *is also non-empty.*

THEOREM 23. *If* $CI(S)$ *is not empty, then* $CI(S) = CN(S)$.

Our final two results are simple consequences of the fact that the continuous orbit $P_x = \{y|\ y = h(x),\ h \in \mathcal{H}(S)\}$ *is connected.*

THEOREM 24. *If* $CI(S)$ $[CN(S)]$ *is non-empty, then* S *is connected.*

Proof. This need only be proved for the case where $CN(S)$. Let $p \in CN(S)$. Then each open neighborhood of p contains a point in every continuous orbit P_x. Therefore $p \in \overline{P}_x$ for each point x. Since P_x is connected, so is $\overline{P}_x$. Hence S is a union of connected subsets, each containing the point p.

Combining Theorems 21 and 24, we have the final result.

THEOREM 25. $CI(S)$ $[CN(S)]$ *is a connected subset of* S.

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