Conclusions. We summarize the results of the preceeding section of this part by the following theorem:

THEOREM 4.4. *There is a countable collection D of mutually disjoint connected subset of* $\mathfrak{I}^2$ *which has properties* (II) *and* (III) *but which does not have property* (I).

This theorem provides a negative answer to problem 2. We point out that the particular collection constructed also is a suitable set for rejecting problem 1. Also, since $\boldsymbol{D}$ is a subcollection of $\boldsymbol{C}$, there is an effective method for constructing $\boldsymbol{D}$.

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Most knots are wild

by

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Let $\mathrm{Emb}(X, R^n)$ denote the topological space consisting of all embeddings of a compact space X into the n-dimensional euclidean space R^n. This is a Baire space ([1]). We will say that *most* embeddings of X in R^n have some given property P if the set of all $f \in \mathrm{Emb}(X, R^n)$ which satisfies this property P contains a dense G_δ.

THEOREM 1. *Most embeddings of the circle in euclidean 3-space are wildly knotted.*

THEOREM 2. *For $n \geqslant 4$ most embeddings of the circle in R^n are unknotted.*

(Note however that knotted embeddings do exist for all $n \geqslant 3$. See Blankinship [2].)

Proof of Theorem 2. We will show that $\mathrm{Emb}(S^1, R^n)$ contains a subset $\mathrm{Emb}(S^1, R^{n-1}) \times F(S^1, R)$ which is a dense G_δ, and consists entirely of unknotted embeddings.

Let $F(S^1, R^n)$ denote the Banach space consisting of all mappings from S^1 to R^n. We will identify $F(S^1, R^n)$ with the product $F(S^1, R^{n-1}) \times$ $\times F(S^1, R)$. Since $n-1 \geqslant 3$, the subset $\mathrm{Emb}(S^1, R^{n-1}) \subset F(S^1, R^{n-1})$ is a dense G_δ. (Hurewicz-Wallman [6], p. 56.) Therefore $\mathrm{Emb}(S^1, R^{n-1}) \times$ $\times F(S^1, R)$ is a dense G_δ in $F(S^1, R^n)$, and hence a fortiori it is a dense G_δ in $\mathrm{Emb}(S^1, R^n)$.

But an argument due to Bing and Klee shows that every

$$(f, g) \in \mathrm{Emb}(S^1, R^{n-1}) \times F(S^1, R) \subset \mathrm{Emb}(S^1, R^n)$$

can be transformed into the standard embedding by an isotopy of R^n. First consider an isotopy of the form

$$h_t(x, y) = \big(x, y + tp(x)\big),$$

([1]) See Lemma 2. E is a *Baire space* if every countable intersection of dense open subsets is dense. A subset $S \subset E$ is called a G_δ if S can be expressed as a countable intersection of open subsets.

where $0 \leqslant t \leqslant 1$, $x \in R^{n-1}$, $y \in R$. Such an isotopy can transform (f, g) into (f, g') where $g'\colon S^1 \to R$ is any desired mapping: it is only necessary to choose $p\colon R^{n-1} \to R$ as an extension of the mapping $f(s) \to g'(s) - g(s)$ from $f(S^1)$ to R.

In particular g' can be chosen as a function having only one local maximum and one local minimum on S^1. But then, according to Milnor ([7], § 4.5), the embedding (f, g') is unknotted. Therefore every $(f, g) \in \mathrm{Emb}(S^1, R^{n-1}) \times F(S^1, R)$ is unknotted; which completes the proof of Theorem 2.

Remark. More generally, let X be a compact polyhedron of dimension $d \geqslant 1$. *If $n \geqslant 3d+1$ then most embeddings of X in R^n are unknotted* (i.e. are ambient isotopic to a standard embedding).

This follows from Bing and Kister [1], together with the argument above which shows that most embeddings can be deformed into a hyperplane of dimension $2d+1$.

The proof of Theorem 1 will be based on the following. Consider an embedded solid torus $\overline{T} \subset R^3$ with interior T; and an embedding

$$k\colon S^1 \to T$$

which has winding number ± 1. (Compare Schubert [9].) In other words we assume that the induced homomorphism $k_*\colon H_1(S^1) \to H_1(T)$ is an isomorphism. The simple closed curve $k(S^1) \subset T$ will be denoted by K.

LEMMA 1. *The complement $R^3 - T$ is a retract of $R^3 - K$.*

(I am indebted to J. Kister for suggesting this formulation.)

Proof. It clearly suffices to show that the boundary torus $T^{\cdot}$ is a retract of $\overline{T} - K$. In other words we must show that the identity map of $T^{\cdot}$ extends to a mapping $\overline{T} - K \to T^{\cdot}$. This is an extension problem of the type which is studied in obstruction theory. (See, for example, Hilton and Wylie [4], § 7.) The only obstruction to the existence of an extension is a cohomology class in the relative cohomology group $H^2(\overline{T} - K, T^{\cdot}; \pi_1(T^{\cdot}))$. We will prove that all of the cohomology groups of $(\overline{T} - K, T^{\cdot})$ are zero, so that there is no obstruction.

First note that the inclusion $K \to \overline{T}$ is a homotopy equivalence. Therefore the relative Čech cohomology groups $H^i(\overline{T}, K)$ are zero. But a duality theorem of the form

$$H^i(\overline{T}, K) \cong H_{3-i}(\overline{T} - K, T^{\cdot})$$

is not difficult to establish. (Compare [8], Lemma 2. If $N \subset T$ is a compact polyhedral neighborhood of K, one can first establish the isomorphism $H^i(\overline{T}, N) \cong H_{3-i}(\overline{T} - N, T^{\cdot})$, and then pass to the direct limit as N shrinks down to K.) Therefore the pair $(\overline{T} - K, T^{\cdot})$ has trivial homology, and hence has trivial cohomology. This completes the proof.

An important consequence of Lemma 1 is the following. Define the *rank* of K (or of T) to be the minimal number of generators for the fundamental group $\pi_1(R^3 - K)$ (or for $\pi_1(R^3 - T)$). Then it follows that

$$\operatorname{rank} K \geqslant \operatorname{rank} T.$$

To prove Theorem 1 we will also need to know that the space $\mathrm{Emb}(S^1, R^3)$ is a Baire space. This is clear since $\mathrm{Emb}(S^1, R^3)$ is a dense G_δ in the Banach space $F(S^1, R^3)$. More generally:

LEMMA 2. *If X is compact and Y is complete metric, then* $\mathrm{Emb}(X, Y)$ *is a Baire space.*

Proof. This follows since $\mathrm{Emb}(X, Y)$ is a G_δ in $F(X, Y)$ (see [6], p. 57 (B)), and hence is a dense G_δ in its closure in $F(X, Y)$ which is a complete metric space.

Proof of Theorem 1. Let $U_r \subset \mathrm{Emb}(S^1, R^3)$ be the open set consisting of all embeddings k such that $k(S^1) \subset T$ with winding number ± 1 for the interior T of some differentiably embedded solid torus, with $\operatorname{rank}(T) \geqslant r$.

This set U_r is dense: Any embedding k can be approximated by a differentiable embedding k', and one can tie a number of small trefoil knots into k' so as to guarantee that its rank is $\geqslant r$. (Compare Fox [5].)

Let $W \subset \mathrm{Emb}(S^1, R^3)$ be the intersection of the dense open sets U_r. For any $k \in W$ we have $k \in U_r$ and hence $\operatorname{rank}\big(k(S^1)\big) \geqslant r$ for all integers r. This implies that $\operatorname{rank}\big(k(S^1)\big) = \infty$ so that k must be wildly knotted.

Concluding remarks. These two theorems raise a number of questions. Is it true that most embeddings of the unit interval $[0, 1]$ in R^3 are wildly knotted? Theorem 1 suggests that this is true without suggesting a proof. What can be said about 2-spheres in 3-space; or more generally about k-spheres in n-space?

A different type of question arises if we ask whether an embedding is knotted "with probability 1". (Such a question is quite different from our Baire space arguments: even in the Baire space R^n a dense G_δ set may have measure zero.) One way to make sense out of this question is to put the probability measure on $F([0, 1], R^n)$ which is associated with Brownian motion. In dimension 3 such a Brownian motion has self-intersections with probability 1. (See [3].) In dimension 4 however, it is an embedding with probability one, hence for $n \geqslant 5$ it follows that a Brownian motion is unknotted with probability 1. (Compare the proof of Theorem 2.) There remains the question as to whether a Brownian motion in 4-space is knotted.

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