Conclusions. We summarize the results of the preceding section of this part by the following theorem:

**Theorem 4.4.** There is a countable collection \( D \) of mutually disjoint connected subset of \( \mathbb{R}^3 \) which has properties (II) and (III) but which does not have property (I).

This theorem provides a negative answer to problem 2. We point out that the particular collection constructed also is a suitable set for rejecting problem 1. Also, since \( D \) is a subcollection of \( C \), there is an effective method for constructing \( D \).

**References**


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**Most knots are wild**

*by J. Milnor (Princeton)*

Let \( \text{Emb}(X, \mathbb{R}^n) \) denote the topological space consisting of all embeddings of a compact space \( X \) into the \( n \)-dimensional euclidean space \( \mathbb{R}^n \). This is a Baire space (1). We will say that most embeddings of \( X \) in \( \mathbb{R}^n \) have some given property \( P \) if the set of all \( f \in \text{Emb}(X, \mathbb{R}^n) \) which satisfies this property \( P \) contains a dense \( G_\delta \).

**Theorem 1.** Most embeddings of the circle in euclidean 3-space are wildly knotted.

**Theorem 2.** For \( n \geq 4 \), most embeddings of the circle in \( \mathbb{R}^n \) are unknotted.

(Note however that knotted embeddings do exist for all \( n \geq 3 \). See Blankinship [2].)

**Proof of Theorem 2.** We will show that \( \text{Emb}(S^1, \mathbb{R}^n) \) contains a subset \( \text{Emb}(S^1, \mathbb{R}^{n-1}) \times \text{F}(S^1, \mathbb{R}^n) \) which is a dense \( G_\delta \), and consists entirely of unknotted embeddings.

Let \( \text{F}(S^1, \mathbb{R}^n) \) denote the Banach space consisting of all mappings from \( S^1 \) to \( \mathbb{R}^n \). We will identify \( \text{F}(S^1, \mathbb{R}^n) \) with the product \( \text{F}(S^1, \mathbb{R}^{n-1}) \times \times \text{F}(S^1, \mathbb{R}) \). Since \( n-1 \geq 3 \), the subset \( \text{Emb}(S^1, \mathbb{R}^{n-1}) \subseteq \text{F}(S^1, \mathbb{R}^{n-1}) \) is a dense \( G_\delta \). (Hurewicz-Wallman [6], p. 56.) Therefore \( \text{Emb}(S^1, \mathbb{R}^{n-1}) \times \times \text{F}(S^1, \mathbb{R}) \) is a dense \( G_\delta \) in \( \text{F}(S^1, \mathbb{R}^n) \), and hence a fortiori it is a dense \( G_\delta \) in \( \text{Emb}(S^1, \mathbb{R}^n) \).

But an argument due to Bing and Klee shows that every

\[
(f, g) \in \text{Emb}(S^1, \mathbb{R}^{n-1}) \times \text{F}(S^1, \mathbb{R}) \subset \text{Emb}(S^1, \mathbb{R}^n)
\]

can be transformed into the standard embedding by an isotopy of \( \mathbb{R}^n \).

First consider an isotopy of the form

\[
h_t(x, y) = (x, y + t \rho(x)),
\]

(1) See Lemma 2. \( E \) is a Baire space if every countable intersection of dense open subsets is dense. A subset \( S \subset E \) is called a \( G_\delta \) if \( S \) can be expressed as a countable intersection of open subsets.
where \( 0 < t \leq 1 \), \( x \in R^{n-1} \), \( y \in R \). Such an isotopy can transform \((f, g)\) into \((f', g')\) where \( g' : S^1 \to B \) is any desired mapping: it is only necessary to choose \( p : R^{n-1} \to B \) as an extension of the mapping \( f(s) \to g'(s) - g(s) \) from \( f(S) \) to \( B \).

In particular \( g' \) can be chosen as a function having only one local maximum and one local minimum on \( S^1 \). But then, according to Milnor ([7], § 4.3), the embedding \((f, g')\) is unknotted. Therefore every \((f, g) \in Emb(S, R^n) \times F(S, B)\) is unknotted; which completes the proof of Theorem 2.

Remark. More generally, let \( X \) be a compact polyhedron of dimension \( d \geq 1 \). If \( n > 3d + 1 \), then most embeddings of \( X \) in \( R^n \) are unknotted (i.e., are ambient isotopic to a standard embedding).

This follows from Bing and Kister [13], together with the argument above which shows that most embeddings can be deformed into a hyperplane of dimension \( 2d + 1 \).

The proof of Theorem 1 will be based on the following. Consider an embedded solid torus \( T \subset R^3 \) with interior \( T \); and an embedding \( k : S^1 \to T \) which has winding number \( \pm 1 \). (Compare Schubert [9].) In other words we assume that the induced homomorphism \( k_* : H_1(S) \to H_1(T) \) is an isomorphism. The simple closed curve \( k(S^1) \subset T \) will be denoted by \( K \).

**Lemma 1. The complement \( R^n - T \) is a retract of \( R^n - K \).**

(I am indebted to J. Kister for suggesting this formulation.)

**Proof.** It clearly suffices to show that the boundary torus \( T \) is a retract of \( T - K \). In other words we must show that the identity map of \( T \) extends to a mapping \( T - K \to T \). This is an extension problem of the type which is studied in obstruction theory. (See, for example, Hilton and Wylie [4], § 7.) The only obstruction to the existence of an isomorphism class in the relative cohomology group \( H^n(T - K, T; \pi_1(T)) \). We will prove that all of the cohomology groups of \( (T - K, T) \) are zero, so that there is no obstruction.

First note that the inclusion \( K \to T \) is a homotopy equivalence. Therefore the relative Čech cohomology groups \( H^*(T, K) \) are zero. But a duality theorem of the form

\[
H^*(T, K) \cong H_{n-*}(T - K, T)
\]

is not difficult to establish. (Compare [8], Lemma 2. If \( N \subset T \) is a compact polyhedral neighborhood of \( K \), one can establish the isomorphism \( H^*(T, N) \cong H_{n-*}(T - N, T) \), and then pass to the direct limit as \( N \) shrinks down to \( K \).) Therefore the pair \((T - K, T)\) has trivial homology, and hence has trivial cohomology. This completes the proof.

An important consequence of Lemma 1 is the following. Define the rank of \( K \) (or of \( T \)) by the minimal number of generators for the fundamental group \( \pi_1(R^n - K) \) (or for \( \pi_1(R^n - T) \)). Then it follows that

\[
\text{rank } K \geq \text{rank } T.
\]

To prove Theorem 1 we will also need to know that the space \( \text{Emb}(S, R^n) \) is a Baire space. This is clear since \( \text{Emb}(S, R^n) \) is a dense \( G_0 \) in the Banach space \( F(S, B^n) \) which is a complete metric space.

**Lemma 2. If \( X \) is compact and \( Y \) is complete metric, then \( \text{Emb}(X, Y) \) is a Baire space.**

**Proof.** This follows since \( \text{Emb}(X, Y) \) is a \( G_0 \) in \( F(X, Y) \) (see [6], p. 57 (1)), and hence is a dense \( G_0 \) in its closure in \( F(X, Y) \) which is a complete metric space.

**Proof of Theorem 1.** Let \( U \subset \text{Emb}(S, R^n) \) be the open set consisting of all embeddings \( k \) such that \( k(S^1) \subset T \) with winding number \( \pm 1 \) for the interior \( T \) of some differentiably embedded solid torus, with \( \text{rank } (T) \geq r \).

This set \( U \) is dense: Any embedding \( k \) can be approximated by a differentiable embedding \( k' \), and one can tie a number of small trefoil knots onto \( k' \) so as to guarantee that its rank is \( \geq r \). (Compare Fox [5].)

Let \( W \subset \text{Emb}(S, R^n) \) be the intersection of the dense open sets \( U_r \). For any \( k \in W \) we have \( k \in U_r \) and hence \( \text{rank } (k(S^1)) \geq r \) for all integers \( r \). This implies that \( \text{rank } (k(S)) = \infty \) so that \( k \) must be wildly knotted.

**Concluding remarks.** These two theorems raise a number of questions. Is it true that most embeddings of the unit interval \([0, 1]\) in \( R^n \) are wildly knotted? Theorem 1 suggests that this is true without suggesting a proof. What can be said about 2-spheres in 3-space; or more generally about \( k \)-spheres in \( n \)-space?

A different type of question arises if we ask whether an embedding is knotted “with probability 1". (Such a question is quite different from our Baire space arguments: even in the Baire space \( R^n \) a dense \( G_0 \) set may have measure zero.) One way to make sense out of this question is to put the probability measure on \( F([0, 1], R^n) \) which is associated with Brownian motion. In dimension 3 such a Brownian motion has self-intersections with probability 1. (See [3].) In dimension 4 however, it is an embedding with probability one, hence for \( n \geq 5 \) it follows that a Brownian motion is unknotted with probability 1. (Compare the proof of Theorem 2.) There remains the question as to whether a Brownian motion in 4-space is knotted.
References


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