est divergente pour chaque \( \varphi \in E \) et qu'elle est convergente pour chaque \( \varphi \in E \). Il nous reste à démontrer que la suite \( \{s_n(\varphi, F_n)\} \) est convergente pour tout \( \varphi \in E \).

En effet, si \( \varphi \in E \), alors \( \varphi \in E_n \) pour tout \( n \). La suite \( \{s_n(\varphi, F_n)\} \) est convergente pour chaque \( \varphi \in E_n \), pour chaque \( n \) et \( \lim_{n \to \infty} s_n(\varphi, F_n) = F(\varphi) \).

En passant dans (37) à la limite avec \( h \), ce qui est permis en vertu de la convergence uniforme de la série

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} s_n(\varphi, F_n),
\]

nous obtiendrons

\[
\lim_{n \to \infty} s_n(\varphi, F) = \sum_{n=1}^{\infty} \frac{1}{2^n} F_n(\varphi) = F(\varphi)
\]

ci qui termine la démonstration.

**Travaux cités**


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Concerning some problems raised by A. Lelek

by

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1. Introduction. A. Lelek raises in [3] a series of questions about fixations in Euclidean n-dimensional space, \( \mathbb{R}^n \). We may ask when each of the following properties hold for a collection \( C \) of subsets of \( \mathbb{R}^n \).

(I) There exists a 0-dimensional compact set \( Z \subseteq \mathbb{R}^n \) such that \( Z \nsubseteq C \).

(II) There is an arc \( A \subseteq \mathbb{R}^n \) such that \( A \cap C = \emptyset \) for every \( C \in C \).

(III) There exists, for every \( \xi > 0 \), a finite sequence \( Z_1, ..., Z_k \) of closed and mutually disjoint subsets of \( \mathbb{R}^n \) such that \( \delta(Z_1) < \xi \) for \( i = 1, ..., k \) and \( \bigcup_{i=1}^{k} Z_i \nsubseteq C \) for all \( C \in C \).

Let \( C^* \) be the union of all sets belonging to \( C \). Denote by \( A(C) \) the set of all points \( p \in \mathbb{R}^n \) such that there is a sequence \( C_1, C_2, ... \) of elements of \( C \) such that \( \{p\} = \lim C_i \) (see [2] for the definition of \( \lim \)). Lelek asks the following questions:

**Problem 1.** Is it true that if \( C^* \) is a bounded subset of the plane and there exists an \( \varepsilon > 0 \) such that \( C \) is a disjoint collection of connected sets of diameter greater than \( \varepsilon \), then (I) holds?

**Problem 2.** Is it true that if \( C^* \) is a bounded subset of the plane and \( C \) is a disjoint collection of connected sets then (III) implies (I)?

**Problem 3.** Is it true that if \( C^* \) is a subset of the plane, \( C \) is a disjoint collection of connected sets and \( \dim A(C) \leq 0 \), then (II) implies (III)?

In this paper, we give negative answers to these questions by constructing two counter-examples. The collections \( C \) and \( D \) defined are collections of subsets of the unit square, \( \mathbb{R}^2 \).

2. Preliminary definitions and results. We first define some subsets of \( \mathbb{R}^2 \) which will be used in both constructions. Let \( R \) be the usual metric on \( \mathbb{R}^2 \); let \( B(x, r) \) be the open ball of radius \( r \)

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about \( x \) in \( \mathcal{P} \). Let \( A = \{(x, y) \in \mathcal{P}^2 \mid x = 0\} \), \( B = \{(x, y) \in \mathcal{P}^2 \mid y = 0\} \), \( T = \{(x, y) \in \mathcal{P}^2 \mid y = 1\} \).

**Definition.** For a positive integer \( n \) and for integers \( i \) and \( j \) such that \( 0 \leq i < 2^{n-1} \) and \( 0 \leq j < 2^{n-1} \), let \( Q_0(i, j) = \{(x, y) \in \mathcal{P}^2 \mid 2^n x < (i+1)2^n \text{ and } 2^n y < (j+1)2^n \} \). Then \( Q_n = \bigcup_{i=0}^{2^n - 1} Q_0(i, j) \) is called a \( n \)-tile of \( \mathcal{P} \).

**Definition.** If \( (Q_1, ..., Q_k) \) is a finite sequence of elements of \( Q_n \), such that
1. \( Q_i \) and \( Q_{i+1} \) intersect in an edge for \( 1 \leq i < k-1 \),
2. \( Q_i \cap T \neq \emptyset \),
3. \( Q_k \cap B \neq \emptyset \),
4. \( Q_i \cap Q_j = \emptyset \) if \( i \neq j \),
then \( Q = \bigcup_{i=1}^{k} Q_i \) is an \( n \)-path through \( \mathcal{P} \).

Notice that for any \( n \)-path \( Q \) through \( \mathcal{P} \), both \( Q \cap T \) and \( Q \cap B \) contain a non-degenerate continuum.

If \( X \) is a set we define \( k(X) \) to be the cardinal number of \( X \).

Theorems 2.1 and 2.2 below are well-known, so no proofs will be given.

**Theorem 2.1.** If \( Z \) is a zero-dimensional subset of \( \mathcal{P} \), then there exists a continuum \( K \) separating \( L \) and \( L \cap K \neq Z \).

**Theorem 2.2.** Let \( Z_1, Z_2, ..., Z_k \) be disjoint closed subsets of \( \mathcal{P} \) such that, for all \( i \), \( Z_i \) does not intersect both \( L \) and \( L \cap T \), nor does it intersect both \( L \) and \( L \cap B \). Then there exists a continuum \( K \) separating \( L \) and \( R \) which is disjoint from \( \bigcup_{i=1}^{k} Z_i \).

Let \( \mathcal{X} \) be the set of all non-degenerate continua contained in \( \mathcal{P} \). Since \( k(\mathcal{X}) = c = \omega(3) \), \( \mathcal{X} \) may be indexed by an ordinal number \( A \) (whose cardinality, therefore, is \( e \)) such that \( k(a) = \alpha \in \mathcal{X} = \{K : a \in A \} \).

**Definition.** If \( x \) and \( y \) are two points in \( \mathcal{P} \), let \( L(x, y) \) be the straight line segment with \( x \) and \( y \) as end points.

**Theorem 2.3.** If \( M \) is a connected point set in \( \mathcal{P} \), \( U \) is an open set containing \( M \), and \( x \) and \( y \) are points of \( M \), then there are points \( x_0, x_1, ..., x_n \) in \( M \) such that \( L(x_0, x_n) \) is contained in \( U \) for \( 0 \leq i \leq n \) and such that if \( 0 < i < j < n \), \( L_i \cap L_{i+1} = \emptyset \) only if \( j = i + 1 \). Moreover, \( G = \bigcup_{i=0}^{n} L_i \) is a continuum contained in \( U \) and is topologically a closed interval which can be linearly ordered from \( x \) to \( y \).

Proof. This follows from [4], Theorem 77, p. 56, and some plane geometry.

**3. Example I.** A. The construction of example I. Let \( \mathcal{P} \) be the set of all \( P \) such that for some \( n, P \) is an \( n \)-path through \( \mathcal{P} \). \( \mathcal{P} \) is a countable set, so we write \( \mathcal{P} = (P_1, P_2, ...) \). For each positive integer \( i \) and each \( a \in A \), we define \( G_a \) so that
- \( G_a \cap P_i \neq \emptyset \),
- \( 0 \leq \beta < a \) and \( G_\beta \subseteq P_i \), then \( G_\beta \cap G_a = \emptyset \),
- \( 0 \leq \beta < a \), \( G_\beta \subseteq G_a \),
- \( G_a \cap B_0 = \emptyset \),
- \( B_0 = \bigcup_{a \in A} G_a \).

The definition is inductive.

For \( a = 0 \). Let \( G_0 = \{a_0\} \) where \( a_0 \in K_0 \) if \( K_0 \subseteq P_1 \); let \( G_0 = \emptyset \) if \( K_0 \) is not contained in \( P_1 \). Then \( G_0 \) satisfies conditions a)-d). Assume that \( G_n \) has been defined for all \( k < n \) and satisfies conditions a)-d).

Let \( G_n = \{a_0\} \) where \( a_0 \in K_n \cap B_0 \); let \( G_n = \emptyset \) if \( K_n \) is not contained in \( P_n \). Conditions a), c), and d) are clearly satisfied for \( G_n \). Suppose that \( K_n \subseteq P_n \). We need only prove \( K_n \subseteq \bigcup_{a \in A} G_a \). But from d) and the induction hypothesis we see that \( k(G_n) < 1 \) for \( k < n \), so \( k(G_n \cap B_0) \neq \emptyset \).

**) Assume that \( a < A \), that \( a > 0 \), and that for all \( \beta \), \( 0 < \beta < a \), and \( a \in A \) for all positive integers \( k \), \( G_a \) has been defined and satisfies conditions a)-d).

**Assertion 3.1.** For all \( a \in A \) and \( 0 < \beta < a \), \( G_\beta \subseteq G_a \).

Proof. For all \( x \in B_0 \) there is a \( k < a \) such that \( x \in C_k \). Let \( \beta(x) \) be the least \( k \) in the well-ordered set \( (\beta : \beta < a) \). Define \( g_\beta : B_0 \rightarrow (\beta : \beta < a) \) by \( g_\beta(x) = \beta(x) \). Suppose \( g_\beta(x) = g_\beta(y) = \beta \). Then \( x, y \in C_k \) but neither \( x \) nor \( y \) is in \( B_0 \). As \( k(G_\beta - B_0) \leq 1 \) or \( k = \beta \). Thus, \( g_\beta \) is a one-to-one mapping, and the first inequality holds. The second inequality is true by our choice of \( A \).

**Assertion 3.2.** For all \( a < A \) and \( 0 < \beta < a \), \( G_\beta \cap G_a = \emptyset \) if \( k < n \).

Proof. Since \( K_n \subseteq P_1 \), \( G_\beta \cap G_a = \emptyset \) if \( k < n \).

Now let \( G_n = \{a_0\} \cup B_0 \), where \( a_0 \in K_n \cap B_0 \) if \( K_n \subseteq P_1 \); let \( G_n = B_0 \) if \( K_n \subseteq P_1 \). Then \( G_n \subseteq P_1 \) for \( B_0 \cap P_1 \) and if \( K_n \subseteq P_1 \), \( a_0 \in P_1 \). That properties c) and d) hold is clear. Since \( B_0 \cap G_a \neq \emptyset \) for all \( k < n \), \( G_n \subseteq P_1 \) by Assertion 3.2, we need only show that \( K_n \cap G_\beta = \emptyset \) if \( K_n \subseteq P_1 \). It suffices to show that there is an \( a_0 \) in \( K_n \cap B_0 \). But \( k(B_0) < c \) by Assertion 3.1, so \( k(G_n) \geq k(G_\beta) > 0 \); that is, the set is not empty. Now assume that \( G_n \) has been defined for \( k > n \) and satisfies a)-d).
Let \( C_n = \{ e_{an} \} \cup B_{an} \), where \( e_{an} = K_n - (\bigcup_{i=1}^{n-1} B_{ai} \cup \bigcup_{i=1}^{n-1} C_{ai}) \) if \( K_n \subseteq P_1 \); let \( C_n = B_{an} \), if \( K_n \not\subseteq P_1 \). Clearly a), c), and d) hold. By (a) assumption, in order to show property b), it suffices to show that if \( K_n \subseteq P_1 \), \( K_n - (\bigcup_{i=1}^{n-1} B_{ai} \cup \bigcup_{i=1}^{n-1} C_{ai}) \) \( \neq 0 \). This may be proved by showing that, for \( i = 1, \ldots, n-1 \), \( k(C_{ai}) < c \), for i that in that case, using Assertion 3.1, \( k(\bigcup_{i=1}^{n-1} B_{ai} \cup \bigcup_{i=1}^{n-1} C_{ai}) < c = k(K_n) \). From d) we know that \( k(C_{ai} - B_{ai}) \leq k(C_{ai}) \) for all positive integers \( i \) and all \( a \) in \( A \). Let \( C = \bigcup_{i=1}^{n-1} C_{ai} \). Let \( C = (C; i = 1, 2, \ldots) \). C is our collection for this example.

B. The properties of example I. (1) \( C^* \) is a bounded subset of the plane.

(2) For all \( i \), \( C_i \cap T \neq 0 \) and \( C_i \cap B \neq 0 \). We have noticed that any \( \eta \)-path intersects \( T \) in a set containing a non-degenerate continuum; that is, there is an \( a \in A \) such that \( K_n \subseteq C_i \cap T \subseteq P_i \). Therefore, \( T \cap C_i \supseteq K_n \cap C_i \). Similarly, \( C_i \cap B \neq 0 \). (3) For all \( i \), \( d(C_i) > 1 \) by (2). (4) \( C \) has property (II), \( T \) is such an arc.

(5) For all \( i \), \( C_i \subseteq P_i \), since \( C_i \subseteq P_i \) for all \( i \).

(6) If \( i \neq j \), \( C_i \cap C_j \) is disjoint. Suppose \( x \in C_i \cap C_j \). Let \( a \) be the least \( \xi \) such that \( x \in C_{ai} \); let \( \beta \) be the least \( \xi \) such that \( x \in C_{aj} \). We may assume without loss of generality that \( a < \beta \). If \( a < \beta \), then \( x \in B_{ai} \cup B_{aj} \), so \( x \in C_{al} \) which is not the case by our choice of \( \beta \). Thus, \( a = \beta \). We may now assume without loss of generality that \( i < j \); for that \( j > i \), and \( x \in C_{ai} \) unless \( x \in B_{ai} \) which we have seen is not true. Therefore, \( i = j \).

(7) \( C \) is connected for all \( i \). If \( C_i \) is not connected for some \( i \), \( C_i = A \cup B \) where \( A \) and \( B \) are mutually separated in \( P_i \). There is, therefore, a closed (and hence compact) set \( F \) separating \( A \) and \( B \) in \( P_i \). If \( F \) contains no continuum, \( F \) would be 0-dimensional ([11], p. 22) and could not separate in the two-dimensional Cantor manifold \( P_i \). Therefore, \( F \) contains a continuum \( K \); that is, \( K \subseteq F \cap P \subseteq C_i \). Thus, \( K \subseteq C_i \) or \( K \subseteq C_i \). But, \( F \cap C_i \cap C_i \) or \( K \cap C_i \). Since \( C_i \neq 0 \), a contradiction.

(8) \( C \) does not have property (III). If \( C \) did have property (III), there would be a finite collection \( Z_1, \ldots, Z_k \) of mutually disjoint closed subsets of \( S^2 \) such that \( \delta(Z_i) < 1/4 \) for all \( i \) and for all \( j \), \( C_j \cap Z_i \neq 0 \). Because of the small diameter involved, no \( Z_i \) intersects both \( B \) and \( T \), nor does any \( Z_i \) intersect both \( B \) and \( R \). By Theorem 2.2, there is a closed connected set \( K \) separating \( B \) and \( T \) which is disjoint from the set \( \bigcup_{i=1}^{n} Z_i \) as \( A \) is normal, there is an open set \( U \) containing \( K \) and not intersecting \( L \cup R \cup \bigcup_{i=1}^{n} Z_i \). Since \( K \) separates \( L \) and \( R \), there is a point \( a \in K \cap T \) and a point \( y \in K \cap \bigcup_{i=1}^{n} Z_i \). By Theorem 2.5, there is a closed connected set \( G \) contained in \( U \) which is topologically a closed interval from \( a \) to \( y \) and which can be linearly ordered from \( a \) to \( y \). Since \( G \subseteq U \), \( G \cap \bigcup_{i=1}^{n} Z_i = 0 \). Let \( \eta = \rho(G, \bigcup_{i=1}^{n} Z_i) \); \( \eta > 0 \) and \( B(\theta, \eta) \cap \bigcup_{i=1}^{n} Z_i = 0 \). Let \( \eta \) be such that \( 1/2^\eta \) is less than \( \eta/3 \). Let \( d = \bigcup_{\rho \in \eta} Q \subset B(\eta, \eta) \). We will prove that there is an \( \eta \)-path \( \rho \) such that \( Q \subseteq \rho \).

We call a finite sequence \((Q_1, \ldots, Q_\ell)\) of elements of \( Q \) admissible provided

\[(a) \quad Q_1 \cap Q_{\ell+1} \text{ intersect in an edge for } i = 1, \ldots, \ell-1,
(b) \quad a \in Q_i,
(c) \quad Q_i \neq Q_j \text{ if } i \neq j.
\]

An admissible sequence \((Q_1, \ldots, Q_\ell)\) covers a point \( x \in T \) provided

\[x \in \bigcup_{i=1}^{\ell} Q_i.
\]

Assertion 3.3. If \( x \in \rho \) and there is an admissible sequence covering \( x \), then \( \rho \) is in \( B(x, 1/2^\eta) \), there is an admissible sequence covering both \( x \) and \( y \).

Proof. Let \((Q_1, \ldots, Q_\ell)\) be an admissible sequence covering \( x \). If this or any other admissible sequence covering \( x \) covers \( y \), the assertion is proved, so we assume that this sequence does not cover \( y \). Let \( \theta \) be the least \( i \) such that \( x \in Q_1 \) then \((Q_1, \ldots, Q_\ell)\) is an admissible sequence covering \( x \). Let \( Q_1, \ldots, Q_\ell \) be the elements of the \( \eta \)-tiling which are distinct from \( Q \), but which intersect \( Q \). Then \( y \in B(x, 1/2^\eta) \subseteq Q_{\ell} \subseteq \bigcup_{i=1}^{\ell} Q_i \subseteq B(x, \eta) \subseteq B(\theta, \eta) \). Let \( \rho' \) be a \( \eta \)-path such that \( y \in \bigcup_{i=1}^{\ell} Q_i \). If \( \rho' \) and \( Q \) have an edge in common, \((Q_1, \ldots, Q_\ell, Q')\) is an admissible sequence covering \( x \) and \( y \). Suppose that \( Q \) and \( Q' \) have only a corner point in common. Then there are two elements of the \( \eta \)-tiling such that each has an edge in common with each of \( Q \) and \( Q' \). If one of these two is not among \( Q_1, \ldots, Q_\ell, \) let \( Q' \) be that one; then \((Q_1, \ldots, Q_\ell, Q', Q'')\) is an admissible sequence covering \( x \) and \( y \). If both are among \( Q_1, \ldots, Q_\ell \), let \( Q' \) be the one of the two with the lowest subscript and let \( Q'' \) be the other; then \((Q_1, \ldots, Q', Q, Q'', \rho')\) is an admissible sequence covering \( x \) and \( y \).\]
4. Example II. The construction of the example. Example II is constructed by choosing a subset $D$ of our collection $C$ of example I. If $n$ is a positive integer, let $T_{\alpha}$ be the set of all $(x, y)$ in $P$ such that $y = 1/2^i$ and $(i + 1)/2^{i-1} \leq x = (i + 3)/2^{i+1}$ for $i$ an integer, $0 \leq i < 2^n$. For all $n$ and $i$, $T_{\alpha}$ is a closed set such that $\delta(T_{\alpha}) = 1/2^{n+1}$. If $i \neq j$, $T_{\alpha} \cap T_{\beta} = \emptyset$. For any positive integer $m$, let $S_m$ be the set of all $n$-paths $P$ through $P$ such that for all positive integers $m$, $1 \leq m \leq n$, there is a non-degenerate continuum contained in $P \cap T_{\alpha}$ for some non-negative integer $i$, $S_m$ is finite for any $n$ and $S_m \subseteq S$, where $S$ is defined as in example I. Let $S' = \bigcup_{m=n}^{\infty} S_m$. Then $S'$ is countable, and $S' = \{P_n, P_{n+1}, \ldots\}$ where $P_n \in S$. Let $D_i = C_n$ for every positive integer $i$; let $D_i = C_n$ for every positive integer $i$. Let $D = (D_i : i \geq 1)$.

The properties of example II. (1) $D$ is a bounded subset of the plane.
(2) $D_i \cap T = \emptyset$ and $D_i \cap B = \emptyset$ for all $i$.
(3) $\delta(D_i) \geq 1$ for all $i$.
(4) $D$ has property (II).
(5) $D_i \subseteq R_i = P_n$ for all $i$.
(6) If $i \neq j$, $D_i$ and $D_j$ are disjoint.
(7) $D_i$ is connected for all $i$.

The above statements are corollaries of the corresponding statements for example I.
(8) $D$ has property (III). Let $\zeta > 0$ be given. Let $n$ be a positive integer such that $1/2^{n+1} < \zeta$. Then, for all $i$, $T_{\alpha} \subseteq \emptyset$.

Assertion 4.1. For all but a finite number of positive integers $i$, there is an integer $i$, $0 \leq i < 2^n - 1$, such that $T_{\alpha} \cap D_i \neq \emptyset$.

Proof: If $x \in S_m$ where $m > n$, then there is an $i$ such that $x \in S_m$, and $x$ is a point such that $P \cap T = \emptyset$ contains a non-degenerate continuum; that is, $P \cap T = \emptyset$ for some $\beta \in \beta$. Now let $P = P_n$ for some $n$, so as $x \in S_n$, by Assertion 3.2 we have $D_i \cap T_{\alpha} = C_n$, $T_{\alpha} \supseteq P_n \cap T_{\alpha} = \emptyset$. This completes the proof, for we have noted that, for all $m$, $S_m$ is finite, hence, $\bigcup_{m=1}^{\infty} S_m$ is finite.

We now make up a finite collection of disjoint closed sets whose diameters are less than $\zeta$ and such that $D_i$ intersects each $\beta$ their union. Each $T_{\alpha}$ is an element of this collection and for each $\beta$ such that $D_i \cap \bigcup_{m=1}^{\infty} S_m = \emptyset$ the set consisting of one point from $T \cap D_j$ is in this collection. Clearly, this collection satisfies the conditions of property (III).
(9) $D$ does not have property (I). Let $Z \subseteq P$ be compact, $\dim Z = 0$.

4. Example III. The construction of the example. Example III is constructed by choosing a subset $D$ of our collection $C$ of example I. If $n$ is a positive integer, let $T_{\alpha}$ be the set of all $(x, y)$ in $P$ such that $y = 1/2^i$ and $(i + 1)/2^{i-1} \leq x = (i + 3)/2^{i+1}$ for $i$ an integer, $0 \leq i < 2^n$. For all $n$ and $i$, $T_{\alpha}$ is a closed set such that $\delta(T_{\alpha}) = 1/2^{n+1}$. If $i \neq j$, $T_{\alpha} \cap T_{\beta} = \emptyset$. For any positive integer $m$, let $S_m$ be the set of all $n$-paths $P$ through $P$ such that for all positive integers $m$, $1 \leq m \leq n$, there is a non-degenerate continuum contained in $P \cap T_{\alpha}$ for some non-negative integer $i$, $S_m$ is finite for any $n$ and $S_m \subseteq S$, where $S$ is defined as in example I. Let $S' = \bigcup_{m=n}^{\infty} S_m$. Then $S'$ is countable, and $S' = \{P_n, P_{n+1}, \ldots\}$ where $P_n \in S$. Let $D_i = C_n$ for every positive integer $i$; let $D_i = C_n$ for every positive integer $i$. Let $D = (D_i : i \geq 1)$.

The properties of example II. (1) $D$ is a bounded subset of the plane.
(2) $D_i \cap T = \emptyset$ and $D_i \cap B = \emptyset$ for all $i$.
(3) $\delta(D_i) \geq 1$ for all $i$.
(4) $D$ has property (II).
(5) $D_i \subseteq R_i = P_n$ for all $i$.
(6) If $i \neq j$, $D_i$ and $D_j$ are disjoint.
(7) $D_i$ is connected for all $i$.

The above statements are corollaries of the corresponding statements for example I.
(8) $D$ has property (III). Let $\zeta > 0$ be given. Let $n$ be a positive integer such that $1/2^{n+1} < \zeta$. Then, for all $i$, $T_{\alpha} \subseteq \emptyset$.
Assertion 4.2. There is a positive integer \( n \) and a non-negative integer \( j \) such that \( S_{n+1} = \{(x, y) \in \mathbb{Z}^2 : (4j+1)|x|^{2^n+1} < x < (4j+3)|x|^{2^n+1} \text{ and } 0 < y < 1/2^n \} \) is disjoint from \( Z \).

Proof. The contrary would imply \( Z \cap R \neq \emptyset \), contradicting \( \dim Z = 0 \).

We now construct some \( P \subset \mathbb{R}^2 \) missing \( Z \). Let \( S_{n+1} \) be disjoint from \( Z \). Then \( T_{n+1} \subset S_{n+1} \cap Z = 0 \). Let \( T \) be \( T_1 \). For every \( i, i = 1, \ldots, n-1 \), let \( T_{i+1} \) be some \( T_i \). Let \( T_n \) be \( T_{n+1} \). For \( i = 0, \ldots, n \), let \( a_i \) be the left hand end point of \( T_i \) and let \( b_i \) be the right hand end point of \( T_i \). Let \( L' = \bigcup_{i=0}^{n+1} L(a_i, b_i) \); let \( L'' = \bigcup_{i=0}^{n+1} L(b_i, a_i) \). Then \( L' \cup L'' \subset \mathbb{R}^2 \) is a simple closed curve. Let \( J \) be that curve together with the bounded domain of \( \mathbb{R}^2 \) enclosed by it.

Now \( J \) may be homeomorphically mapped onto \( \mathbb{R} \) in such a way that \( L' \) goes onto \( L', L'' \) goes onto \( T \), \( T \) goes onto itself, and \( T_k \) goes onto \( T_k \). By Theorem 2.1, there is a closed connected set \( K' \) separating \( L' \) from \( K' \) and not intersecting \( Z \cap T \). Since \( Z \cap T \) is disjoint closed sets, there is an open set \( U \) containing \( K' \) and not intersecting \( Z \cap L'' \). Since \( K' \) separates \( L' \) and \( L'' \), \( \exists \) \( y_k \cap T_k \). By Theorem 2.3, there is a closed connected set \( K \) contained in \( U \) which is topologically a closed interval from \( x_k \) to \( y_k \) and can be linearly ordered from \( x_k \) to \( y_k \). Let \( \eta = \eta(K, Z \cup L' \cup L'') \subset 1/2^{n+1} \). Then \( B(\eta, Z \cap T_k) = 0 \). Let \( m \) be an integer such that \( 1/2^{m+1} < \eta/3 \). Then \( \eta > 2 \).

We will show that there is an element of \( S_{n+1} \) contained in \( B(K, \eta) \cup S_{n+1} \). Let \( Q = (Q \subset \mathbb{R}^2, Q \subset B(K, \eta) \cup S_{n+1}) \). If we call a finite sequence \( (Q_1, \ldots, Q_n) \) of elements of \( \mathbb{R}^2 \) admissible provided it satisfies conditions (a), (b), and (c) preceding Assertion 3.3 and if we say that such a sequence covers a point provided that point is in the union of the elements of the sequence, then we may prove by the same method used to prove Assertion 3.4 that

Assertion 4.3. There is an admissible sequence \( (Q_1, \ldots, Q_n) \) covering \( y_k \).

Assertion 4.4. If \( i \) is an integer greater than \( n + 1 \), there is an integer \( k \) such that \( T_{n+1} \subset S_{n+1} \).

Proof. We can show that \( T_{n+k} \cap S_{n+1} \) is in \( S_{n+1} \) if \( k \) is an integer no smaller than \( 2 \). To do this, we need only show that

\[
\frac{4p+1}{2^n+1} < \frac{4(2^{n+1}+1)+1}{2^{n+3}+1} < \frac{4(2^{n+1}+1)+3}{2^{n+4}+1} < \frac{4p+3}{2^{n+2}},
\]

that \( 2^{n+2}+1 < 2^{n+8}-1 \), and that \( 1/2^{n+8} < 1/2^n \). We see that these inequalities hold by an easy computation.

For each integer \( i \) such that \( n+2 \leq i \leq m \), let \( x_i \) be the midpoint of some \( T_{n+1} \) which is contained in \( S_{n+1} \). Let \( x_n = x_{n+1} \) be

\[
(4p+1)|x|^{2^n+1} - 1/2^{n+1}, 1/2^n \]

which is in \( T_{n+1} \cap S_{n+1} \). Let \( x_{n+1} \) be that point of \( B \), which has the same first coordinate as \( x_n \).

Let \( (Q_1, \ldots, Q_n) \) be a sequence such as is guaranteed by Assertion 4.3. We construct a finite sequence of elements of \( \mathbb{R}^2 \), \( (Q_1, \ldots, Q_n) \), such that

(a) \( Q_i \subset S_{n+1} \) for \( 1 \leq i \leq t \);

(b) \( Q_i \cap S_{n+1} \) is an edge for \( 1 \leq i \leq t-1 \); \( Q_t \) and \( Q_t \cap S_{n+1} \) intersect in an edge,

(c) \( x_n \in Q_t \), \( x_{n+1} \in Q_t \), and \( x_i \in Q_t \) for \( n+1 \leq i \leq m+1 \),

(d) \( Q_i \neq Q_j \) if \( i \neq j \).

Because of our choice of the \( x_i \)'s, each \( x_i \) is on the bottom edge of at least one element of \( S_n \) and on the top edge of at least one element of \( S_n \) if we picture \( J \) in the usual way, with the point \( (0, 0) \) in the lower left corner and the point \( (1, 1) \) in the upper right corner. Since \( m \geq n + 2 \), any element of \( S_n \) which contains an interior point of \( S_{n+1} \) is contained in \( S_{n+1} \). Let \( Q_i \) be that element of \( S_{n+1} \) which is directly below \( Q_t \) and which has an edge in common with \( Q_t \). Clearly, \( x_n \in Q_t \) for \( x_n \in S_{n+1} \) which includes the common edge of \( Q_t \) and \( Q_{n+1} \). We easily see that we can construct an \( L \)-shaped path of elements of \( S_{n+1} \) from \( x_n \) to \( x_{n+1} \), from \( x_{n+1} \) to \( x_m \), and so on, ending with one from \( x_m \) to \( x_{m+1} \) such that each path joins the preceding one in such a way that all of the requirements (a)-(d) are satisfied. For example, to construct a path of the described type from \( x_n \) to \( x_{n+1} \), assuming that one exists to \( x_{n+1} \) with \( Q_m(\epsilon, 2^{\eta+1}) \) the last element of the path, we first choose \( Q_m(\epsilon, 2^{\eta+1}-1) \). We choose successively \( Q_m(\epsilon+1, 2^{\eta+1}-1) \), \( Q_m(\epsilon+2, 2^{\eta+1}-1) \), and so on, using the plus sign if the first coordinate of \( x_m \) is larger than that of \( x_n \), the minus sign if it is not, until we choose the first \( Q_m(\epsilon, 2^{\eta+1}-1) \) such that \( x_{n+1} \in Q_m(\epsilon, 2^{\eta+1}-1) \); we then choose \( Q_m(\epsilon, 2^{\eta+1}-2), \ldots, Q_m(\epsilon, 2^{\eta+1}-1) \), completing the desired path.

From the method of constructing the sequences involved, we note that the sequence \( (Q_1, \ldots, Q_m) \) satisfies the requirements in the definition of an \( m \)-path so that \( Q = \bigcup_{i=0}^m Q_i \) is an \( m \)-path. We now need only prove that \( Q \) is in \( S_n \). However, we see that for \( 1 \leq i \leq n \), \( Q \cap T_i \) contains a non-degenerate continuum as \( T_i \cap Q \) separates \( T \cap \mathbb{R}^2 \) and \( B \cap Q \) in the Cantor manifold \( Q \). On the other hand, because of the size, shape, and orientation of the sets involved, as \( x_i \in T_i \) for \( 1 \leq i \leq m \) and as \( x_i \in Q_t \), \( T_i \cap Q \) contains a non-degenerate continuum. Thus, \( Q \) is in \( S_n \) and \( Q \subset B(K, \eta) \cap S_{n+1} \).

Therefore we have an element \( R \) of \( J \) which contains no point of \( Z \), for \( Q \) is such an element. But \( D_i \cap Z_i \subset R_i \cap Z \subset Z \cap S_{n+1} \) \( \cap Z = 0 \), so (9) is proved.
Conclusions. We summarize the results of the preceding section of this part by the following theorem:

**Theorem 4.4.** There is a countable collection $D$ of mutually disjoint connected subset of $3$ which has properties (II) and (III) but which does not have property (I).

This theorem provides a negative answer to problem 2. We point out that the particular collection constructed also is a suitable set for rejecting problem 1. Also, since $D$ is a subcollection of $G$, there is an effective method for constructing $D$.

**References**


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**Most knots are wild**

by

J. Milnor (Princeton)

Let $\text{Emb}(X, R^n)$ denote the topological space consisting of all embeddings of a compact space $X$ into the $n$-dimensional euclidean space $R^n$. This is a Baire space (1). We will say that most embeddings of $X$ in $R^n$ have some given property $P$ if the set of all $f \in \text{Emb}(X, R^n)$ which satisfies this property $P$ contains a dense $G_\delta$.

**Theorem 1.** Most embeddings of the circle in euclidean 3-space are wildly knotted.

**Theorem 2.** For $n \geq 4$ most embeddings of the circle in $R^n$ are unknotted.

(Note however that knotted embeddings do exist for all $n \geq 3$. See Blankinship [2].)

**Proof of Theorem 2.** We will show that $\text{Emb}(S^1, R^n)$ contains a subset $\text{Emb}(S^1, R^{n-1}) \times F(S^1, R^1)$ which is a dense $G_\delta$, and consists entirely of unknotted embeddings.

Let $F(S^1, R^n)$ denote the Banach space consisting of all mappings from $S^1$ to $R^n$. We will identify $F(S^1, R^n)$ with the product $F(S^1, R^{n-1}) \times \times F(S^1, R^1)$. Since $n-1 \geq 3$, the subset $\text{Emb}(S^1, R^{n-1}) \subset F(S^1, R^{n-1})$ is a dense $G_\delta$. (Hurewicz-Wallman [6], p. 56.) Therefore $\text{Emb}(S^1, R^{n-1}) \times \times F(S^1, R^1)$ is a dense $G_\delta$ in $F(S^1, R^n)$, and hence a fortiori it is a dense $G_\delta$ in $\text{Emb}(S^1, R^n)$.

But an argument due to Bing and Klee shows that every $(f, g) \in \text{Emb}(S^1, R^{n-1}) \times F(S^1, R^1) \subset \text{Emb}(S^1, R^n)$

can be transformed into the standard embedding by an isotopy of $R^n$.

First consider an isotopy of the form

$$h_t(x, y) = (x, y + f_t(x)).$$

(1) See Lemma 2. $E$ is a Baire space if every countable intersection of dense open subsets is dense. A subset $S \subset E$ is called a $G_\delta$ if $S$ can be expressed as a countable intersection of open subsets.