est divergente pour chaque $\varphi \in E$ et qu'elle est convergente pour chaque $\varphi \notin E$. Il nous reste à démontrer que la suite $\{\tilde{s}_k(\varphi, F)\}$ est convergente pour tout $\varphi \notin E$.

En effet, si $\varphi \notin E$, alors $\varphi \notin E_n$ pour tout n. La suite $\{\tilde{s}_k(\varphi, F_n)\}$ est convergente pour chaque $\varphi \notin E_n$ pour chaque n et $\lim\limits_{k\to\infty} \tilde{s}_k(\varphi, F_n) = \tilde{F}_n(\varphi)$. En passant dans (37) à la limite avec k, ce qui est permis en vertu de la convergence uniforme de la série

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tilde{s}_k(\varphi, F_n),$$

nous obtiendrons

$$\lim_{k\to\infty} \tilde{s}_k(\varphi, F) = \sum_{n=1}^{\infty} \frac{1}{2^n} \tilde{F}_n(\varphi) = \tilde{F}(\varphi)$$

ce qui termine la démonstration.

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Concerning some problems raised by A. Lelek

by

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1. Introduction. A. Lelek raises in [3] a series of questions about fixations in Euclidean n-dimensional space, $\mathcal{E}^n$. We may ask when each of the following properties hold for a collection $\mathbf{C}$ of subsets of $\mathcal{E}^n$.

(I) *There exists a 0-dimensional compact set $Z \subseteq \mathcal{E}^n$ such that $Z \cap C \neq 0$ for every $C \in \mathbf{C}$.*

(II) *There is an arc $A \subseteq \mathcal{E}^n$ such that $A \cap C \neq 0$ for every $C \in \mathbf{C}$.*

(III) *There exists, for every $\zeta > 0$, a finite sequence $Z_1, \ldots, Z_k$ of closed and mutually disjoint subsets of $\mathcal{E}^n$ such that $\delta(Z_i) < \zeta$ for $i = 1, \ldots, k$ and $(\bigcup_{i=1}^{k} Z_i) \cap C \neq 0$ for all $C \in \mathbf{C}$.*

Let $\mathbf{C}^*$ be the union of all sets belonging to $\mathbf{C}$. Denote by $A(\mathbf{C})$ the set of all points $p \in \mathcal{E}^n$ such that there is a sequence $C_1, C_2, \ldots$ of elements of $\mathbf{C}$ such that $\{p\} = \operatorname{Lim} C_i$ (see [2] for the definition of Lim). Lelek asks the following questions:

Problem 1. *Is it true that if $\mathbf{C}^*$ is a bounded subset of the plane and there exists an $\varepsilon > 0$ such that $\mathbf{C}$ is a disjoint collection of connected sets of diameter greater than ε, then* (I) *holds?*

Problem 2. *Is it true that if $\mathbf{C}^*$ is a bounded subset of the plane and $\mathbf{C}$ is a disjoint collection of connected sets then* (III) *implies* (I)?

Problem 3. *Is it true that if $\mathbf{C}^*$ is a subset of the plane, $\mathbf{C}$ is a disjoint collection of connected sets and $\dim A(\mathbf{C}) \leqslant 0$, then* (II) *implies* (III)?

In this paper, we give negative answers to these questions by constructing two counter-examples. The collections $\mathbf{C}$ and $\mathbf{D}$ defined are collections of subsets of the unit square, $\mathfrak{J}^2$.

2. Preliminary definitions and results. We first define some subsets of $\mathfrak{J}^2$ which will be used in both constructions. Let ϱ be the usual metric on $\mathfrak{J}^2$; let $B(x, r)$ $\big(\overline{B}(x, r)\big)$ be the open (closed) ball of radius r

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about x in $\mathfrak{J}^2$. Let $L = \{(x, y) \in \mathfrak{J}^2\colon\ x = 0\}$, $R = \{(x, y) \in \mathfrak{J}^2\colon\ x = 1\}$, $B = \{(x, y) \in \mathfrak{J}^2\colon\ y = 0\}$, $T = \{(x, y) \in \mathfrak{J}^2\colon\ y = 1\}$.

DEFINITION. For a positive integer n and for integers i and j such that $0 \leqslant i \leqslant 2^n - 1$ and $0 \leqslant j \leqslant 2^n - 1$, let $Q_n(i, j) = \{(x, y) \in \mathfrak{J}^2\colon\ i2^{-n} \leqslant x \leqslant (i+1)2^{-n},\ j2^{-n} \leqslant y \leqslant (j+1)2^{-n}\}$. Then $\mathfrak{Q}_n = \{Q_n(i, j)\colon\ 0 \leqslant i \leqslant 2^n - 1;\ 0 \leqslant j \leqslant 2^n - 1\}$ is called an *n-tiling* of $\mathfrak{J}^2$.

DEFINITION. If $(Q_1, \ldots, Q_k)$ is a finite sequence of elements of $\mathfrak{Q}_n$ such that

1. Q_i and Q_{i+1} intersect in an edge for $1 \leqslant i \leqslant k-1$,
2. $Q_1 \cap T \neq 0$,
3. $Q_k \cap B \neq 0$,
4. $Q_i \neq Q_j$ if $i \neq j$,

then $Q = \bigcup_{i=1}^{k} Q_i$ is an *n-path through* $\mathfrak{J}^2$.

Notice that for any n-path Q through $\mathfrak{J}^2$, both $Q \cap T$ and $Q \cap B$ contain a non-degenerate continuum.

If X is a set we define $k(X)$ to be the cardinal number of X.

Theorems 2.1 and 2.2 below are well-known, so no proofs will be given.

THEOREM 2.1. *If Z is a zero-dimensional subset of $\mathfrak{J}^2$, then there exists a continuum K separating L and R, $K \cap Z = 0$.*

THEOREM 2.2. *Let $Z_1, \ldots, Z_k$ be disjoint closed subsets of $\mathfrak{J}^2$ such that, for all i, Z_i does not intersect both L and R, nor does it intersect both B and T. Then there exists a continuum K separating L and R which is disjoint from $\bigcup_{i=1}^{k} Z_i$.*

Let $\mathfrak{K}$ be the set of all non-degenerate continua contained in $\mathfrak{J}^2$. Since $k(\mathfrak{K}) = c = k(\mathfrak{J})$, $\mathfrak{K}$ may be indexed by an ordinal number A (whose cardinality, therefore, is c) such that $k(\{a \in A\colon\ a < \beta\}) < c$ for every $\beta \in A$: $\mathfrak{K} = \{K_a\colon\ a \in A\}$. If $K \in \mathfrak{K}$, $k(K) = c$.

DEFINITION. If x and y are two points in $\mathcal{E}^2$, let $L(x, y)$ be the straight line segment with x and y as end points.

THEOREM 2.3. *If M is a connected point set in $\mathcal{E}^2$, U is an open set containing M, and x and y are points of M, then there are points $x_1, \ldots, x_n$ in M such that $L_i = L(x_i, x_{i+1})$ is contained in U for $0 \leqslant i \leqslant n$ where $x_0 = x$ and $x_{n+1} = y$ and such that if $0 \leqslant i < j \leqslant n$, $L_i \cap L_j \neq 0$ only if $j = i+1$. Moreover, $G = \bigcup_{i=1}^{n} L_i$ is a continuum contained in U and is topologically a closed interval which can be linearly ordered from x to y.*

Proof. This follows from [4], Theorem 77, p. 56, and some plane geometry.

3. Example I. A. The construction of example I. Let $\mathfrak{P}$ be the set of all P such that for some n, P is an n-path through $\mathfrak{J}^2$. $\mathfrak{P}$ is a countable set, so we write $\mathfrak{P} = \{P_1, P_2, \ldots\}$. For each positive integer i and each a in A, we define C_{ia} so that

a) $C_{ia} \subseteq P_i$,

b) If $\beta \leqslant a$ and $K_\beta \subseteq P_i$, then $K_\beta \cap C_{ia} \neq 0$,

c) If $0 \leqslant \beta \leqslant a$, $C_{i\beta} \subseteq C_{ia}$,

d) Let $B_{ia} = \bigcup_{\beta < a} C_{i\beta}$. Then $k(C_{ia} - B_{ia}) \leqslant 1$,

The definition is inductive.

For $a = 0$. Let $C_{10} = \{a_{10}\}$ where $a_{10} \in K_0$ if $K_0 \subset P_1$; let $C_{10} = 0$ if K_0 is not contained in P_1. Then C_{10} satisfies conditions a)-d). Assume that C_{k0} has been defined for all $k < n$ and satisfies conditions a)-d). Let $C_{n0} = \{a_{n0}\}$ where $a_{n0} \in K_0 - \bigcup_{k<n} C_{k0}$ if $K_0 \subset P_n$; let $C_{n0} = 0$ if K_0 is not contained in P_n. Conditions a), c), and d) are clearly satisfied for C_{n0}. Suppose that $K_0 \subseteq P_n$. We need only prove $K_0 - \bigcup_{k<n} C_{k0} \neq 0$ to satisfy b), but from d) and the induction hypothesis we see that $k(C_{k0}) \leqslant 1$ for $k < n$, so $k(K_0 - \bigcup_{k<n} C_{k0}) = c \neq 0$.

(*) *Assume that $a \in A$, that $a > 0$, and that for all β, $0 \leqslant \beta < a$, and for all positive integers k, $C_{k\beta}$ has been defined and satisfies conditions* a)-d).

ASSERTION 3.1. $k(B_{ia}) \leqslant k(\{\beta \in A\colon\ \beta < a\}) < c$ *for all positive integers i.*

Proof. For all $x \in B_{ia}$ there is a $\lambda < a$ such that $x \in C_{i\lambda}$. Let $\beta(x)$ be the least such λ in the well-ordered set $\{\beta \in A\colon\ \beta < a\}$. Define g_i: $B_{ia} \to \{a \in A\colon\ \beta < a\}$ by $g_i(x) = \beta(x)$. Suppose $g_i(x) = g_i(y) = \beta$. Then $x, y \in C_{i\beta}$ but neither x nor y is in $B_{i\beta}$. As $k(C_{i\beta} - B_{i\beta}) \leqslant 1$, $x = y$. Thus, g_i is a one-one mapping, and the first inequality holds. The second inequality is true by our choice of A.

ASSERTION 3.2. *If $\beta < a$ and $K_\beta \subseteq P_i$, then $K_\beta \cap B_{ia} \neq 0$.*

Proof. Since $K_\beta \subseteq P_i$, $K_\beta \cap B_{ia} = K_\beta \cap \bigcup_{\lambda<a} C_{i\lambda} \supseteq K_\beta \cap C_{i\beta} \neq 0$ by b).

Now let $C_{1a} = \{a_{1a}\} \cup B_{1a}$ where $a_{1a} \in K_a - \bigcup_{i=1}^{\infty} B_{ia}$ if $K_a \subseteq P_1$; let $C_{1a} = B_{1a}$ if $K_a \nsubseteq P_1$. Then $C_{1a} \subseteq P_1$, for $B_{1a} = \bigcup_{\beta<a} C_{1\beta} \subseteq P_1$ and if $K_a \subseteq P_1$, a_{1a} is in P_1. That properties c) and d) hold is clear. Since $B_{1a} \cap K_\beta \neq 0$ for $\beta < a$ if $K_\beta \subseteq P_1$ by Assertion 3.2, we need only show that $K_a \cap C_{1a} \neq 0$ if $K_a \subseteq P_1$. It suffices to show that there is an a_{1a} in $K_a - \bigcup_{i=1}^{\infty} B_{ia}$. But $k(B_{ia}) < c$ by Assertion 3.1, so $k(K_a - \bigcup_{i=1}^{\infty} B_{ia}) > 0$; that is, the set is not empty. Now assume that C_{ka} has been defined for $k > n$ and satisfies a)-d).

Let $C_{n\alpha} = \{a_{n\alpha}\} \cup B_{n\alpha}$ where $a_{n\alpha} \in K_\alpha - (\bigcup_{i=1}^{\infty} B_{i\alpha} \cup \bigcup_{i=1}^{n-1} C_{i\alpha})$ if $K_\alpha \subseteq P_n$; let $C_{n\alpha} = B_{n\alpha}$ if $K_\alpha \nsubseteq P_n$. Clearly a), c), and d) hold. By (*) assumption, in order to show property b), it suffices to show that if $K_\alpha \subseteq P_n$, $K_\alpha - (\bigcup_{i=1}^{\infty} B_{i\alpha} \cup \bigcup_{i=1}^{n-1} C_{i\alpha}) \neq 0$. This may be proved by showing that, for $i = 1, \ldots, n-1$, $k(C_{i\alpha}) < c$, for in that case, using Assertion 3.1, $k(\bigcup_{i=1}^{\infty} B_{i\alpha} \cup \bigcup_{i=1}^{n-1} C_{i\alpha}) < c = k(K_\alpha)$. From d) we know that $k(C_{i\alpha} - B_{i\alpha}) \leqslant 1$ for $i < n$ and from c) we know that $B_{i\alpha} \subseteq C_{i\alpha}$ for $i < n$. Thus, $k(C_{i\alpha}) = k\big(B_{i\alpha} \cup (C_{i\alpha} - B_{i\alpha})\big) \leqslant k(B_{i\alpha}) + k(C_{i\alpha} - B_{i\alpha}) < c + 1 = c$.

We have now defined $C_{i\alpha}$ for all positive integers i and all α in A. Let $C_i = \bigcup_{\alpha \in A} C_{i\alpha}$. Let $\boldsymbol{C} = \{C_i\colon i = 1, 2, \ldots\}$. $\boldsymbol{C}$ is our collection for this example.

B. The properties of example I. (1) *$\boldsymbol{C}^*$ is a bounded subset of the plane.*

(2) *For all i, $C_i \cap T \neq 0$ and $C_i \cap B \neq 0$.* We have noticed that any n-path intersects T in a set containing a non-degenerate continuum; that is, there is an $\alpha \in A$ such that $K_\alpha \subseteq P_i \cap T \subseteq P_i$. Therefore, $T \cap C_i \supseteq K_\alpha \cap C_i \supseteq K_\alpha \cap C_{i\alpha} \neq 0$ by property b) of $C_{i\alpha}$. Similarly, $C_i \cap B \neq 0$.

(3) *For all i, $\delta(C_i) \geqslant 1$* by (2).

(4) *$\boldsymbol{C}$ has property* (II), for T is such an arc.

(5) *For all i, $C_i \subseteq P_i$*, since $C_{i\alpha} \subseteq P_i$ for all α.

(6) *If $i \neq j$, C_i and C_j are disjoint.* Suppose $x \in C_i \cap C_j$. Let α be the least $\lambda \in A$ such that $x \in C_{i\lambda}$; let β be the least $\lambda \in A$ such that $x \in C_{j\lambda}$. We may assume without loss of generality that $\alpha \leqslant \beta$. If $\alpha < \beta$, then $\beta > 0$, and $x \in B_{i\beta}$, so $x \notin C_{j\beta}$ unless $x \in B_{j\beta} = \bigcup_{\lambda<\beta} C_{j\lambda}$ which is not the case by our choice of β. Thus, $\alpha = \beta$. We may now assume without loss of generality that $i \leqslant j$, for that if $j > i$, then $j > 1$, and $x \notin C_{j\beta}$ unless $x \in B_{j\beta}$ which we have seen is not true. Therefore, $i = j$.

(7) *C_i is connected for all i.* If C_i is not connected for some i, $C_i = A \cup E$ where A and E are mutually separated in P_i. There is, therefore, a closed (and hence compact) set F separating A and E in P_i. If F contains no continuum, F would be 0-dimensional ([1], p. 22) and could not separate in the two-dimensional Cantor manifold P_i ([1], p. 93). Therefore, F contains a continuum K; that is, $K \subseteq F \subseteq P_i \subseteq \mathfrak{J}^2$. Thus, $K \in \mathcal{K}$, or for some $\alpha \in A$, $K = K_\alpha$. But, $F \cap C_i \supseteq K_\alpha \cap C_i \supseteq K_\alpha \cap C_{i\alpha} \neq 0$, a contradiction.

(8) *C does not have property* (III). If $\boldsymbol{C}$ did have property (III), there would be a finite collection $Z_1, \ldots, Z_k$ of mutually disjoint closed subsets of $\mathfrak{J}^2$ such that $\delta(Z_i) < 1/4$ for all i and for all j, $C_j \cap \bigcup_{i=1}^{k} Z_i \neq 0$. Because of the small diameter involved, no Z_i intersects both B and T, nor does any Z_i intersect both L and R. By Theorem 2.2, there is a closed connected set K separating L and R which is disjoint from the closed set $\bigcup_{i=1}^{k} Z_i$. As $\mathfrak{J}^2$ is normal, there is an open set U containing K and not intersecting $L \cup R \cup \bigcup_{i=1}^{k} Z_i$. Since K separates L and R, there is a point x_0 in $K \cap T$ and a point y_0 in $K \cap B$. By Theorem 2.3, there is a closed connected set G contained in U which is topologically a closed interval from x_0 to y_0 and which can be linearly ordered from x_0 to y_0. Since $G \subseteq U$, $G \cap \bigcup_{i=1}^{k} Z_i = 0$. Let $\eta = \varrho(G, \bigcup_{i=1}^{k} Z_i)$; $\eta > 0$ and $B(G, \eta) \cap \bigcup_{i=1}^{k} Z_i = 0$. Let n be such that $1/2^n$ is less than $\eta/3\sqrt{2}$. Let $\mathfrak{Q} = \{Q \in \mathfrak{Q}_n\colon Q \subset B(G, \eta)\}$. We will prove that there is an n-path Q such that $Q \subseteq \cup \mathfrak{Q}$.

We call a finite sequence $(Q_1, \ldots, Q_t)$ of elements of $\mathfrak{Q}$ *admissible* provided

(a) Q_i and Q_{i+1} intersect in an edge for $i = 1, \ldots, t-1$,

(b) $x_0 \in Q_1$,

(c) $Q_i \neq Q_j$ if $i \neq j$.

An admissible sequence $(Q_1, \ldots, Q_t)$ *covers* a point $x \in \mathfrak{J}^2$ provided $x \in \bigcup_{i=1}^{t} Q_i$.

Assertion 3.3. *If $x \in G$ and there is an admissible sequence covering x, then if y is in $B(x, 1/2^{n+1})$, there is an admissible sequence covering both x and y.*

Proof. Let $(Q_1, \ldots, Q_t)$ be an admissible sequence covering x. If this or any other admissible sequence covering x covers y, the assertion is proved, so we assume that this sequence does not cover y. Let r be the least i such that $x \in Q_i$; then $(Q_1, \ldots, Q_r)$ is an admissible sequence covering x. Let $Q_1', \ldots, Q_8'$ be the elements of the n-tiling which are distinct from Q_r but which intersect Q_r. Then $y \in B(x, 1/2^{n+1}) \subseteq Q_r \cup \bigcup_{i=1}^{8} Q_i' \subseteq B(x, \eta) \subseteq B(G, \eta)$; as $y \notin Q_r$, $y \in \bigcup_{i=1}^{8} Q_i'$. Let Q^* be a Q_i' such that $y \in Q_i'$. If Q^* and Q_r have an edge in common, $(Q_1, \ldots, Q_r, Q^*)$ is an admissible sequence covering x and y. Suppose that Q_r and Q^* have only a corner point in common. Then there are two elements of the n-tiling such that each has an edge in common with each of Q_r and Q^*. If one of these two is not among $Q_1, \ldots, Q_r$, let Q' be that one; then $(Q_1, \ldots, Q_r, Q', Q^*)$ is an admissible sequence covering x and y. If both are among $Q_1, \ldots, Q_r$, let Q' be the one of the two with the lowest subscript and let Q'' be the other; then $(Q_1, \ldots, Q', Q_r, Q'', Q^*)$ is an admissible sequence covering x and y.

ASSERTION 3.4. *There is an admissible sequence* $(Q_1, \ldots, Q_k)$ *covering* y_0.

Proof. Suppose not. We notice immediately that there is an admissible sequence covering x_0. Then, if we consider G as ordered from x_0 to y_0, there is a last point $x \in G$ which can be covered by an admissible sequence, for the intersection of G and the union of all elements of all admissible sequences is a closed subset of G. Suppose x is not y_0. Clearly, there is a point $y \in G \cap B(x, 1/2^{n+1})$ which follows x. By Assertion 3.3, y may be covered by an admissible sequence, contradicting the choice of x.

ASSERTION 3.5. *There is an n-path* Q *such that* $Q \subseteq \cup \mathfrak{Q}$.

Proof. Let $(Q_1, \ldots, Q_k)$ be an admissible sequence covering y_0. Then $Q = \bigcup_{i=1}^{k} Q_i$ is an n-path through $\mathfrak{J}^2$ which is contained in $\cup \mathfrak{Q}$.

If Q is an n-path through $\mathfrak{J}^2$ such that $Q \subseteq \cup \mathfrak{Q} \subseteq B(G, \eta)$, then $Q \cap \bigcup_{i=1}^{k} Z_i = 0$ and there is a j such that $Q = P_j$. But $C_j \subseteq P_j$, so $C_j \cap \bigcup_{i=1}^{k} Z_i = 0$, so the proof of (8) is complete.

(9) *C does not have property* (I) by [3], Theorem 2.

(10) $\dim A(\mathbf{C}) = -1$. As the diameter of every C in $\mathbf{C}$ is large, if $\operatorname{Lim} C_i$ exists for any sequence of elements of $\mathbf{C}$, it contains more than one point. Thus $A(\mathbf{C}) = 0$.

C. Conclusions. We may summarize the results of section B of this part by the following theorem:

THEOREM 3.6. *There is a countable collection* $\mathbf{C}$ *of mutually disjoint connected subsets of* $\mathfrak{J}^2$ *such that properties* (I) *and* (III) *do not hold for* $\mathbf{C}$ *while property* (II) *does hold for* $\mathbf{C}$, *such that* $\dim A(\mathbf{C}) \leqslant 0$, *and such that* $\delta(C) > \frac{1}{2}$ *for all* C *in* $\mathbf{C}$.

This theorem provides an answer to questions 1 and 3.

We point out that a countable collection $\mathbf{C}$ having the desired properties may be effectively constructed, although we have not done so. The construction rests on the fact that there are connected sets in $\mathfrak{J}^2$ which intersect both T and B but which do not separate L and R. In each n-path, $n = 1, 2, \ldots$, we construct such a connected set C whose closure D is a continuum having the following properties:

(a) $D = \bigcup \mathfrak{D}$ where $\mathfrak{D}$ is a collection of arcs, only a finite number of which have diameter greater than ε for any $\varepsilon > 0$.

(b) $\mathfrak{D}$ is an arc with respect to its members.

(c) C is composed of one element of each member of $\mathfrak{D}$.

The details of this construction are extremely involved. Since a complete construction would contain many pages of tedious arguments and definitions, we have ommitted the construction.

4. Example II. The construction of the example. Example II is constructed by choosing a subset $\mathbf{D}$ of our collection $\mathbf{C}$ of example I. If n is a positive integer, let T_{ni} be the set of all (x, y) in $\mathfrak{J}^2$ such that $y = 1/2^n$ and $(4i+1)/2^{n+2} \leqslant x \leqslant (4i+3)/2^{n+2}$ for i an integer, $0 \leqslant i \leqslant 2^n-1$. For all n and i, T_{ni} is a closed set such that $\delta(T_{ni}) = 1/2^{n+1}$. If $i \neq j$, $T_{ni} \cap T_{nj} = 0$. For any positive integer n, let $\mathfrak{T}_n$ be the set of all n-paths P through $\mathfrak{J}^2$ such that for all positive integers m, $1 \leqslant m \leqslant n$, there is a non-degenerate continuum contained in $P \cap T_{mi}$ for some non-negative integer i. $\mathfrak{T}_n$ is finite for any n and $\mathfrak{T}_n \subseteq \mathfrak{T}$ where $\mathfrak{T}$ is defined as in example I. Let $\mathfrak{T}' = \bigcup_{n=2}^{\infty} \mathfrak{T}_n \subseteq \mathfrak{T}$. Then $\mathfrak{T}'$ is countable, and $\mathfrak{T}' = \{P_{k_1}, P_{k_2}, \ldots\}$ where $P_{k_i} \in \mathfrak{T}$. Let $R_i = P_{k_i}$ for every positive integer i; let $D_i = C_{k_i}$ for every positive integer i. Let $\mathbf{D} = \{D_i\colon i \geqslant 1\}$.

The properties of example II. (1) *$\mathbf{D}^*$ is a bounded subset of the plane.*

(2) *$D_i \cap T \neq 0$ and $D_i \cap B \neq 0$ for all i.*

(3) *$\delta(D_i) \geqslant 1$ for, all i.*

(4) *$\mathbf{D}$ has property* (II).

(5) *$D_i \subseteq R_i = P_{k_i}$ for all i.*

(6) *If $i \neq j$, D_i and D_j are disjoint.*

(7) *D_i is connected for all i.*

The above statements are corollaries of the corresponding statements for example I.

(8) *D has property* (III). Let $\zeta > 0$ be given. Let n be a positive integer such that $1/2^n < \zeta$. Then, for all i, T_{ni} is a closed set such that $\delta(T_{ni}) < \zeta$.

ASSERTION 4.1. *For all but a finite number of positive integers j, there is an integer i, $0 \leqslant i \leqslant 2^n-1$, such that $T_{ni} \cap D_j \neq 0$.*

Proof. If $P \in \mathfrak{T}_m$ where $m \geqslant n$, then there is an i such that $P \cap T_{ni}$ contains a non-degenerate continuum; that is, $P \cap T_{ni} \supseteq K_\beta$ for some $\beta \in A$. Now $P = P_{k_j}$ for some j, so as $K_\beta \subseteq P_{k_j}$, by Assertion 3.2 we have $D_j \cap T_{ni} = C_{k_j} \cap T_{ni} \supseteq B_{k_j\beta} \cap T_{ni} \supseteq B_{k_j\beta} \cap K_\beta \neq 0$. This completes the proof, for we have noted that, for all m, $\mathfrak{T}_m$ is finite, hence, $\bigcup_{m<n} \mathfrak{T}_m$ is finite.

We now make up a finite collection of disjoint closed sets whose diameters are less than ζ and such that D_j intersects for all j their union. Each T_{ni} is an element of this collection and for each j such that $D_j \cap \bigcup_{i=1}^{2^n-1} T_{ni} = 0$ the set consisting of one point from $T \cap D_j$ is in this collection. Clearly, this collection satisfies the conditions of property (III).

(9) *D does not have property* (I). Let $Z \subseteq \mathfrak{J}^2$ be compact, $\dim Z = 0$.

Assertion 4.2. *There is a positive integer n and a non-negative integer j such that $S_{nj} = \{(x, y) \in \mathfrak{I}^2: (4j+1)/2^{n+2} \leqslant x \leqslant (4j+3)/2^{n+2}$ and $0 \leqslant y \leqslant 1/2^n\}$ is disjoint from Z.*

Proof. The contrary would imply $Z \supseteq B$, contradicting $\dim Z = 0$.

We now construct some $P \in \mathfrak{S}'$ missing Z. Let S_{np} be disjoint from Z. Then $T_{np} \cap Z \subseteq S_{np} \cap Z = 0$. Let T be T_0. For every i, $i = 1, \ldots, n-1$, let T_i be some T_{ik}. Let T_n be T_{np}. For $i = 0, \ldots, n$, let a_i be the left hand end point of T_i and let b_i be the right hand end point of T_i. Let $L' = \bigcup_{i=1}^{n-1} L(a_i, a_{i+1})$; let $R' = \bigcup_{i=1}^{n+1} L(b_i, b_{i+1})$. Then $L' \cup T \cup R' \cup T_n$ is a simple closed curve. Let J be that curve together with the bounded domain of $\mathcal{E}^2$ enclosed by it.

Now J may be homeomorphically mapped onto $\mathfrak{I}^2$ in such a way that L' goes onto L, R' goes onto R, T goes onto itself, and T_n goes onto B. By Theorem 2.1, there is a closed connected set K' separating L' from R' and not intersecting $Z \cap J$. Since Z and K' are disjoint closed sets, there is an open set U containing K' and not intersecting $Z \cup L' \cup R'$. Since K' separates L' and R', there is a point x_0 in $K' \cap T$ and a point y_0 in $K' \cap T_n$. By Theorem 2.3, therefore, there is a closed connected set K contained in U which is topologically a closed interval from x_0 to y_0 and which can be linearly ordered from x_0 to y_0. Let $\eta = \varrho(K, Z \cup L' \cup R') \leqslant 1/2^{n+1}$. Then $B(K, \eta) \cap Z = 0$. Let m be an integer such that $1/2^m < \eta/3\sqrt{2}$. Then $m \geqslant n+2$.

We will show that there is an element of $\mathfrak{S}_m$ contained in $B(K, \eta) \cup S_{np}$. Let $\mathfrak{Q}' = \{Q \in \mathfrak{S}_m: Q \subseteq B(K, \eta) \cap J\}$. If we call a finite sequence $(Q_1, \ldots, Q_t)$ of elements of $\mathfrak{Q}'$ admissible provided it satisfies conditions (a), (b), and (c) preceeding Assertion 3.3 and if we say that such a sequence covers a point provided that point is in the union of the elements of the sequence, then we may prove by the same method used to prove Assertion 3.4 that

Assertion 4.3. *There is an admissible sequence $(Q_1, \ldots, Q_t)$ covering y_0.*

Assertion 4.4. *If i is an integer greater than $n+1$, there is an integer k such that $T_{ik} \subset S_{np}$.*

Proof. We can show that $T_{n+k, 2^{k-1}(2p+1)}$ is in S_{np} if k is an integer no smaller than 2. To do this, we need only show that

$$\frac{4p+1}{2^{n+2}} \leqslant \frac{4\big(2^{k-1}(2p+1)\big)+1}{2^{n+k+2}} \leqslant \frac{4\big(2^{k-1}(2p+1)\big)+3}{2^{n+k+2}} \leqslant \frac{4p+3}{2^{n+2}},$$

that $2^{k-1}(2p+1) \leqslant 2^{n+k}-1$, and that $1/2^{n+k} \leqslant 1/2^n$. We see that these inequalities hold by an easy computation.

For each integer i such that $n+2 \leqslant i \leqslant m$, let x_i be the midpoint of some T_{ik_i} which is contained in S_{np}. Let $x_n = y_0$; let x_{n+1} be $\big((4p+1)/2^{n+2} + 1/2^{n+4}, 1/2^{n+1}\big)$ which is in $T_{n+1,2p} \cap S_{np}$. Let x_{m+1} be that point of B which has the same first coordinate as x_m.

Let $(Q_1, \ldots, Q_k)$ be a sequence such as is guaranteed by Assertion 4.3. We construct a finite sequence of elements of $\mathfrak{S}_m$, $(Q_1', \ldots, Q_t')$ such that

(a) $Q_i' \subseteq S_{np}$ for $1 \leqslant i \leqslant t$,

(b) Q_i' and Q_{i+1}' intersect in an edge for $1 \leqslant i \leqslant t-1$; Q_1' and Q_k intersect in an edge,

(c) $x_n \in Q_1'$, $x_{m+1} \in Q_t'$, and $x_i \in \bigcup_{j=1}^{t} Q_j'$ for $n \leqslant i \leqslant m+1$,

(d) $Q_i' \neq Q_j'$ if $i \neq j$.

Because of our choice of the x_i's, each x_i is on the bottom edge of at least one element of $\mathfrak{S}_m$ and on the top edge of at least one element of $\mathfrak{S}_m$ if we picture $\mathfrak{I}^2$ in the usual way, with the point $(0, 0)$ in the lower left corner and the point $(1, 1)$ in the upper right corner. Since $m \geqslant n+2$, any element of $\mathfrak{S}_m$ which contains an interior point of S_{np} is contained in S_{np}. Let Q_1' be that element of $\mathfrak{S}_m$ which is directly below Q_k and which has an edge in common with Q_k. Clearly, $x_n \in Q_1'$ for $x_n \in T_n$ which includes the common edge of Q_k and Q_1'. We easily see that we can construct an L-shaped path of elements of $\mathfrak{S}_m$ from x_n to x_{n+1}, from x_{n+1} to x_{n+2}, and so on, ending with one from x_m to x_{m+1} and such that each path joins the preceeding one in such a way that all of the requirements (a)-(d) are satisfied. For example, to construct a path of the described type from x_r to x_{r+1}, assuming that one exists to x_r with $Q_m(s, 2^{m-r})$ the last element of the path, we first choose $Q_m(s, 2^{m-r}-1)$. We choose successively $Q_m(s\pm 1, 2^{m-r}-1)$, $Q_m(s\pm 2, 2^{m-r}-1)$, and so on, using the plus sign if the first coordinate of x_{r+1} is larger than that of x_r, the minus sign if it is not, until we choose the first $Q_m(v, 2^{m-1}-1)$ such that $x_{r+1} \in Q_m(v, 2^{m-r-1}-1)$; we then choose $Q_m(v, 2^{m-r}-2), \ldots, Q_m(v, 2^{m-r-1}-1)$, completing the desired path.

From the method of constructing the sequences involved, we note that the sequence $(Q_1, \ldots, Q_k, Q_1', \ldots, Q_t')$ satisfies the requirements in the definition of an m-path so that $Q = \bigcup_{i=1}^{k} Q_i \cup \bigcup_{i=1}^{t} Q_i'$ is an m-path. We now need only prove that Q is in $\mathfrak{S}_m$. However, we see that for $1 \leqslant i \leqslant n$, $Q \cap T_i$ contains a non-degenerate continuum as $T_i \cap Q$ separates $T \cap Q$ and $B \cap Q$ in the Cantor manifold Q. On the other hand, because of the size, shape, and orientation of the sets involved, as $x_i \in T_{ik_i}$ for $n \leqslant i \leqslant m$ and as $x_i \in Q$, $T_{ik_i} \cap Q$ contains a non-degenerate continuum. Thus, Q is in $\mathfrak{S}_m$ and $Q \subseteq B(K, \eta) \cap S_{np}$.

Therefore we have an element R_i of $\mathfrak{S}'$ which contains no point of Z, for Q is such an element. But $D_i \cap Z \subseteq R_i \cap Z = Q \cap Z \subseteq \big(B(K, \eta) \cap S_{np}\big) \cap Z = 0$, so (9) is proved.

Conclusions. We summarize the results of the preceeding section of this part by the following theorem:

THEOREM 4.4. *There is a countable collection D of mutually disjoint connected subset of* $\mathfrak{I}^2$ *which has properties* (II) *and* (III) *but which does not have property* (I).

This theorem provides a negative answer to problem 2. We point out that the particular collection constructed also is a suitable set for rejecting problem 1. Also, since $\boldsymbol{D}$ is a subcollection of $\boldsymbol{C}$, there is an effective method for constructing $\boldsymbol{D}$.

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Most knots are wild

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Let $\operatorname{Emb}(X, R^n)$ denote the topological space consisting of all embeddings of a compact space X into the n-dimensional euclidean space R^n. This is a Baire space [1]. We will say that *most* embeddings of X in R^n have some given property P if the set of all $f \in \operatorname{Emb}(X, R^n)$ which satisfies this property P contains a dense G_δ.

THEOREM 1. *Most embeddings of the circle in euclidean* 3*-space are wildly knotted.*

THEOREM 2. *For $n \geqslant 4$ most embeddings of the circle in R^n are unknotted.*

(Note however that knotted embeddings do exist for all $n \geqslant 3$. See Blankinship [2].)

Proof of Theorem 2. We will show that $\operatorname{Emb}(S^1, R^n)$ contains a subset $\operatorname{Emb}(S^1, R^{n-1}) \times F(S^1, R)$ which is a dense G_δ, and consists entirely of unknotted embeddings.

Let $F(S^1, R^n)$ denote the Banach space consisting of all mappings from S^1 to R^n. We will identify $F(S^1, R^n)$ with the product $F(S^1, R^{n-1}) \times F(S^1, R)$. Since $n-1 \geqslant 3$, the subset $\operatorname{Emb}(S^1, R^{n-1}) \subset F(S^1, R^{n-1})$ is a dense G_δ. (Hurewicz-Wallman [6], p. 56.) Therefore $\operatorname{Emb}(S^1, R^{n-1}) \times F(S^1, R)$ is a dense G_δ in $F(S^1, R^n)$, and hence a fortiori it is a dense G_δ in $\operatorname{Emb}(S^1, R^n)$.

But an argument due to Bing and Klee shows that every

$$(f, g) \in \operatorname{Emb}(S^1, R^{n-1}) \times F(S^1, R) \subset \operatorname{Emb}(S^1, R^n)$$

can be transformed into the standard embedding by an isotopy of R^n. First consider an isotopy of the form

$$h_t(x, y) = \big(x, y + tp(x)\big),$$

[1] See Lemma 2. E is a *Baire space* if every countable intersection of dense open subsets is dense. A subset $S \subset E$ is called a G_δ if S can be expressed as a countable intersection of open subsets.