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The completeness of logic with the added quantifier "there are uncountably many"

by

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The language L_1 is obtained from an ordinary first order language L, having countably many symbols, by adding a new quantifier Q, to be read "there are uncountably many ... such that ...". Mostowski [10] (1) raised the completeness problem for L_1, i.e., the question whether the set V_1 of (semantically) logically valid sentences of L_1 is recursively enumerable.

The compactness of the language L_1 has been established in Theorem 3.4 of the preceding article by G. Fuhrken. The purpose of this note is to point out that the two results and the argument employed by Fuhrken to prove compactness also lead at once to a positive answer to Mostowski's question.

One of these two results is (2.2), Fuhrken's first normal form theorem for L_1. When modified as in the second (but not the first) part of the Remark following its proof, this theorem gives us the following information: Let L' be the first order language obtained from L by adding two new unary predicates U and W_1 and one new ternary predicate. We can define a recursive function correlating with each sentence σ of L_1 a sentence σ' of L' in such a way that:

(A) *σ has a model if and only if σ' has a model $\mathfrak{A}$ of power $\aleph_1$ in which $U^{(\mathfrak{A})}$ has power at most $\aleph_0$.*

The second result, (1.7), is easily derived from the proof of the author's 'Löwenheim-Skolem theorem for two cardinals' (Theorem 6.2 of [9]). (However, this fact for the case when T is incomplete and its significance were only observed recently, by Fuhrken.) Let L'' be the first order language obtained from L' by adding one new unary predicate W_2. (1.7) describes a certain recursive set Σ of sentences, such that, for any sentence δ of L':

(B) *δ has a model $\mathfrak{A}$ of power $\aleph_1$ in which $U^{(\mathfrak{A})}$ has power at most $\aleph_0$, if and only if $\Sigma \cup \{\delta\}$ is consistent.*

(1) The terminology of the preceding paper by G. Fuhrken will be used; numbers refer to its theorems, numbers in brackets to its bibliography.

Now, if we are given any sentence σ of L_1, then, by (A) and (B), σ is logically valid if and only if $\{(\sim\sigma)'\} \cup \Sigma$ is inconsistent, i.e. if and only if $\sim(\sim\sigma)'$ can be derived formally from Σ. This establishes the completeness of L_1:

THEOREM. *V_1 is recursively enumerable.*

From the Theorem and the Compactness Theorem (3.4) for L_1 follows at once the

COROLLARY. *If S is any recursively enumerable set of L_1-sentences, then the set of (semantical) logical consequences of S is recursively enumerable.*

It should be noted that the Theorem given us effectively a procedure for enumerating a certain set of sentences. However, our knowledge that this set coincides with V_1 is of course based on set theory, since V_1 can only be defined in set theory, and indeed on the Axiom of Choice, which is used is the proof of (B).

In conclusion, we shall describe somewhat roughly still another consequence of (A) and (B). Suppose that the definition of the set V_1 has been formalized in the set theory of Gödel's monograph on the Continuum Hypothesis. By replacing throughout this definition the notion of arbitrary set by that of constructible set (in the sense of Gödel's monograph) we obtain the definition of a second set V_1^c. In other words V_1^c is the V_1 of a man who considers only constructible sets. Now, suppose we obtain the definition of the set $V_1'^c$ analogously from the following definition of the set V_1':

$$V_1' = \{\sigma / \sim(\sim\sigma)' \text{ is derivable from } \Sigma\}.$$

Since the definition of V_1' involves only elementary number theory, which is unchanged by the passage to constructible sets, we can conclude (as has been remarked in general by Kreisel) that $V_1'^c = V_1'$. We argued above from (A) and (B) that $V_1 = V_1'$. Carrying out the same argument within the universe of constructible sets we conclude that $V_1^c = V_1'^c$. Hence the

THEOREM. $V_1 = V_1^c$.

Thus, if we can show on the basis of ordinary set theory by assuming the Generalized Continuum Hypothesis that a particular L_1-sentence is logically valid, then we can also do so without that assumption. The author was in fact led to the Theorems above in part by an attempt to eliminate the G. C. H. from the proof of a certain recent result of C. C. Chang (a generalization of Beth's theorem on definability). The method just outlined does indeed work for one part of Chang's result. However, it would take too long to describe here the details.

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Sur l'ensemble des points de divergence des séries entières continues sur la circonférence du cercle de converegnce

par

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J. Śladkowska [3] a démontré que pour un ensemble arbitraire $E \subset \Phi$, où E est de classe $G_{\delta\sigma}$ et Φ est un ensemble de classe F_σ de mesure logarithmique zéro, il existe une fonction continue et périodique dont la série de Fourier est divergente sur l'ensemble E et convergente sur son ensemble complémentaire.

Me basant sur ce résultat, je construis dans ce travail une série entière continue ayant des propriétés semblables.

Je vois d'abord rappeler certaines définitions et notations:

1. Un ensemble E de nombres réels a pour mesure logarithmique *zéro*, si pour un nombre arbitraire $\varepsilon > 0$ il existe une suite dénombrable de segments ouverts recouvrant l'ensemble E, de longueurs l_j, $l_j < 1$ $(j = 1, 2, \ldots)$ telle que

$$\sum_{j=1}^{\infty} \frac{1}{\log(1/l_j)} < \varepsilon.$$

2. Nous appelons l'ensemble E de nombres réels *périodique de période a*, si sa fonction caractéristique est périodique de période a.

3. Nous désignerons par $s_k(\varphi, f)$ [resp. $\tilde{s}_k(\varphi, f)$] la $k^{\text{ième}}$ somme partielle de la série de Fourier (resp. de la série conjuguée de Fourier) de la fonction $f(\varphi)$ au point φ.

4. Par $S_k(\varphi, H)$, en abrégé $S_k(\varphi)$, nous désignerons la $k^{\text{ième}}$ somme partielle de la série entière $\sum_{k=0}^{\infty} a_k z^k = H(z)$ (convergente pour $|z| < 1$) en $z = e^{i\varphi}$. Lorsque la fonction $H(z)$ peut être prolongée à une fonction continue dans $|z| \leqslant 1$ (et de valeurs finies), une telle série est dite *série entière continue*.

Dans la suite nous définissons pour $|\varphi| \leqslant \pi$ et $a < \pi/2$ la fonction

$$f(\varphi, a, m) = \begin{cases} (a - |\varphi|)\sin m|\varphi| & \text{pour} \quad |\varphi| \leqslant a, \\ 0 & \text{pour} \quad |\varphi| > a. \end{cases}$$