to show that $R^*$ is a regular ring. Given $(a, \phi) \in R^*$, determine $a \in M$ and $b \in K$ as in Lemma 3. Then we have

$$(a, \phi)(b, \sigma)(a, \phi) = (a, \phi),$$

and since $(b, \sigma) \in R^*$, $R^*$ is in fact regular. This completes the proof of Theorem 1 (4).

References


(*) As an alternative to (i) we could identify $(a, \phi)$ and $(a', \phi')$ in $R^*$ if and only if for all $x \in E$, $ax + \phi x = a'x + \phi' x$. Our theorem holds with this new $R^*$ (in general different from the previous $R^*$). The new $R^*$ can be identified with a subring of the ring of left endomorphisms of $B$ and, if $B$ has an identity, this new $R^*$ will coincide with $B$.

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Skolem-type normal forms for first-order languages with a generalized quantifier

by

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Let $L$ be a first-order language with countably many non-logical constants and let $\alpha$ be an ordinal number. With $L$ and $\alpha$ we associate a new language $L_\alpha$ which is obtained from $L$ by adding to the symbols of $L$ a new quantifier $Q$ which is read "there are at least $\alpha$". Let $V_\alpha$ be the set of all sentences of $L_\alpha$ which are logically valid (where $Q$ is counted among the logical symbols). Mostowski raised the question (*) whether $V_\alpha$ is axiomatizable. For $\alpha = 0$ he showed that the answer is negative (provided $L$ has a sufficient supply of non-logical constants). One can show (see [3] and [5]) that for the majority of ordinals Mostowski's proof cannot be adapted. In fact for $\alpha = 1$ the answer is positive as shown by Vaught (see [6]).

Another natural question to ask is the following: What is the relationship between the various $V_\alpha$'s? (Note that the formation rules for $L_\alpha$ are independent of the particular $\alpha$). Here we obtain as partial results:

1. $V_0 \subseteq V_\alpha$ for any ordinal $\alpha$ for which $\kappa_{\alpha}$ is regular;
2. $V_\alpha \subseteq V_\beta$ for any ordinal $\alpha$.

The following negative results are known (assuming that $L$ has a sufficient supply of non-logical constants):

3. $V_\alpha \nsubseteq V_{\alpha + 1}$ for any ordinal $\alpha > 0$;
4. $V_\alpha \nsubseteq V_\beta$ for any limit ordinal $\alpha$ and any successor ordinal $\beta$;
5. $V_\alpha \nsubseteq V_\beta$ for any ordinals $\alpha$ and $\beta$ for which $\kappa_\alpha$ is regular and $\kappa_\beta$ is singular;

* The paper is mainly based on § 2 of Part I of the author's Doctoral Dissertation [2]. The main results have been summarized in [6]. The author takes the opportunity to express his gratitude to his Thesis Adviser Professor Robert L. Vaught and to Professor William Craig for the stimulation and help received. It was in Professor William Craig's Seminar conducted at the University of California in Berkeley in 1961 that the author learned about the problems treated in this paper and obtained the first results in this direction.

(*) In [10]; see also [11].
(6) \( V_\alpha \nsubseteq V_\beta \) for any ordinal \( \alpha \) and \( \beta \) such that (i) \( m < \kappa \), implies \( 2^m < \kappa \), for all cardinals \( m \); while (ii) \( m < \kappa_\gamma \) and \( \kappa_\gamma < 2^m \), for some cardinal \( m \).

This leaves the following conjectures:

(A) \( V_\alpha = V_\beta \) whenever

(b) \( \alpha \) and \( \beta \) are both successor ordinals;

or

(c) \( \alpha \) and \( \beta \) are both singular; or

(d) \( \alpha \) and \( \beta \) are both weakly but not strongly inaccessible ordinals; or

(E) \( V_\alpha \subseteq V_\beta \) whenever

(a) \( \alpha \) is a successor cardinal or \( \kappa_\alpha \) is a singular; and

(b) \( \beta \) is a weakly inaccessible cardinal.

(1) and (2) are obtained from results of first-order model theory with the help of Skolem-type normal forms. By a Skolem-type normal form of a sentence we mean, roughly, a sentence which may contain additional predicates, in which the quantifiers occur only in certain special contexts, and which has the property that it is satisfiable if and only if the original sentence is satisfiable. The method may be used to establish other results, e.g. a compactness theorem for \( L_\alpha \) is obtained. For a further application, see the following note by Vaught (this volume, pp. 363-364).

1. Preliminaries. Let \( L \) have as non-logical constants countably many predicates; the list of (individual) variables is \( v_0, v_1, \ldots \); the (primitive) sentential connectives are negation and conjunction; the (primitive) quantifier is the existential quantifier; the identity predicate is the only logical predicate.

The symbols of \( L_\alpha \) are the symbols of \( L \) together with the new quantifier. The syntactical notions like being a formula, being a subformula, being a sentence of \( L_\alpha \) are defined in the usual way with the obvious modifications. In particular:

(1.1) For every formula \( \phi \) of \( L_\alpha \) and every variable \( v_\alpha \) there is a unique formula \( Q_{v_\alpha} \phi \) of \( L_\alpha \), \( Q_{v_\alpha} \phi \) in turn determines \( v_\alpha \) and \( \phi \), and can only obtained in this way.

(1.2) A variable occurs free in \( Q_{v_\alpha} \phi \) if, and only if, it occurs free in \( \phi \) and is distinct from \( v_\alpha \).

The notion of an assignment \( \sigma \) over a relational system \( \mathcal{H} \) (of appropriate similarity type) satisfying a formula of \( L_\alpha \) in \( \mathcal{H} \); and a sentence of \( L_\alpha \), being true in \( \mathcal{H} \) is defined in the usual way with the additional stipulation:

(1\#) For the model-theoretic notions see [12], [13], [14], and [9].

(1.3) \( a \) satisfies \( Q_{v_\alpha} \phi \) in \( \mathcal{H} \) if there are at least \( \kappa_\alpha \) elements \( x \in A \), for which \( a(x) \phi \) satisfies \( \phi \) in \( \mathcal{H} \), where \( a(x) \phi \) is the assignment over \( \mathcal{H} \) which differs from \( a \) at most in the \( n \)-th place and has the value \( x \) at this place.

Let \( I \) be a set of sentences of \( L_\alpha \). The following two properties hold:

(1.4) If \( I \) has a model of power \( \kappa_\gamma \), then \( I \) has a model of every power \( \kappa_\gamma \) for which \( \alpha < \gamma \).

(1.5) For every sentence \( \psi \) of \( L_\alpha \) there is a first-order sentence \( \chi \) of \( L_\alpha \) with the property that for every relational system \( \mathcal{H} \), \( \mathcal{H} \) is a model of \( \phi \wedge Q_{v_\alpha} \psi \) if and only if \( \mathcal{H} \) is a model of \( \chi \). Let \( I' \) be the set of all \( \chi \) for \( \phi \) in \( I \). \( I \) and \( I' \) have the same models of power less than \( \kappa_\alpha \). Note that \( \chi \) and \( I' \) do not depend on the particular \( \alpha \).

(1.4) is in essence to be found already in [10]. It is a downward Löwenheim-Skolem theorem and can be proved by adapting the proof of the corresponding first-order theorem given in [13], obtaining at the same time a stronger version of (1.4). (1.5) is a consequence of the observation that in a relational system of power less than \( \kappa_\alpha \), the \( \kappa_\alpha \)-quantifier acts trivially, i.e. no assignment satisfies any formula \( Q_{v_\alpha} \).

We shall make use of the following theorems of model theory about first-order theories \( T \) with countably many non-logical constants:

(1.6) Assume that among the non-logical constants of \( T \) there is a unary predicate \( v \); if \( T \) has an infinite model \( \mathcal{H} \) for which \( |\mathcal{H}| < |A| \), then \( T \) has an infinite model \( \mathcal{H} \) for which \( |\mathcal{H}| < |A| \).

This is Vaught's Löwenheim-Skolem theorem form two cardinals. For a proof of (1.6) see [9]: This theorem is stated there for complete \( T \) only, but this special case easily implies the general one.

(1.7) Assume \( T \) and \( U \) are as in (1.6). Let \( W \) be a new unary predicate and \( A \) the set of all sentences

\[ \wedge v_0 \ldots \wedge v_{\gamma-1} [Wv_\alpha \wedge \ldots \wedge Wv_{\gamma-1} \rightarrow [\varphi \leftrightarrow \varphi^{W'}]] \]

where \( \varphi \) is any formula of the language of \( T \) having no other free variables than \( v_0, \ldots, v_{\gamma-1} \), and \( \varphi^{W'} \) is obtained from \( \varphi \) by relativising quantifiers to \( W \). If \( T \vdash \chi \wedge \left[ \forall v_\alpha (Wv_\alpha \rightarrow Wv_\beta) \right] \wedge \left[ \exists v_\alpha (Wv_\alpha \rightarrow \sim Wv_\beta) \right] \) is consistent, then \( T \) has a model \( \mathcal{H} \) for which \( |\mathcal{H}| < |A| = \kappa_\alpha \), and conversely, if \( T \) has an infinite model \( \mathcal{H} \) for which \( |\mathcal{H}| < |A| \), then the above set of sentences is consistent.

This is a lemma underlying the proof of (1.6), though it is not stated explicitly in [9].

\[ \text{(1\#) } U^{\mathcal{H}} \text{ is the interpretation of } U \text{ in } \mathcal{H}; [X] \text{ is the cardinality of } X. \]

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(1.8) Assume that among the non-logical constants of T there is a binary predicate R, and that for every model \( \mathcal{M} \) of T, \( E^\mathcal{M} \) is a linear ordering of \( A \). Assume furthermore that \( \kappa_0 \) is regular. If T has a model of power \( \kappa_0 \), every proper initial R-segment of which is of smaller power, then T has a model of power \( \kappa_0 \) every proper R-segment of which is countable.

This generalizes (1.6). The proof is similar to the proof of (1.6) and runs, roughly, like follows:

First observe that T has a model \( \mathcal{M}_0 \) which has a proper elementary extension \( \mathcal{M}_0 \) such that 'all old elements precede all new elements'. (This corresponds in the proof of (1.6) to a model of T with a proper elementary extension 'having the same \( U^T \).') In fact, if \( a = 0 \) take for \( \mathcal{M}_0 \) the model given by hypothesis and for \( \mathcal{M}_0 \) any proper elementary extension of \( \mathcal{M}_0 \). (Note that \( E^\mathcal{M}_0 \) is of type \( \alpha_0 \).)

(1.10) Assume that T and R are as in (1.8). Let \( 0 \) be a (weakly) inaccessible ordinal (greater than \( \alpha_0 \)). If \( \mathcal{M} \) is a model of T for which \( E^\mathcal{M} \) is of type \( \beta_0 \), then \( \mathcal{M} \) has an elementary subsystem \( \mathcal{B} \) which has the property that (i) \( E^\mathcal{B} \) has a type which is an initial ordinal cofinal with \( \omega_1 \), and (ii) \( B \) is an initial R-segment of \( \mathcal{M} \).

This is a rudimentary form of a theorem in [1]. The proof is similar to the argument in the beginning of the proof of (1.8) where the case \( a > 0 \) is considered. The only difference is that one has to choose \( X_0, X_1, \ldots \) such that they are initial segments which correspond to initial ordinals and that they form a properly increasing chain.

2. The Skolem-type normal forms.\(^{(3)}\) The first Skolem-type normal form is designed for the study of \( L_\alpha \) when \( \alpha \) is a successor ordinal, say \( \alpha = \beta + 1 \). The idea behind the normal form is as follows:

Let \( \mathcal{M} \) be a model of power \( \kappa_0 \). Choose a subset \( U \) of A of power \( \kappa_0 \). Instead of saying 'the set \( S \) has power at least \( \kappa_0 \)' we shall say 'there is a biunique correspondence between a subset of \( S \) and \( A \)'. Instead of saying 'the set \( S \) has power less than \( \kappa_0 \)' we say 'there is a biunique correspondence between \( S \) and a subset of \( U \).'

Furthermore, the collection of correspondences is replaced by one ternary relation which 'parametrizes' these correspondences.

Definition (2.1). Let \( L_\alpha^* \) be a language obtained from \( L_\alpha \) by adding to the non-logical constants a new unary predicate \( U \) and a new ternary predicate \( F \).

With every set \( \Sigma \) of sentences of \( L_\alpha \) we associate a set \( \Sigma^* \) of first-order sentences of \( L_\alpha^* \). If \( \Sigma \) is finite, \( \Sigma^* \) will be finite.

First, with every formula \( \varphi \) of \( L_\alpha \) we associate (by recursion) a first-order formula \( \varphi^* \) of \( L_\alpha^* \) having the same free variables as \( \varphi \).

(i) If \( \varphi \) is quantifier-free, then \( \varphi^* = \varphi \).

(ii) If \( \varphi \) is of the form \( \psi \land \psi \), then \( \varphi^* = \psi \land \psi^* \).

(iii) If \( \varphi \) is of the form \( \psi \lor \psi \), then \( \varphi^* = \psi^* \lor \psi \).

(iv) If \( \varphi \) is of the form \( \psi \land \psi \), then \( \varphi^* = \psi \land \psi^* \).

For a further Skolem-type normal form, see [1].
(vii) For every subformula of \( \Sigma \) of the form \( Q_{\alpha} \psi \), the universal closure of
\[
\forall \delta_{\alpha} \exists \delta_{\alpha+1} \exists \delta_{\alpha}[\delta_{\alpha}^\ast \land F_{\delta_{\alpha}} \land \delta_{\alpha+1}^\ast] \forall \delta_{\alpha} \exists \delta_{\alpha+1} \exists \delta_{\alpha}[\delta_{\alpha} \rightarrow \delta_{\alpha+1} \land F_{\delta_{\alpha}} \land \delta_{\alpha+1}^\ast]
\]
where \( \psi \) is the first variable of all variables in \( \phi^\ast \).

Note that \( \Sigma^* \) does depend on the particular \( \alpha \), \( \Sigma^* \cup \{ Q_{\alpha} \psi = \psi, \neg Q_{\alpha} \psi \} \) may be called a Skolem-type normal form of \( \Sigma \).

**Theorem (2.2).** Assume that \( \alpha \) is a successor ordinal, say \( \alpha = \beta + 1 \). Let \( \Sigma \) be a set of sentences of \( L_\alpha \), and let \( \Sigma^* \cup \{ \neg Q_{\alpha} \psi, Q_{\alpha} \psi \} \) be the set of first-order sentences associated with \( \Sigma \) by (2.1). Under these assumptions, a relational system \( \mathfrak{A} \) of power \( \kappa_n \) is a model of \( \Sigma^* \cup \{ \neg Q_{\alpha} \psi, Q_{\alpha} \psi \} \) for some unary relation \( X \) and some ternary relation \( Y \) over \( A = (\mathfrak{A}, X, Y) \) is a model of \( \Sigma^* \cup \{ \neg Q_{\alpha} \psi \} \) if \( X \) is not in power \( \kappa_n \).

**Proof.** Assume first that \( \mathfrak{A} \) is a model of \( \Sigma \) of power \( \kappa_n \). Choose a substructure \( X \) of \( A \) of power \( \kappa_n \). Let \( A \) be the set of all subsets of \( \alpha \) for which there is a subformula \( \phi_{\alpha} \psi \) of \( \Sigma \) and an assignment \( \alpha \) over \( \mathfrak{A} \) such that \( S = \{ (\alpha \in A \mid a(n/e) \text{ satisfies } \phi \} \) in \( \mathfrak{A} \). \( A \) is at most of the power of \( A \).

Let \( J \) be a set of biquine references such that for every \( S \in \mathfrak{A} \) there is a \( j \in J \) with domain \( S \) and range \( A \), or there is a \( j \in J \) with domain \( S \) and range included in \( X \). We can choose \( J \) to be at most of the power of \( A \).

Let \( J \) be a biquine function from \( J \) into \( A \). Finally, set \( Y = \{ (\alpha(j), \alpha \mid j(x)) \mid j \in J \text{ and } \alpha \in \text{ dom } j \} \) and \( \mathfrak{B} = (\mathfrak{A}, X, Y) \). We shall show that \( \mathfrak{B} \) is a model of \( \Sigma^* \cup \{ \neg Q_{\alpha} \psi \} \).

**Remark.** Definition (2.1) and Theorem (2.2) can be modified in various ways.

First, instead of using the universe of \( \mathfrak{A} \) as the standard set of power \( \kappa_n \), we can choose an arbitrary subset of \( A \) of power \( \kappa_n \) for this purpose. This requires the addition of a new unary predicate \( \forall \alpha \) and a change in the definition of \( \phi^\ast \). In particular condition (7) of (2.1) has to be changed by replacing
\[
\forall \delta_{\alpha} \exists \delta_{\alpha+1} \exists \delta_{\alpha}[\phi_{\alpha}^\ast \land F_{\delta_{\alpha}} \land \delta_{\alpha+1}^\ast] \land F_{\delta_{\alpha}} \land \delta_{\alpha+1}^\ast
\]
and a similar change in (vii). With this changed notion of normal form (2.3) can be stated for relational systems of power at least \( \kappa_n \) for those of power \( \kappa_n \) only (4).

Secondly, one can drop the requirement that the sets \( U \) and \( V \) serving as standards for the cardinality of sets are chosen from the subsets of \( A \), thus admitting to enlarge the universe. This requires a further unary predicate \( \forall \delta_{\alpha} \psi \) denoting the old universe. It also requires a further change in the definition of \( \phi^\ast \). In particular condition (7) of (2.1) has to be changed by replacing \( \forall \delta_{\alpha} \psi \) by \( \forall \delta_{\alpha}[\exists W_{\alpha} \land \psi^\ast] \), and similar changes in (7) and (vii). With this changed notion of normal form (2.3) can be stated for arbitrary relational systems, \( \mathfrak{B} \) is, however, no longer obtained as a reduction but as a relativized reduction. This change is also necessary if one wants to extend the results to languages with uncountably many

(*) This form was mentioned in [4].
The second Skolem-type normal form is designed for the study of $L_0$ where $\alpha$ is any ordinal. The idea behind this normal form is as follows: Let $\Phi$ be a model of power $\kappa$. Choose a linear ordering of $A$ such that every proper initial segment has power less than $\kappa$. Instead of saying "the set $S$ has power at least $\kappa$", we shall say "there is a biunique correspondence between a subset of $S$ and $A$"; instead of saying "the set $S$ has power less than $\kappa$" we say "there is a biunique correspondence between $S$ and a proper initial segment of $\mathbb{N}$". The collection of correspondences is again replaced by a ternary relation.

**Definition (2.3).** Let $L^*_\Phi$ be the language obtained from $L_0$ by adding to the non-logical constants a binary predicate $R$ and a ternary predicate $S$.

With every set $\Sigma$ of sentences of $L_\Phi$ we associate a set $\Sigma^*$ of first-order sentences of $L^*_\Phi$. If $\Sigma$ is finite, $\Sigma^*$ will be finite.

First, with every formula $\varphi$ of $L_\Phi$ we associate a first-order formula $\varphi^*$ of $L^*_\Phi$ having the same free variables as $\varphi$.

(i) as in (2.1);

(ii) $\Sigma^*$ is the set of all $\varphi^*$ with $\varphi$ in $\Sigma$ together with the following sentences:

(iii) as in (2.1);

(iv) for every subformula $\varphi$ of the form $\psi_1 \land \psi_2$, the universal closure of

$$\forall v_1 [\forall v_2 (\forall v_3 (\forall v_4 (\forall v_5 (\forall v_6 (\forall v_7 (\forall v_8 (\forall v_9 (\forall v_{10}) ...) ...) ...) ...) ...) ...)]$$

where $v_0$ is the first variable after all variables in $\varphi^*$;

(v) a sentence saying that $R$ is a linear ordering of the universe.

Note that $\Sigma^*$ does not depend on the particular $\alpha$. $\Sigma^* \cup \{Q_\beta \forall v_\alpha \land \forall v_\beta \forall v_\gamma \}$ may be called a *Skolem-type normal form* of $\Sigma$.

**Theorem (2.4).** Let $\Sigma$ be a set of sentences of $L_\Phi$ and $\Sigma^*$ the set of first-order sentences associated with $\Sigma$ by (2.3). A relational system $\mathfrak{B}$ of power $\kappa$ is a model of $\Sigma$ if and only if for some binary relation $X$ and some ternary relation $Y$ over $A$, $\mathfrak{B}(X, Y)$ is a model of $\Sigma^* \cup \{Q_\beta \forall v_\alpha \land \forall v_\beta \forall v_\gamma \}$. $X$ may be even chosen to be a well-ordering.

The proof of (2.4) is similar to the proof of (2.2).

**Remark.** First, the remarks following the proof of (2.2) can be repeated with suitable modifications. Secondly, if $\kappa$ is regular, the definition of $\Sigma^*$ can be simplified by using the observation that a subset of the corresponding ordered set is of power less than $\kappa$ if and only if the subset is bounded.

### 3. Applications

**Theorem (3.1).** Assume that $\kappa$ is regular, and let $\Sigma$ be a set of sentences of $L_\Phi$. If $\Sigma$ has a model, then $\Sigma$ has a model when considered as a set of sentences of $L_\Phi$.

**Proof.** If $\Sigma$ has a model of power less than $\kappa$, then the assertion follows from (1.5) and the first-order L"owenheim-Skolem theorem. If $\Sigma$ has a model of power at least $\kappa$, then by (1.4) $\Sigma$ has a model of power $\kappa$. We may therefore assume that we have a model $\mathfrak{B}$ of $\Sigma$ of power $\kappa$.

Let $\Sigma^*$ be the set of first-order sentences associated with $\Sigma$ by (2.3). By (2.4) there are $X$ and $Y$ such that $(\mathfrak{B}, X, Y)$ is a model of $\Sigma^*$ and $X$ is a well-ordering of $A$ of type $\omega$. From (1.8) taking $\Sigma^*$ for $\Sigma$, we obtain a model $(\mathfrak{B}, X', Y')$ of $\Sigma^*$ of power $\kappa$ and where $X'$ is a linear ordering of $B$ every proper initial segment of which is countable. Applying (2.4) again we see that $\mathfrak{B}$ is a model of $\Sigma$ considered as a set of sentences of $L_\Phi$.

**Remark.** (3.1) can be regarded as a strengthened version of (1.6). Conversely, for the case $\alpha$ a successor ordinals (3.1) could have been obtained using (1.6) instead of (1.8).

**Theorem (3.2).** Let $\Sigma$ be a set of sentences of $L_\Phi$. If $\Sigma$ has a model, then $\Sigma$ has a model when considered as a set of sentences of $L_\Phi$, where $\alpha$ is any ordinal.

**Proof.** If $\Sigma$ has a model of power less than $\kappa$, we argue as in the beginning of the proof of (3.1). Otherwise, as there we shall assume that we have a model $\mathfrak{B}$ of $\Sigma$ of power $\kappa$.

Let $\Sigma^*$ be the set of first-order sentences associated with $\Sigma$ by (2.3). By (2.4) there are $X$ and $Y$ such that $(\mathfrak{B}, X, Y)$ is a model of $\Sigma^*$ and $X$ is a linear ordering of $A$ of type $\omega$. Let $\mathcal{T}$ be the first-order theory of $(\mathfrak{B}, X, Y)$. By (1.9) $\mathcal{T}$ has a model $(\mathfrak{B}, X', Y')$ of power $\kappa$ for which $X'$ is a linear ordering of $B$ every proper initial segment of which is of smaller power. $(\mathfrak{B}, X', Y')$ is also a particular model of $\Sigma^*$. Now considering $\Sigma^*$ as obtained from a set of sentences of $L_\Phi$ and applying again (2.4) we obtain the required model $\mathfrak{B}$ of $\Sigma$.

**Theorem (3.3).** Assume that $\alpha$ is a (weakly) inaccessible ordinal (greater than $\omega$). Let $\Sigma$ be a set of sentences of $L_\Phi$. If $\Sigma$ has a model, then there is an ordinal $\beta$, $\alpha < \beta$ and $\beta$ cofinal with $\alpha$, and $\Sigma$ has a model when considered as a set of sentence of $L_\Phi$.

The proof uses (2.3) and (2.4) as well as (1.10) and is similar to the preceding proofs.

**Theorem (3.4).** Let $\Sigma$ be a set of sentences of $L_\Phi$. If every finite subset of $\Sigma$ has a model, then $\Sigma$ has a model.

(*) This theorem as well as its corollary 3.5 is due to Dana Scott.
Proof. If some finite subset of $\Sigma$ has only countable models, then every finite subset of $\Sigma$ has a countable model and in this case the assertion follows with the help of (1.5) from the first-order compactness theorem together with the Löwenheim-Skolem theorem. We shall therefore assume that every finite subset of $\Sigma$ has an uncountable model and hence by (1.4) a model of power $\kappa$. Let $\Sigma'$ be the set of first-order sentences associated with $\Sigma$ by (2.1). We are going to apply (1.7) taking $\Sigma'$ for $T$. First, we observe that every finite subset of $T$ is contained in the $* \Gamma$ of some finite subset of $\Sigma$. Furthermore, every finite subset of $\Delta$ is contained in the corresponding set for some finite subset of $T$. Since by assumption every finite subset of $\Sigma$ has a model ($\mathbb{M}, X, \mathcal{Y}$) of power $\kappa$, by (2.2), every finite subset of $T$ has a model ($\mathbb{M}, X, \mathcal{Y}$) of power $\kappa$, with countable $\mathcal{Y}$, and hence by (1.7) every finite subset of $T \cup \Delta \cup \{\bigwedge \mathbb{M}[\mathcal{U} \rightarrow \mathcal{V}], \bigvee \mathbb{M}[\mathcal{U} \rightarrow \mathcal{V} \mathbb{M}] \}$ has a model. By the first-order compactness theorem the above set itself has a model, and hence by (1.7) $\Sigma'$ has a model ($\mathbb{M}, X, \mathcal{Y}$) of power $\kappa$ with countable $\mathcal{Y}$. By (2.2) $\mathbb{M}$ is a model of $\Sigma$.

Remark. (3.1) is a compactness theorem. Using ultraproducts one can show that $L_0$ is compact for every $\alpha$ for which $m_\alpha < \kappa$, for each $n \in \omega$, implies $\bigvee \mathbb{M}[m_\alpha < \kappa, n \in \omega]$. (*)

As corollaries of (3.1) and (3.2) we obtain immediately the assertions (1) and (3) of the introduction. (3.3) yields as a corollary.

(3.5) Assume that $\alpha$ is a (weakly) inaccessible ordinal (greater than $\omega$). There is an ordinal $\beta$, $\beta < \alpha$, and $\alpha$ cofinal with $\omega$, such that $V \subseteq V_\alpha$.

To prove (3.5) from (3.3) one has to apply (3.3) not to $L_\alpha$—the $\beta$ would depend on the sentence to be falsified—but to a language obtained from $L_\alpha$ by dublication of the non-logical constants so that the sentences which are not logically valid "can be written with disjoint predicates" and thus can be falsified in one model.

From (3.4) one obtains as a further corollary that the ordering of the natural numbers is not characteristic (up to isomorphism) in $L_\alpha$ as a relativized subset.

4. Some counterexamples. We shall indicate, briefly, some examples of sentences of $L_\alpha$ which show the non-implications (3) to (6) from the introduction. None of the examples originates with the author. They are not the simplest ones; they are rather chosen because their critical properties are obvious.

Let $R$ and $G$ be binary predicates and $P$ be a ternary predicate.

(4.1). Let $\phi$ be a first-order sentence saying that $R$ is a linear ordering without last element. The sentence $\forall \phi \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$ is satisfiable as a sentence of $L_\alpha$ if and only if $\alpha \neq 0$.

(4.2). Let $\psi$ be a first-order sentence saying that $R$ is a linear ordering of the universe. Let furthermore $\psi$ be the conjunction of the following sentences:

(i) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$;
(ii) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$;
(iii) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$.

The sentence $\forall \phi \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$ is satisfiable as a sentence of $L_\alpha$ if and only if $\alpha$ is a successor ordinal. The sentence $\psi$ says that fixing the first argument in $F$ yields a biunique function (I) whose domain contains the predecessors of this element (ii) and whose range is uniformly bounded (iii).

(4.3). Let $\phi$ be as in (4.2). Let furthermore $\psi$ be the conjunction of the following sentences:

(i) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$;
(ii) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$;
(iii) $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$.

The sentence $\forall \phi \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$ is satisfiable as a sentence of $L_\alpha$ if and only if $\alpha$ is singular. (The sentence says that $\mathbb{M}$ is a function (1) whose domain is bounded (ii) while its range is cofinal in the universe (iii).)

(4.4). The sentence $\forall \phi_0 \forall \neg Q_0 \exists \neg P_0 \forall \psi_0$ is satisfiable as a sentence of $L_\alpha$ if and only if, for some cardinal $\gamma < \kappa$, $\kappa < \gamma^\omega$.

References

The completeness of logic with the added quantifier
"there are uncountably many"

by

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The language $L'$ is obtained from an ordinary first order language $L$, having countably many symbols, by adding a new quantifier $Q$, to be read
"there are uncountably many... such that...". Mostowski [10](!) raised the completeness problem for $L'$, i.e., the question whether the set $V_4$ of (semantically) logically valid sentences of $L'$ is recursively enumerable.

The compactness of the language $L'$ has been established in Theorem 3.4 of the preceding article by G. Fuhrken. The purpose of this note is to point out that the two results and the argument employed by Fuhrken to prove compactness also lead at once to a positive answer to Mostowski's question.

One of these two results is (2.2), Fuhrken's first normal form theorem for $L'$. When modified as in the second (but not the first) part of the Remark following its proof, this theorem gives us the following information: Let $L'$ be the first order language obtained from $L$ by adding two new unary predicates $U$ and $W_1$ and one new ternary predicate $W_2$. We can define a recursive function correlating with each sentence $\sigma$ of $L'$ a sentence $\sigma'$ of $L'$ in such a way that:

(A) $\sigma$ has a model if and only if $\sigma'$ has a model $\mathcal{M}$ of power $\kappa$, if in which $U^{(\kappa)}$ has power at most $\kappa$.

The second result, (1.7), is easily derived from the proof of the author's 'Lowenheim-Skolem theorem for two cardinals' (Theorem 6.2 of [9]). (However, this fact for the case when $T$ is incomplete and its significance were only observed recently, by Fuhrken.) Let $L''$ be the first order language obtained from $L'$ by adding one new unary predicate $W_2$. (1.7) describes a certain recursive set $\Sigma$ of sentences, such that, for any sentence $\delta$ of $L'$:

(B) if $\delta$ has a model $\mathcal{M}$ of power $\kappa$, in which $U^{(\kappa)}$ has power at most $\kappa$, if and only if $\Sigma \cup \{\delta\}$ is consistent.

(!) The terminology of the preceding paper by G. Fuhrken will be used; numbers refer to its theorems, numbers in brackets to its bibliography.