There also exists a $g \in G(S)$ defined as follows:

$$g(x) = f_{i,i+2}^{(1,2)}(x) \text{ for all } i \text{ and all } x \in F(R_i^{(1)}),$$
$$g(x) = f(x) \text{ elsewhere.}$$

Now suppose $x \in F(R_0^{(0)})$. Then $f(x) \in F(R_1^{(1)})$ and $gf(x) \in F(R_3^{(2)})$; on the other hand, $g(x) \in F(R_1^{(1)})$ and $fg(x) \in F(R_2^{(2)})$. Hence (i) holds.

COROLLARY 3. *If $S$ is an ordering relation and if $G(S)$ is non-Abelian, then*

$$\varkappa\big(G(S)\big) \geqslant 2^{\aleph_0}.$$

Proof. By Theorem 2, $S$ is representable in the form (22) above. For each function $h$ on the set of integers to $\{0, 1\}$ we use (22) above to define an automorphism $f_h$ of $S$:

$$f_h(x) = x \text{ for } x \in F(T_1) \cup F(T_2),$$
$$f_h(x) = f_U^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(U^{(j)}),$$
$$f_h(x) = f_V^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(V^{(j)}),$$
$$f_h(x) = f_{i,i+h(j)}^{(j,j+1)}(x) \text{ for all } i, j \text{ and all } x \in F(R_i^{(j)}).$$

Clearly, $f_h \neq f_{h'}$ for $h \neq h'$.

We note in conclusion that every non-Abelian $G(S)$ has as a subgroup the automorphism group of a relation of type $(\omega^* + \omega)^2$.

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# On the imbedding of a regular ring in a regular ring with identity


by

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**1. Introduction.** Throughout this note we shall suppose that $R$ is an associative ring, regular in the sense of von Neumann (this means that for every $a \in R$, $axa = a$ for some $x \in R$). We shall prove the following theorem:

THEOREM 1. *A regular ring is isomorphic to a two-sided ideal of a regular ring with identity.*

A special case of this theorem has been established previously by Kohls [3]. Also, Johnson [2] has shown that, for a certain class of rings which includes all regular rings, each of the rings is isomorphic to a subring of a regular ring with identity.

In this note we shall not require familiarity with the theory of regular rings.

Our procedure is to imbed the regular ring $R$ in the ring of all pairs $(a, \varrho)$ with $a \in R$ and $\varrho$ from a suitable commutative regular ring $M$ with identity such that $R$ is an algebra over $M$. If every non-zero element of $R$ has the same additive order, necessarily a prime or infinity, then we can choose $M$ as the prime field of the corresponding characteristic (and begin our proof with section 4). In the most general case we shall need the following result:

THEOREM 2. *There exists a commutative regular ring $M$ with identity such that every regular ring $R$ is an algebra over $M$.*

First we shall construct this ring $M$ and then turn each regular ring into an algebra over $M$. Finally, we verify the main result, i.e. Theorem 1.

**2. Construction of $M$.** Let $p_1, \ldots, p_i, \ldots$ be the set of rational primes in some order, and $K_{p_i}$ the prime field of characteristic $p_i$. Define $K$ as the complete direct sum of the $K_{p_i}$:

$$K = \sum_{p_i}{}^{*} K_{p_i},$$

that is, $K$ consists of all vectors $(\varrho_1, \dots, \varrho_i, \dots)$ with $\varrho_i \in K_{p_i}$, where equality, addition and multiplication are defined componentwise. $K$ is obviously a ring (moreover, a regular ring) with the identity $(1_{p_1}, \dots, 1_{p_i}, \dots)$ where $1_{p_i}$ denotes the identity of $K_{p_i}$.

Define $M$ as consisting of all $\varrho = (\varrho_1, \dots, \varrho_i, \dots) \in K$ such that, for some pairs of integers $s, t$ with $t > 0$, we have

(1) $$t\varrho_i = s1_{p_i} \quad \text{for almost all (1) } p_i\,.$$

The number $s/t$ is uniquely determined by $\varrho$. In fact, if $t_1\varrho_i = s_1 1_{p_i}$ for almost all $p_i$ for some rational number $s_1/t_1$, then $(ts_1 - t_1 s)\varrho_i = 0$ for almost all $i$. Hence either $ts_1 - t_1 s \equiv 0 \pmod{p_i}$ for infinitely many $p_i$, in which case $ts_1 - t_1 s = 0$, $s/t = s_1/t_1$; or $\varrho_i = 0$ for almost all $i$, in which case $p_i | s, s_1$ for almost all $i$ and $s = 0 = s_1$. Thus we may associate with $\varrho \in M$ the rational number $s/t$:

$$\varphi(\varrho) = \frac{s}{t}\,.$$

If $\varrho \in M$, $\sigma \in M$, and $\varphi(\varrho) = s/t$, $\varphi(\sigma) = s'/t'$, then $\varrho \pm \sigma \in M$ and

$$\varphi(\varrho \pm \sigma) = \frac{s}{t} \pm \frac{s'}{t'} = \varphi(\varrho) \pm \varphi(\sigma)\,, \qquad \varphi(\varrho\sigma) = \frac{ss'}{tt'} = \varphi(\varrho)\varphi(\sigma)\,.$$

For, if $t\varrho_i = s1_{p_i}$ and $t'\sigma_i = s'1_{p_i}$ for almost all $p_i$, then $tt'(\varrho_i \pm \sigma_i) = (st' \pm ts')1_{p_i}$ and $tt'(\varrho_i\sigma_i) = (ss')1_{p_i}$ for almost all $p_i$. Obviously, $1 = (1_{p_1}, \dots, 1_{p_i}, \dots) \in M$ with $\varphi(1) = 1$ is the identity in $M$. Thus $M$ is a subring of $K$ (containing the discrete direct sum of the prime fields) with identity and $\varphi$ is a ring homomorphism (2) of $M$ into the rational number field $K_0$. (Moreover, $\varphi$ is surjective. For, if $s/t \in K_0$, then $t \not\equiv 0 \pmod{p_i}$ for almost all $p_i$ and for these $p_i$ there exist solutions $\xi_i = \varrho_i \in K_{p_i}$ of the equations $t\xi_i = s1_{p_i}$. If we choose arbitrarily the remaining $\varrho_i$, then manifestly $\varrho = (\varrho_1, \dots, \varrho_i, \dots) \in M$ with $\varphi(\varrho) = s/t$.)

We claim that $M$ is regular. If $\varrho = (\varrho_1, \dots, \varrho_i, \dots) \in M$ and $\varphi(\varrho) = s/t$, then define $\sigma_i = \varrho_i^{-1}$ or $= 0$ according as $\varrho_i \neq 0$ or $= 0$. In this case $\sigma = (\sigma_1, \dots, \sigma_i, \dots) \in M$. For, if $s = 0$, then almost all $\varrho_i = 0$ and almost all $\sigma_i = 0$. If $s \neq 0$, then almost all $\varrho_i$ satisfy $t\varrho_i = s1_{p_i}$, and therefore almost all $\sigma_i$ will satisfy $t1_{p_i} = s\sigma_i$. Furthermore, $(\varrho\sigma)_i$ is $0$ or $1_{p_i}$ according as $\varrho_i = 0$ or not; thus $\varrho\sigma\varrho = (\varrho_1, \dots, \varrho_i, \dots) = \varrho$, and $M$ is regular.

Thus the $M$ we have constructed is a commutative regular ring with identity.

**3. Proof of Theorem 2.** Let $R$ be an arbitrary regular ring. For the elements $a$ of $R$ and for all $\varrho \in M$ we shall define a multiplication $\varrho a$ so that $R$ will be an algebra over $M$; that is,

(1) *Almost all* means all with a finite number of exceptions.

(2) The *kernel* of $\varphi$ is the discrete direct sum of the prime fields.



(a) $\varrho a \in R$, $1a = a$,
(b) $\varrho(a+b) = \varrho a + \varrho b$,
(c) $(\varrho + \sigma)a = \varrho a + \sigma a$,
(d) $(\varrho\sigma)a = \varrho(\sigma a)$,
(e) $\varrho(ab) = (\varrho a)b = a(\varrho b)$

for all $\varrho, \sigma \in M$ and $a, b \in R$.

If $n$ is an integer $\geqslant 1$, $R_n$ will denote the set of $a \in R$ satisfying $na = a + \dots + a = 0$ ($n$ addenda), and $\bar{R}_n$ will denote the set of $b \in R$ satisfying $b = nc$ for some $c \in R$. Obviously, $R_n$ and $\bar{R}_n$ are two-sided ideals of $R$ and, if $m$ is a divisor of $n$, then $R_n \supseteq R_m$ and $\bar{R}_m \supseteq \bar{R}_n$.

LEMMA 1 (3). *Let $R$ be a regular ring. Then*

A. $R = R_n \oplus \bar{R}_n$ *for every* $n$ (4);

B. $\bar{R}_{mn} = \bar{R}_m \cap \bar{R}_n$ *for all integers* $m, n$;

C. $\bar{R}_n = R_m \oplus \bar{R}_{mn}$ *if* $m$ *and* $n$ *are relatively prime integers*;

D. $R_n = R_{p_1} \oplus \dots \oplus R_{p_k}$ *if* $p_1, \dots, p_k$ *are the distinct prime factors of* $n$;

E. *if* $t \geqslant 1$, *then for each* $a \in \bar{R}_t$ *there is one and only one* $b \in \bar{R}_t$ *such that* $a = tb$ (*this unique* $b$ *will be denoted by* $t^{-1} \circ a$).

A. Suppose that $a \in R$. Then for some $x \in R$, $na = (na)x(na)$. Set

$$a_n = a - naxa\,, \qquad \bar{a}_n = naxa\,.$$

We have $a = a_n + \bar{a}_n$, where $a_n \in R_n$, since $na_n = na - naxna = 0$, and obviously $\bar{a}_n \in \bar{R}_n$. Thus $R_n$ and $\bar{R}_n$ generate $R$. Suppose that $b \in R_n \cap \bar{R}_n$. Then $b = nc$ for some $c \in R$. Hence, for some $y \in R$, $b = byb = (nc)yb = cx(nb) = 0$. This proves A. Henceforth we shall denote the components of $a$ in $R_n$, $\bar{R}_n$ by $a_n$, $\bar{a}_n$, respectively.

B. Clearly, $\bar{R}_{mn} \subseteq \bar{R}_m \cap \bar{R}_n$. On the other hand, if $a \in \bar{R}_m \cap \bar{R}_n$, then $a = mb = nc$ for some $b, c \in R$. Hence, for some $x \in R$, $a = axa = (mb)x(nc) = mn(bxc)$, so $a \in \bar{R}_{mn}$. This proves B.

C. Since $\bar{R}_{mn} \subseteq \bar{R}_n$, A gives $\bar{R}_n = (\bar{R}_n \cap R_{mn}) \oplus \bar{R}_{mn}$. We show that $\bar{R}_n \cap R_{mn} = R_m$. For suitable integers $u, v$ we have $um + vn = 1$. If $a \in R_m$ then $a = (um + vn)a = n(va) \in \bar{R}_n$; thus $R_m \subseteq \bar{R}_n$ and so $R_m \subseteq \bar{R}_n \cap R_{mn}$. If $b \in \bar{R}_n \cap R_{mn}$ then $\bar{b}_m = b - b_m \in \bar{R}_n$ and $\bar{b}_m \in \bar{R}_m$ imply, in view of B, that $\bar{b}_m \in \bar{R}_m \cap \bar{R}_n = \bar{R}_{mn}$. Hence $\bar{b}_m = b - b_m \in R_{mn}$ implies $\bar{b}_m = 0$ and $b = b_m \in R_m$. This proves C.

D. By A, for every $l \geqslant 1$, $R = R_p \oplus \bar{R}_p = R_{p^l} \oplus \bar{R}_{p^l}$. By B, $\bar{R}_{p^l} = \bar{R}_p$, and so $R_p \subseteq R_{p^l}$ implies $R_{p^l} = R_p$. A repeated application of C yields for $n = p_1^{l_1} \dots p_k^{l_k}$

$$R = R_{p_1^{l_1}} \oplus \dots \oplus R_{p_k^{l_k}} \oplus \bar{R}_n = R_{p_1} \oplus \dots \oplus R_{p_k} \oplus \bar{R}_n\,.$$

This, together with $R_{p_j} \subseteq R_n$ and A, gives D.

(3) Most of this lemma could be omitted by making use of the description of additive groups of regular rings (see [1], Theorem 74.2).

(4) We write $R = A \oplus B$ if the ring $R$ is the direct sum of its ideals $A$ and $B$.

E. If $a \in \bar{R}_t$ then for some $x, y \in R$ we have $a = tx = (tx)y(tx)$; so $a = tb$ with $b = t(xyx) \in \bar{R}_t$. On the other hand, if $a = tb'$ with $b' \in \bar{R}_t$, then $b - b' \in R_t^{\cdot} \cap \bar{R}_t = 0$ and $b = b'$.

This completes the proof of Lemma 1.

Now we have come to the definition of $\varrho a$. We write explicitly $\varrho = (\varrho_1, \ldots, \varrho_i, \ldots)$ and choose $s, t$ so that (1) holds. Then we select an arbitrary finite set of primes $p_1, \ldots, p_k$ with the proviso that among them all the primes occur for which (1) fails to hold. By A and D of Lemma 1 we may write

(2) $$a = a_{p_1} + \ldots + a_{p_k} + \bar{a}_{p_1 \ldots p_k},$$

and we define

(3) $$\varrho a = \varrho_1 a_{p_1} + \ldots + \varrho_k a_{p_k} + t^{-1} \circ (s\bar{a}_{p_1 \ldots p_k}).$$

First of all observe that $R_p$ is evidently a vector space over the prime field $K_p$ of characteristic $p$, and therefore in (3) the product $\varrho_i a_{p_i}$ has a well-defined meaning. Because of E, the last addend in (3) is defined (if $s \neq 0$, $t$ has all its prime factors included in $p_1, \ldots, p_k$), and it is clear that $\varrho a$, as defined in (3), belongs to $R$. We need to show that in definition (3):

(*) the choice of the finite set of primes is irrelevant,

(**) the value of $\varrho a$ is not changed through the use of a different pair $s_1, t_1$ in place of $s, t$ (provided $s_1/t_1 = s/t$).

If we take a new prime $p_{k+1}$ then by C of Lemma 1 we have $\bar{a}_{p_1 \ldots p_k} = a_{p_{k+1}} + \bar{a}_{p_1 \ldots p_k p_{k+1}}$. Since $t\varrho_{k+1} = s1_{p_{k+1}}$ we have $s\bar{a}_{p_1 \ldots p_k} = t\varrho_{k+1} a_{p_{k+1}} + s\bar{a}_{p_1 \ldots p_k p_{k+1}}$, whence

$$t^{-1} \circ (s\bar{a}_{p_1 \ldots p_k}) = \varrho_{k+1} a_{p_{k+1}} + t^{-1} \circ (s\bar{a}_{p_1 \ldots p_k p_{k+1}}),$$

since division by $t$ in $\bar{R}_{p_1 \ldots p_k} (\subseteq \bar{R}_t)$ is unique by E. Repetition of this argument shows (*). To prove (**), choose the set $p_1, \ldots, p_k$ to include all primes for which (1) fails to hold for either $s, t$ or $s_1, t_1$, and let $n = p_1 \ldots p_k$. Then $ts_1 = t_1 s$ implies $ts_1 \bar{a}_n = t_1 s \bar{a}_n$, $s_1 \bar{a}_n = t^{-1} \circ (t_1 s \bar{a}_n) = t_1 (t^{-1} \circ (s\bar{a}_n))$, and hence $t_1^{-1} \circ (s_1 \bar{a}_n) = t^{-1} \circ (s\bar{a}_n)$ [here we use the fact that if $b \in \bar{R}_t$ then for any integer $r$, $rb \in \bar{R}_t$ and $t^{-1} \circ (rb) = r(t^{-1} \circ b)$].

We need to verify (a)-(e). Given $\varrho, \sigma \in M$, choose the set $p_1, \ldots, p_k$ sufficiently large to include all primes for which neither $\varrho$ nor $\sigma$ satisfies (1). Taking the corresponding decompositions (2) for $a$ and $b \in R$, we may prove (a)-(e) componentwise. This, being a straightforward computation, will be omitted. This establishes Theorem 2.

**4. Proof of Theorem 1.** We begin with the following lemmas.

LEMMA 2. *If $a$ is an element of a regular ring $R$, then there exists an idempotent $e \in R$ such that $ae = ea = a$.*



We have for some $x \in R$, $a = axa$. Put $f = ax$; then $f$ is an idempotent and $fa = a$. There is a $y \in R$ such that $a - af = (a - af)y(a - af)$. Again, $g = y(a - af)$ is an idempotent, and $(a - af)g = a - af$, $gf = 0$. Set $e = f + g - fg$. Then $e$ is an idempotent, since

$$e^2 = (f+g-fg)(f+g-fg) = f(f+g-fg) + (g-fg)g = f+g-fg = e.$$

Furthermore,

$$ea = (f+g-fg)(fa) = fa = a,$$

$$ae = a(f+g-fg) = af + (a-af)g = af + a - af = a.$$

This proves Lemma 2.

LEMMA 3. *If $a$ is an element of a regular ring $R$ and $\varrho$, $\sigma$ are elements of $M$ (the ring of Theorem 2) such that $\varrho\sigma\varrho = \varrho$, then there exists a $b \in R$ satisfying*

(4) $$aba + \varrho(ab + ba) + \sigma a^2 + 2\varrho\sigma a + \varrho^2 b = a.$$

Note. If $R$ has an identity 1 and if we write $\varrho$ for $\varrho \cdot 1$, then (4) is equivalent to $(a+\varrho)(b+\sigma)(a+\varrho) = a + \varrho$.

To prove Lemma 3, choose an idempotent $e \in R$ such that $ea = ae = a$; $e$ exists by Lemma 2. Then for some $x \in R$ we have $(a + \varrho e)x(a + \varrho e) = a + \varrho e$. This equation persists if $x$ is replaced by $exe$, since $(a + \varrho e)e = a + \varrho e = e(a + \varrho e)$. We may therefore assume that $x = ex = xe$. Now set $b = x - \sigma e$. We then have

$$(a + \varrho e)(b + \sigma e)(a + \varrho e) = a + \varrho e.$$

Formula (4) follows since $ae = a = ea$ and $be = b = eb$. The proof of Lemma 3 is completed.

Now let $R$ be an arbitrary regular ring, and let $R^*$ be the set of all couples $(a, \varrho)$ $(a \in R,\ \varrho \in M)$ with equality, addition and multiplication defined as follows:

(i) $(a, \varrho) = (a', \varrho')$ if and only if $a = a'$ and $\varrho = \varrho'$;

(ii) $(a, \varrho) + (a', \varrho') = (a + a', \varrho + \varrho')$;

(iii) $(a, \varrho)(a', \varrho') = (aa' + \varrho' a + \varrho a', \varrho\varrho')$.

By Theorem 2, $R$ is an algebra over $M$, and therefore the identities are satisfied which ensure that $R^*$ is in fact an associative ring with $(0, 1)$ as identity, and

$$a \to (a, 0)$$

is an isomorphism of $R$ with a two-sided ideal of $R^*$. The calculation is straightforward and will therefore be left to the reader. Finally, we need to show that $R^*$ is a regular ring. Given $(a, \varrho) \in R^*$, determine $\sigma \in M$ and $b \in R$ as in Lemma 3. Then we have

$$(a, \varrho)(b, \sigma)(a, \varrho) = (a, \varrho),$$

and since $(b, \sigma) \in R^*$, $R^*$ is in fact regular. This completes the proof of Theorem 1 [5].

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(5) As an alternative to (i) we could identify $(a, \varrho)$ and $(a', \varrho')$ in $R^*$ if and only if for all $x \in R$, $ax+\varrho x = a'x+\varrho' x$. Our theorem holds with this new $R^*$ (in general different from the previous $R^*$). The new $R^*$ can be identified with a subring of the ring of left endomorphisms of $R$ and, if $R$ has an identity, this new $R^*$ will coincide with $R$.


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# Skolem-type normal forms for first-order languages with a generalized quantifier *


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Let $L$ be a first-order language with countably many non-logical constants and let $\alpha$ be an ordinal number. With $L$ and $\alpha$ we associate a new language $L_\alpha$ which is obtained from $L$ by adding to the symbols of $L$ a new quantifier $Q$ which is read "there are at least $\aleph_\alpha$...". Let $V_\alpha$ be the set of all sentences of $L_\alpha$ which are logically valid (where $Q$ is counted among the logical symbols).

Mostowski raised the question [1] whether $V_\alpha$ is axiomatizable. For $\alpha = 0$ he showed that the answer is negative (provided $L$ has a sufficient supply of non-logical constants). One can show (see [3] and [5]) that for the majority of ordinals Mostowski's proof cannot be adapted. In fact for $\alpha = 1$ the answer is positive as shown by Vaught (see [16]).

Another natural question to ask is the following: What is the relationship between the various $V$'s? (Note that the formation rules for $L_\alpha$ are independent of the particular $\alpha$.) Here we obtain as partial results:

(1) $V_1 \subseteq V_\alpha$ *for any ordinal* $\alpha$ *for which* $\aleph_\alpha$ *is regular;*

(2) $V_\alpha \subseteq V_0$ *for any ordinal* $\alpha$.

The following negative results are known (assuming that $L$ has a sufficient supply of non-logical constants):

(3) $V_0 \nsubseteq V_\alpha$ *for any ordinal* $\alpha > 0$;

(4) $V_\alpha \nsubseteq V_\beta$ *for any limit ordinal* $\alpha$ *and any successor ordinal* $\beta$;

(5) $V_\alpha \nsubseteq V_\beta$ *for any ordinals* $\alpha$ *and* $\beta$ *for which* $\aleph_\alpha$ *is regular and* $\aleph_\alpha$ *is singular;*

* The paper is mainly based on § 2 of Part I of the author's Doctoral Dissertation [2]. The main results have been summarized in [6]. The author takes the opportunity to express his gratitude to his Thesis Advisor Professor Robert L. Vaught and to Professor William Craig for the stimulation and help received. It was in Professor William Craig's Seminar conducted at the University of California in Berkeley in 1961 that the author learned about the problems treated in this paper and obtained the first results in this direction.

(1) In [10]; see also [11].