fect subsets of D such that for each t, $D \epsilon S(E_t)$. We notice that all of the sets mentioned are effectively given. The set D is the set we set out to construct.

The fact, mentioned in the introduction, that this set does not have non-zero σ-finite Hausdorff measure for any Hausdorff measure is clear. To have non-zero measure, each of the subsets E_t would have to have non-zero measure, and since there are uncountably many of them in D and they are pairwise disjoint, D would have non-σ-finite measure.

References

[1] E. Best, *A closed dimensionless linear set*, Proc. Edin. Math. Soc. Series 2, vol. 6, part II, pp. 105-108.

[2] A. Denjoy, *Memoire sur la derivation et son calcul inverse*, Paris 1954.

[3] P. R. Halmos, *Measure theory*, New York 1950.

[4] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. 99 (1928), pp. 134-141.

Reçu par la Rédaction le 7. 7. 1962

On the genus of an n-connected graph

by

F. Harary* and Y. Kodama*

1. Introduction. The *connectivity* $\varkappa(G)$ of a graph G is the smallest number of points whose removal results either in a disconnected graph or in the graph with one point and no lines. Graph G is *n-connected* if $\varkappa(G) \geqslant n$. For $n \geqslant 1$, an *n-component* of G is a maximal n-connected subgraph. Thus a 1-component of G is a (connected) component; a 2-component of G is called a *block* of G (by Battle, Harary, Kodama and Youngs [1] or Harary [3], and a nonseparable subgraph by Whitney [6]); and a 3-component of G will be called a *brick* of G.

The *genus* $\gamma(G)$ of G is the smallest integer n such that G is imbeddable in the orientable surface S_n whose genus $\gamma(S_n)$ is n. In [1], we (Battle, Harary, Kodama and Youngs) proved that the genus of any graph is the sum of the genuses of its blocks. Our present object is to study the genus of a graph in terms of its bricks, and in general of its n-components.

The problem is so complicated that we restrict our study in this note to the case where an n-connected graph G is the union of two $(n+1)$-components, B and C. We will see that the number of points in $B \cap C$ is exactly n, that $\gamma(G) \leqslant \gamma(B)+\gamma(C)+n-1$, and by an example that this inequality is best possible. Let $v_1, v_2, \ldots, v_n$ be the set of points in $B \cap C$ and call G_{ij} the graph obtained by adding line $v_i v_j$ to G. Then we will prove that, if $\gamma(G_{ij}) > \gamma(G)$ for all $1 \leqslant i < j \leqslant n$, then $\gamma(G) = \gamma(B)+\gamma(C)+n-1$. This last equation is specialized to the case where B and C are bricks, i.e., $n = 2$.

2. Results. We will present one lemma, one theorem, one corollary, and several examples.

Remark 1. Let an n-connected graph G be the union of two $(n+1)$-components B and C. Then the number of points of $B \cap C$ is exactly n. Moreover, the set of lines of $B \cap C$ consists of all lines of G whose end points are in $B \cap C$.

* * This work was supported in part by the National Science Foundation under grants NSF G-17771 and G-20637.

It is obvious that the number of points of $B \cap C$ is $\geqslant n$. For if $B \cap C$ has $< n$ points, then $\varkappa(G) < n$. But if the number of points of $B \cap C$ is $> n$, then $\varkappa(G) \geqslant n+1$. Therefore $B = C = G$.

LEMMA. *If an n-connected graph G is the union of two $(n+1)$-components B and C, then*

$$\gamma(G) \leqslant \gamma(B) + \gamma(C) + n - 1 . \tag{1}$$

Proof. Let M and N be orientable 2-manifolds of genuses $\gamma(B)$ and $\gamma(C)$ and imbed B and C in M and N, respectively. There exist disjoint disks $D_1, \ldots, D_n$ in M such that the frontier $\mathrm{Fr}(D_i)$ is a circle and $\mathrm{Fr}(D_i) \cap B = v_i$ for $i = 1, \ldots, n$. If we remove the interior points of $\bigcup_{i=1}^{n} D_i$ from M, we get an orientable 2-manifold M_0 with n boundaries. By the same construction we obtain from N an orientable 2-manifold N_0 with n boundaries. If we identify the boundaries of M_0 and N_0 so that the orientations of M_0 and N_0 are preserved, we get an orientable manifold A. It is easy to verify that $\gamma(A) = \gamma(M) + \gamma(N) + n - 1$. Let G_0 be the image of $B \cup C$ under this identification. It is obvious that G is a subgraph of G_0. Since $G_0 \subset A$, we get $\gamma(G) \leqslant \gamma(G_0) \leqslant \gamma(M) + \gamma(N) + n - 1 = \gamma(B) + \gamma(C) + n - 1$.

In the proof of the lemma, the only hypothesis which was used is that $G = B \cup C$ and $B \cap C$ contains exactly n points. Hence we may state the following stronger form of the lemma:

If a graph G is the union of two subgraphs B and C whose intersection contains exactly n points, then $\gamma(G) \leqslant \gamma(B) + \gamma(C) + n - 1$.

Remark 2. Let an n-connected graph G be the union of two $(n+1)$-components B and C. By Remark 1, $B \cap C$ contains exactly n points $v_1, \ldots, v_n$. Let G_{ij}, $1 \leqslant i < j \leqslant n$ be the graph obtained by adding line $v_i v_j$ to G. Then

$$\gamma(G) \leqslant \gamma(G_{ij}) \leqslant \gamma(G) + 1 . \tag{2}$$

THEOREM. *Let an n-connected graph G be the union of two $(n+1)$-components B and C. Let $v_1, \ldots, v_n$ be the set of points of $B \cap C$. Call G_{ij} the graph obtained by adding line $v_i v_j$ to G. If $\gamma(G_{ij}) = \gamma(G) + 1$ for $1 \leqslant i < j \leqslant n$, then*

$$\gamma(G) = \gamma(B) + \gamma(C) + n - 1 . \tag{3}$$

Proof. By the lemma, it is enough to prove that $\gamma(G) \geqslant \gamma(B) + \gamma(C) + n - 1$. Let us imbed G in an orientable 2-manifold M of genus $\gamma(G)$. Take a triangulation T of M such that the 1-skeleton of T contains G and the 0-skeleton of T contains the set of points of G. Let T_0 be the second barycentric subdivision of T. Since B is $(n+1)$-connected, there exists a component U of $(M - C)$ such that $U \cup \{v_1, \ldots, v_n\} \supset B$.

Consider the open star $V = \mathrm{St}(\mathrm{Fr}\, U, T_0)$ of the frontier $\mathrm{Fr}(U)$ of U. Let k be the number of disjoint circles J_i which form the intersection $\mathrm{Fr} V \cap U$. Then the set $U \cap V$ consists of k disjoint open cylinders $H_1, \ldots, H_k$. For any $j = 1, \ldots, n$, the intersection $\overline{H}_j \cap (\bigcup_{i=1}^{n} v_i) \neq \emptyset$, where $\overline{H}_j$ is the closure of H_j in M. For, assume $\overline{H}_j \cap (\bigcup_{i=1}^{n} v_i) = \emptyset$. Then the circle J_j does not intersect G. Therefore, if we cut M at J_j, we get an orientable 2-manifold N with two boundaries L_1 and L_2. Attach two disks D_1 and D_2 to L_1 and L_2. Then we get an orientable 2-manifold N_0. It is obvious that $\gamma(N_0) = \gamma(M) - 1$. Since $G \subset N_0$, this contradicts $\gamma(M) = \gamma(G)$. Next, we shall prove that, for each j, $\overline{H}_j \cap (\bigcup_{i=1}^{n} v_i)$ is exactly one point. Suppose that $\overline{H}_j \cap (\bigcup_{i=1}^{n} v_i) = \bigcup_{i=1}^{m} v_i$, $m > 1$. Since H_j is an open cylinder and $\overline{H}_j$ is the union of the closures of all components of $H_j - G$, there exists a component W of $H_j - G$ such that $\overline{W}$ contains two points v_a and v_b of $\bigcup_{i=1}^{n} v_i$. Then $\overline{W}$ contains a line L which connects v_a and v_b such that $\mathrm{Fr} W \cap L = v_a \cup v_b$. This contradicts the hypothesis that $\gamma(G_{ab}) > \gamma(G)$. Let us denote by H_u^i, $u = 1, \ldots, a_i$, those open cylinders H_j such that $v_i \in \overline{H}_j$ and J_u^i is the circle of $\mathrm{Fr}\, \overline{H}_u^i$ for $i = 1, \ldots, n$, where $\sum_{i=1}^{n} a_i = k$. Now cut M at $\bigcup_{j=1}^{k} J_j = \bigcup_{u,i} J_u^i$. We get two orientable 2-manifolds F_1 and F_2 with k boundaries such that $M = F_1 \cup F_2$. If we attack k disks D_j^i, $j = 1, \ldots, k$, to F_i, we get an orientable 2-manifold M_i, $i = 1, 2$.

The next step in the proof of the theorem involves the verification of equation (3). This can be accomplished by a calculation using the Euler characteristic (as in Ringel [5], pp. 56-57). But we prefer to emply a technique which exploits the Mayer-Vietoris sequence as in Eilenberg and Steenrod [2].

Considering the Mayer-Vietoris sequence of the triple $(M; F_1, F_2)$, we find

$$2\gamma(G) = k_1 + k_2 , \tag{4}$$

where k_i is the rank of the homology group $H_1(F_i)$ with coefficients in the rational field, and also

$$2\gamma(M_i) = k_i - (k - 1) \quad (i = 1, 2) . \tag{5}$$

Since $C \subset M_1$, we get

$$\gamma(C) \leqslant \gamma(M_1) . \tag{6}$$

Let X_i, $i = 1, \ldots, n$, be n 2-spheres. We remove a_i disjoint open disks from X_i to get an orientable 2-manifold Y_i with a_i boundaries, $i = 1, \ldots, n$. Let A_u^i, $u = 1, \ldots, a_i$, be the boundaries of Y_i. Identify the boundaries J_u^i in F_2 with A_u^i in $\bigcup_{i=1}^{n} Y_i$ in such a way that the identification preserves the orientations induced by those of F_2 and each Y_i. Call M_0 the orientable 2-manifold so obtained. Consider the Mayer-Vietoris sequence of the triple $(M_0; F_2, \bigcup_{i=1}^{n} Y_i)$:

$$\ldots \to H_2(F_2) + H_2(\bigcup_{i=1}^{n} Y_i) \to H_2(M_0) \to H_1(\bigcup_{j=1}^{k} J_j) \to$$
$$H_1(F_2) + H_1(\bigcup_{i=1}^{n} Y_i) \to H_1(M_0) \to \hat{H}_0(\bigcup_{j=1}^{k} J_i) \to$$
$$\hat{H}_0(F_2) + \hat{H}_0(\bigcup_{i=1}^{n} Y_i) \to \hat{H}_0(M_0) \to \ldots$$

where all homology groups have coefficients in the rational field. The groups $H_2(F_2)$, $H_2(\bigcup_{i=1}^{n} Y_i)$, $\hat{H}_0(M_0)$ and $\hat{H}_0(F_2)$ are zero. Since the rank of $H_1(Y_i)$ is $a_i - 1$ and $\sum_{i=1}^{n} a_i = k$, the rank of $H_1(\bigcup_{i=1}^{n} Y_i) = k - n$. Thus, we get the following relation:

(7) $$2\gamma(M_0) = k + k_2 - 2n + 1 .$$

From the relations (4), (5) and (7)

(8) $$\gamma(G) = \gamma(M_1) + \gamma(M_0) + n - 1 .$$

Now take an interior point w_i in Y_i, $i = 1, \ldots, n$. Since T_0 is the second barycentric subdivision, every line of B incident on v_i cuts exactly one circle J_u^i and cuts it exactly once. Therefore, if we join w_i to each point of $B \cap (\bigcup_{u=1}^{a_i} J_u^i)$ by lines $\{L_\alpha\}$ in Y_i such that, if L_α and L_β are lines joining w_i to different points of $B \cap (\bigcup_{u=1}^{a_i} J_u^i)$ with $L_\alpha \cap L_\beta = w_i$, we have an imbedding of B in M_0. Thus we have proved that

(9) $$\gamma(B) \leqslant \gamma(M_0) .$$

From (6), (8) and (9),

$$\gamma(G) \geqslant \gamma(B) + \gamma(C) + n - 1 ,$$

completing the proof.

There is a stronger form of the theorem, analogous with the stronger form of the lemma stated above:

Let G be a graph which is the union of its subgraphs B and C. Let $B \cap C$ be a graph with n points $v_1, \ldots, v_n$ and let $B - (B \cap C)$ be connected. Then, if $\gamma(G_{ij}) = \gamma(G) + 1$ for $1 \leqslant i < j \leqslant n$,

$$\gamma(G) = \gamma(B) + \gamma(C) + n - 1 .$$

COROLLARY. *Let G be a graph which is the union of two subgraphs B and C such that $B \cap C$ is a graph with two points v_1 and v_2 and $B - (B \cap C)$ is connected. Denote by G', B' and C' the graphs obtained by adding one line joining v_1 and v_2 to G, B and C, respectively. If*

(i) $\gamma(B') = \gamma(B)$ *and* $\gamma(C') = \gamma(C)$,

then

(ii) $\gamma(B) + \gamma(C) \geqslant \gamma(G)$,

which in turn implies

(iii) $\gamma(G') = \gamma(G)$.

Proof. That (ii) implies (iii) is any easy consequence of the theorem. To prove that (i) implies (ii), imbed B' and C' in orientable 2-manifolds M_1 and M_2 with genuses $\gamma(B')$ and $\gamma(C')$. Let L_1 and L_2 be lines joining v_1 and v_2 in B' and C'. There exist closed disks D_i in M_i, $i = 1, 2$, such that $L_i \subset D_i$ and $L_i \cap D_i = B \cap D_1 = C \cap D_2 = v_1 \cup v_2$. By the same method as in the proof of the lemma we get the inequality (ii).

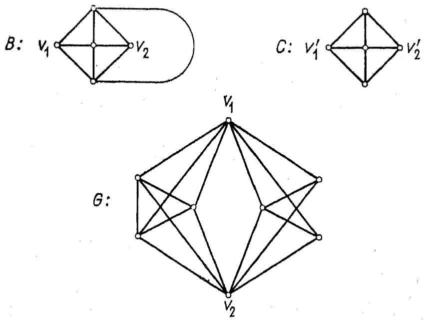

Fig. 1

In Figure 1, we show a block G with two bricks B and C which serves to illustrate the following assertion.

EXAMPLE 1. There exists a 2-connected graph (block) G which is the union of two 3-components (bricks) B and C such that $\gamma(G') = \gamma(G)$ and $\gamma(G) = \gamma(B) + \gamma(C) + 1$, where G' is the graph obtained by adding the line joining the two points of $B \cap C$. This example shows that the converse of the theorem does not generally hold. In Figure 1, both bricks B and C are planar, but $\gamma(G) = \gamma(G') = 1$ since both G and G' contain a homeomorph of K_5, the complete graph with 5 points.

EXAMPLE 2. There exists a 2-connected graph G which is the union of two 3-components B and C such that

$$\gamma(G) = \gamma(B) + \gamma(C), \quad \gamma(B') > \gamma(B) \quad \text{and} \quad \gamma(C') = \gamma(C).$$

This example means that the converses of the theorem and the part (i) of the corollary do not generally hold.

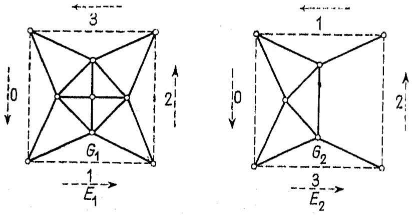

Fig. 2

Let us take two rectangles E_1 and E_2 which contain graphs G_1 and G_2 respectively, as shown in Figure 2 (compare [4]).

Let us identify the sides with the same number in E_1 and E_2, preserving orientation. By [5], this identification results in a torus. Let f be the identification map. Put $f(G_2) = B$, $f(G_1) = C$ and $G = B \cup C$. Since C contains a subgraph homeomorphic to K_5, it follows from the

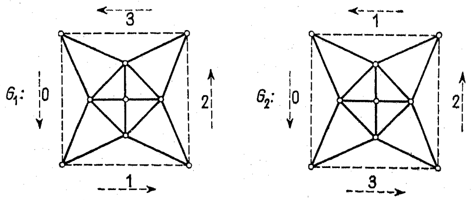

Fig. 3

classical theorem of Kuratowski that $\gamma(C) = 1$. Since B in Figure 2 is isomorphic to B in Example 1 and B' is K_5, $\gamma(B) = 0$ and $\gamma(B') = 1$. It is easy to show that $\gamma(C') = \gamma(C) = 1$. But here,

$$\gamma(G) = \gamma(B) + \gamma(C).$$

EXAMPLE 3. There exists a 2-connected graph G which is the union of two 3-components B and C such that $\gamma(B') = \gamma(B)$, $\gamma(C') = \gamma(C)$ and $\gamma(B) + \gamma(C) > \gamma(G)$.

This example means that the converse of the second part of the corollary does not hold. In Example 2, replace the graph G_2 by a copy G_3 of the graph G_1, as in Figure 3. Put $B = f(G_1)$, $C = f(G_3)$ and $G = B \cup C$. We have the relations $\gamma(B') = \gamma(B) = \gamma(C') = \gamma(C) = 1$ and $\gamma(G) < \gamma(B) + \gamma(C)$.

These three examples show that the inequality in the lemma is best possible in general.

References

[1] J. Battle, F. Harary, Y. Kodama and J. W. T. Youngs, *Additivity of the genus of a graph*, to appear in Bull. Amer. Math. Soc. 68 (1962), pp. 565-568.

[2] S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton 1952.

[3] F. Harary, *The maximum connectivity of a graph*, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), pp. 1142-1146.

[4] D. E. Cohen, F. Harary and Y. Kodama, *On the embedding of complete graphs in orientable surfaces*, Mathematika 10 (1963), pp. 79-83.

[5] G. Ringel, *Farbungsprobleme auf Flächen und Graphen*, Berlin 1959.

[6] H. Whitney, *Nonseparable and planar graphs*, Trans. Amer. Math. Soc. 34 (1932), pp. 339-362.

Reçu par la Rédaction le 1. 8. 1962