2. Fall. $D_1 \cap B_2$ ist nicht zusammenhängend. Dann besteht dieser Durchschnitt aus einer Kante von $D_1$ und dem dieser Kante gegenüberliegenden Eckpunkt $p$. $L$ sei der kleinste Teilbogen von $B_1$, der $K$ und $p$ enthält. Die Endpunkte von $L$ sind $p$ und ein Endpunkt $q$ von $K$, und die Strecke $[p,q]$ ist eine Kante von $D_1$. Wir betrachten nun den Bogen $B = (B_1 \setminus L) \cup [p,q]$. Es ist nicht schwierig nachzuweisen, daß $B$ zu $B_2$ gehört. Um zu zeigen, daß $B$ die Röhre $T_4$ durchdringt, bemerkten wir folgendes: $B$ kann höchstens so viele Teilbogen besitzen, die eine von $T_4$ verschiedene Röhre eines zu $T_4$ gehörenden Würfels durchlaufen, wie $B_1$ selbst. Da jedoch $r(B_2) = r_0$ war und $r(B) \geq r_0$ sein muß, folgt daraus unmittelbar, daß $B$ ebensolche Teilbogen enthält, die $T_4$ durchlaufen, wie $B_1$ selbst. Da natürlich $d(B) < d(B_2)$ ist, ist der Beweis hiermit beendet.

On a class of order-types generalizing ordinals

by

M. Slater (Chicago)

1. Introduction. The class $\Omega$ of ordinals $T$ may be characterized by the following properties:
(i) The right half of every Dedekind cut of $T$ has a first term,
(ii) $T$ has a first term.

It does not appear that anyone has investigated those order types for which (i) holds but (ii) does not. In this paper normal forms are given for such types, their possible factorizations are investigated, and various properties generalizing those of ordinals are given.

2. Definitions and notation. Let $T = (T, <)$ be a linearly ordered set. We define a Dedekind cut of $T$ to be a pair $(A, B)$ such that $A \neq \emptyset \neq B$, $A \cup B = T$, and for $a \in A$, $b \in B$, $a < b$. We write $T = A + B$.

We say a cut $(A, B)$ is of type $J$, $G$, $L$ or $R$ according as $A$, $B$, $A \cup B$ or $A \cap B$ is linearly ordered.

Ordinals will be denoted by Greek letters. If
$$a = \alpha_1 + \cdots + \alpha_n a_0$$
in Cantor normal form (see [3], p. 320), we shall typically write $a = (a_1, a_2, \ldots, a_n)$. We define functions $\varphi$ and $\psi$ on $\Omega$ by setting
$$\varphi 0 = +\infty, \quad \varphi \infty = -\infty,$$
$$\varphi a = a_0 \quad (a > 0), \quad \varphi a = a_0 \quad (a > 0).$$

We shall use the following valuation-like properties of $\varphi$:
(i) $\varphi(a + b) = \max(\varphi a, \varphi b)$.
(ii) $\varphi(\alpha \beta) = \varphi a + \varphi b$.
(iii) $a + \beta = \beta$ iff $\varphi a < \varphi b$. 

Repou par la Rédaction le 19. 1. 1963
4. Normal forms for R\(\beta\) types. We shall use the notation
\[\sum_{\alpha} (\lambda_\alpha, I_\alpha)\] or \[\sum_{\lambda} (\lambda_n, I_n)\] for
\[\sum_{\alpha \leq \lambda} \rho_{\alpha} I_\alpha = \ldots + \rho_{\beta} I_\beta + \rho_{\gamma} I_\gamma + \rho_{\lambda} I_\lambda,
\]
where \(\lambda_n < \lambda_1 < \lambda_2 < \ldots\) and each \(I_n > 0\). The letter \(\lambda\) will always denote
\[\lim_{n} \lambda_n.
\]

**Theorem 2.** An ordered set is R\(\beta\) iff it is of one of the following forms:
1. \(\omega^*\alpha + \bar{a}\), \(\varphi a < \mu\).
2. \(\Sigma (\lambda_n, I_n)\).
3. \(\omega^* \alpha a + \bar{a}\), \(\varphi a > \mu\).
4. \(\Sigma (\lambda_n, I_n) + \bar{a}\), \(\varphi a \geq \lambda\).

**Proof.** It is clear that an ordered set of any of these forms is R\(\beta\).

Conversely, define a function \(f\) on an initial segment of \(\omega\) inductively
by setting \(f(0) = s_0\), where \(s_0 \in T\) is arbitrary; \(f(1) = s_1\), where \(s_1 < s_0\) and
\[\beta_1 = \varphi (s_1, s_0) \geq \varphi f(s_0) = \beta_0,
\]
if such an \(s_1\) exists, and in general \(f(n) = s_n\), where
\[\beta_n = \varphi (s_n, s_{n-1}) \geq \beta_{n-1}
\]
provided such an \(s_n\) exists. If not, we terminate the construction, and have \(Df = n\).

Suppose first \(f\) is defined on all \(n\). Then by Theorem 1e we have
\[T = \sum_{n \leq \alpha} (S_{n+1}, S_n) + R(S_{n+1})\]
\[(A)\]
\[= \sum_{n \leq \alpha} (\omega^* b_n + \gamma_n) + \beta_n \leq \beta_1 \leq 
\]

Suppose \(B = \{s_\alpha\}\) is ultimately constant. Let \(N\) be the first index
beyond which \(B\) is constant. Then for some \(\mu\)
\[\{S_{\alpha+1}, S_\alpha\} = \omega^* b_\alpha + \gamma_\alpha \quad \text{for} \quad n \geq N; \quad \beta_{N-1} < \mu.
\]
Thus
\[T = \sum_{n \geq N} (\omega^* b_n + \gamma_n) + \sum_{n < N} (\omega^* b_n + \gamma_n)
\]
\[= \sum_{n < N} (\omega^* b_n) + a = \omega^* \alpha a + a
\]
for some \(a\) such that \(\varphi a = \max(\varphi, \beta_{N-1}) < \mu\). So in this case \(T\) may be
written in the form 1.
If $B$ contains a properly increasing subsequence we amalgamate terms in (A) to reach an expression

$$T = \sum_{m \in \omega} (\omega \omega m_0 + \lambda_m) \cdot \varphi \theta < \mu_0 < \mu_{n+1}.$$ 

If $\lambda_m = \kappa_n + \kappa_c \cdot \varphi \kappa < \mu_{n+1} \leq \mu_n (n > 0)$, we may replace each $\lambda_m$ by $\kappa_n (n > 0)$ without changing the sum:

$$T = \sum_{m \in \omega} (\omega \omega m_0 + \kappa_n) + \lambda_n.$$ 

On expanding each term $\omega \omega m_0 + \kappa_n$ and $\lambda_n$ into its own normal form, we find $T$ is expressible in the form 2.

Suppose next $f$ is defined not on all $\omega$, but only on $\omega + 1$. We define a function $\xi$ on a subset of $\omega$ inductively by setting

$$T_0 = T, \quad T_{n+1} = T_n = f(N), \quad \xi(0) = \varphi R[N].$$

Suppose we have defined $T_i, f_i, N_i, t_i, \xi(i)$ for all $i < m$, where $m > 1$. Then we set

$$T_m = T_{m-1} - R[t_{m-1}].$$

We may assume $T$ is not an ordinal, so that $T_m$ is non-empty. We define $f_m$ for $T_m$ exactly as we defined $f$ for $T$. If $\xi = \omega$ we do not define $\xi(m)$, and set $\xi = m$. If $\xi = \omega$ is finite, we set

$$D[t_m] = N_m + 1, \quad t_m = f_d(N_m), \quad \xi(m) = \varphi[R[t_m], t_{m-1}].$$

By the definition of $f_m$, we have $\xi(0) > \xi(1) > ...$ So we have constructed a descending sequence of ordinals, defined on $D$. Thus $D = \rho$ is finite, and $f_\rho$ must be defined on all $\omega$. So by what we have already proved

$$T_\rho = \omega \omega^\rho + \sum (\lambda_n, \mu_n)$$

and $T = \omega \omega^\rho + \alpha$ where $\alpha = \beta + R[t_\rho]$ or $T = \sum (\lambda_n, \mu_n) + \alpha'$ where $\alpha' = R[t_\rho]$. Suppose in the second case $\varphi \alpha' < \lambda$. Then $\varphi \alpha' < \mu_n$ for some $n$, since $\lambda$ is of the second kind. But then we can extend the definition of $f_\rho$ to $N_\rho + 1$ by setting $f_\rho(N_\rho + 1)$ to be any point to the left of the final segment of $\sum (\lambda_n, \mu_n) + \alpha'$ of $T$. Thus $\varphi \alpha' > \lambda$ and in the second case $T$ is of the form 4. A similar argument shows that in the first case $\varphi \alpha > \mu$, so that $T$ is of the form 3.

**Theorem 3.** No $R \beta$ type is of more than one of the five forms.

**Proof.** Given any properly descending infinite sequence $\Sigma; \kappa_0 > \kappa_1 > ...$ in $T$, we define functions $RS$ and $\varphi S$ on $\omega$ by setting $RS(\xi) = R[\kappa]$, and $\varphi S(\xi) = \varphi R[\kappa]$.

Consider the assertions $A; RS$ is ultimately constant; $B; \varphi S$ is ultimately constant. If $T$ is of type 1 (2) it satisfies $A$ for no $\xi_1, 2$ for all $\xi_1$; if $T$ is of type 3 or 4 it satisfies $A$ for all $\xi_1; B$ for all $S$. Furthermore, if $T$ is of type 3 it has an initial segment of type 1, and if $T$ is of type 4 every initial segment is of type 2 or 4. There are no $S$ if $T$ is an ordinal; otherwise these properties are mutually exclusive, so that the theorem is proved.

**Theorem 4.** The expression for an $R \beta$ type $T$ of type 1, 2, 3 is unique. If $T$ is of type 4, its expression is unique modulo any final segment of $\Sigma(\lambda_n, \mu_n)$.

**Proof.** Suppose one expression for $T$ is 1, 2, 3 or 4, as in Theorem 2. Let $S$ be a given infinite descending sequence in $T$.

1. $RS(m)$ is for sufficiently large $m$ of the form $\omega^\rho + \alpha$. This determines $\alpha$ and $\mu$.
2. Given any $\alpha$ we may find an $m$ such that $\varphi S(m) > \lambda_n$. Then

$$RS(m) = \gamma + \sum (\lambda_n, \mu_n)$$

in normal form. Since normal form is unique, this determines $\lambda_n$ and $\mu_n$ for $\gamma < \alpha$.

3. We may find a $N$ beyond which $RS$ is constant and $LS$ of type 1. The constant must be $\alpha$ and the initial segments of the form $\omega^\rho + \beta$. This determines $\alpha$ and (by 1) $\mu$.

4. We determine $\alpha$ as in 3. Let $s$ be such that $R[s] = \alpha$ and $L[s] = \sum (\lambda_n, \mu_n)$. If also $T = \sum (\lambda_n, \mu_n) + \alpha$, we must have $L[s] = \sum (\lambda_n, \mu_n) + \beta$ (for some $m > n$ and $\beta$) of type 2. On expanding $\beta$ in normal form and using 2, we see that for some $r > 0$ and all $n > r$, where

$$\lambda_n = \lambda_n + r, \quad \mu_n = \mu_n + r, \quad \lambda_n = \lambda_n + r, \quad \mu_n = \mu_n + r.$$ 

Since any final segment of $\Sigma(\lambda_n, \mu_n)$ may be absorbed in $\alpha$, it is clear that this is best possible.

5. **Order-preserving maps.** It is well known that an order-preserving map of an ordinal onto itself must be the identity. For $R \beta$ sets we have

**Theorem 5.** Let $A$ be an ordered set whose order type $T$ is $R \beta$. Then the group $G(A)$ of order-preserving maps of $A$ onto itself is the trivial group unless $T$ is of type 3, in which case it is the infinite cyclic group.

**Proof.** 1. Let $C$ be $R \beta, \delta \neq 0$. Then $C + \delta = C$ iff $C = \omega \cdot \omega^\rho + \alpha$ is of type 1, and $\delta = \omega^\rho + \alpha$ for some $\rho > 0$.

2. Suppose now $f$ is an order-preserving map of $A$ onto itself, other than the identity. We may assume without loss of generality that $f(b) = a < b$ for some $b$. 

18*
Then \( L(a) \) and \( L(b) \) have the same \( R \) \( \times \) order type. But \( L(b) = L(a) + \{ [a, b), \text{ so, by } 1, \text{ is of type } 1 \}. \) Thus \( T \) must be of type 1 or 3.

3. Let us identify each \( a \in A \) with the cut \( (L(a), B(a)) \). On identifying \( A \) with its order type \( \omega^* + \alpha \), we may write the cuts unambiguously in the symbolic form

(i) \[ a = (\omega^{(\alpha + \beta)})^{(+ \beta)} \quad (\beta < \alpha, \beta > 0) \]

or

(ii) \[ a = (\omega^{(\alpha + \beta)})^{(+ \beta)} \quad (\beta < \alpha, \beta > 0) \]

4. \( T \neq \omega^\alpha + \alpha \). For if \( b = (\omega^{(\alpha + \beta)})^{(+ \beta)} \), then \( a = f(b) \) is of type \( i \), where we must have \( r = n; \beta = \beta \). So \( a = b \).

5. If \( f \) is determined by its effect on \( c = (\omega^{\alpha + \beta}) \). For if \( f(c) = a \), then \( B(c) \) is of type \( a \), and \( f \) on \( B(c) \) is determined by the properties of ordinals quoted above, while \( L(a) \) is of type \( \omega^\alpha + \alpha \), so that \( f \) on \( L(a) \) is determined by \( c \).

6. Suppose now \( f(c) = a \) given by (i) or (ii) above. Then \( \beta = 0 \) and \( \alpha > \alpha \), so that \( T \) is of type 3. The mapping \( f \rightarrow r \) (if \( a \) is of type \( (i) \)) or \( f \rightarrow r \) (if \( a \) is of type \( (ii) \)) sets up an isomorphism between \( G(A) \) and the infinite cyclic group. This isomorphism is brought out graphically if we write

\[ T = \omega^\alpha + a = \omega^\alpha (\omega^{\alpha + \alpha})^{(+ \beta)} + (\omega^{\alpha + \beta}) + a \]

COROLLARY. An ordered group \( H \) whose order type \( T \) is \( R \) \( \times \) is trivial or the infinite cyclic group.

For its regular representation is a subgroup of \( G(T) \).

In contrast to Theorem 5 we have

**Theorem 6.** If \( T \) and \( T' \) are \( R \) \( \times \) groups, there is an order-preserving map \( f \) of \( T \) into \( T' \) iff the appropriate condition in the diagram is satisfied:

<table>
<thead>
<tr>
<th>( T' ) of type</th>
<th>12</th>
<th>34</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) of type</td>
<td>12</td>
<td>( \lambda &lt; \lambda' )</td>
<td>( \lambda &lt; \lambda' )</td>
</tr>
<tr>
<td>34</td>
<td>( \lambda &lt; \lambda' )</td>
<td>( \lambda &lt; \lambda' )</td>
<td>( \alpha &lt; \alpha' )</td>
</tr>
<tr>
<td>0</td>
<td>( \lambda &lt; \lambda' )</td>
<td>( \alpha &lt; \alpha' )</td>
<td></td>
</tr>
</tbody>
</table>

Here we take \( \lambda = \mu + 1 \) if \( T \) is of type 1 or 3, and \( T = \alpha \) if \( T \) is of type 0.

If \( T' = T \) it is easy to construct \( f \neq id. \) so that \( f(t) \leq t \) for all \( t \in T \). If \( T \) is of type 0, and \( g \neq id. \) so that \( g(t) \geq t \) for all \( t \in T \). If \( T \) is not finite or \( \omega^* \) (compare [2]).

**6. Factorization of \( R \) \( \times \) sets.** We shall find it convenient to reclassify \( R \) \( \times \) sets \( T \) in seven types, and associate with each \( T \) ordinals \( \alpha \) and \( \lambda \) as follows:

- **Group I:** \( T = \omega^\alpha + a; \ \lambda = \mu + 1 \).
  - \( \alpha < \mu \).
  - \( \alpha < \lambda \leq \mu \).
  - \( \lambda \leq \mu \).

- **Group II:** \( T = \sum \{ \lambda_i \} + a; \ \lambda = \lim \lambda_i \).
  - \( \alpha = 0 \).
  - \( \alpha < \lambda \leq \mu \).
  - \( \lambda \leq \mu \).

If \( T \) is an ordinal, we assign it to type 0, and take

\[ a = T, \ \ \lambda = \omega^\alpha. \]

It is desirable to have \( \alpha(T) \neq 0. \) We therefore redefine \( \alpha(T) \) to be \( \omega^\beta \) (of type 4) and \( \omega^\alpha \) (of type 1, \( \alpha(T) = 0 \)). We shall say \( T \) is \( R \) \( \times \) iff \( T \) is \( R \) \( \times \) and of type \( i (i = 0, 1, ..., 6) \). We may regard types 1, 2, 3 as degenerate cases of types 4, 5, 6, respectively.

**Theorem 7.** Every \( R \) \( \times \) set is of one and only one of the seven types. Its expression in the form (B) is unique (modulo any final segment of \( \Sigma \) if \( T \) is of type 5 or 6). In particular, \( \alpha \) and \( \lambda \) are well-defined functions of \( T \).

This follows from Theorems 2, 3, 4.

**Theorem 8.** Let \( T = PB \) be a non-trivial factorization of an ordered set \( T \neq 0 \). Then \( T \) is \( R \) \( \times \) iff \( P \) is an ordinal and \( Q \) is \( R \) \( \times \) (\( 0 = 1, ..., 6) \).

Proof. In one direction this needs only a direct verification. Conversely, let \( P \) and \( Q \) be ordered sets. Then the Dedekind cuts of \( PQ \) are precisely the pairs \( (PA, PB) \) where \( (A, B) \) is a cut of \( Q \), and the pairs \( (PL(a) + C, D + PB(a)) \), where \( a \in Q \) and \( (C, D) \) is a cut of \( P \).

Suppose now \( T = PQ \) is \( R \) \( \times \). If \( C, D \) is a cut of type \( L \) or \( G \) in \( P \), then so is \( (PL(a) + C, D + PB(a)) \) in \( T \). Such cuts exist since \( P \neq 0, 1; \) \( Q \neq 0. \) Thus \( P \) is \( R \) \( \times \). \( P \) has no first term, then every cut \( (PA, PB) \) in \( T \) is of type \( L \) or \( G \). Such cuts exist, since \( P \neq 0, 1 \). So \( P \) is an ordinal.

If the cut \( (A, B) \) of \( Q \) is of type \( G \), then so is \( (PA, PB) \). If \( (A, B) \) is of type \( L \), then \( (PA, PB) \) is of type \( L \) or \( G \). So \( Q \) is \( R \) \( \times \). That \( Q \) is of the same type as \( T \) follows from Theorem 7 and the first part of this theorem.
6.1. We first examine the factorization of $R_1$, $2$, $3$, $4$ types.

**Theorem 9.** Let $T$ be of type $1$ or $2$ or $3$ or $4$ be given by
\[ \omega^\alpha + \alpha, \quad \omega^\alpha + \beta, \quad \omega^\alpha + \sum (\lambda_\alpha, \lambda_\beta) + \beta, \]
respectively, where $0 < \alpha < \beta < \epsilon$, and $\delta = (\lambda_\alpha, \lambda_\beta, r)$; $\alpha = (a_1, a_2, s)$. Then $T$ has a unique irreducible right factor $S$, given by
\[ \omega^\alpha, \quad \omega^\alpha \cdot \omega^\beta \cdot \beta + 1, \quad \omega^\alpha + \omega^\beta, \quad \sum (\lambda_\alpha, \lambda_\beta) + \delta + 1, \]
respectively, where $\alpha = (-\beta + \delta, a_1, a_2, r)$; $\omega^\alpha = (-\omega + a_1, a_2, s)$. We have $T = \gamma \delta$ if
\[ \gamma = \omega^\alpha + \alpha \quad \text{for some } m > 0, \quad \gamma = \beta, \quad \gamma = \beta, \quad \text{respectively.} \]

**Corollary.** The irreducible $R_1$, $2$, $3$, $4$ sets are precisely those of the forms
\[ \omega^\alpha, \quad \omega \cdot (\omega^\alpha \cdot \omega^\beta + 1), \quad \omega^\alpha \cdot \omega^\beta + 1, \quad \sum (\lambda_\alpha, \lambda_\beta) + \omega^\alpha \cdot \omega^\beta + 1, \]
respectively, where $\omega > 0$.

**Proof.** If $T$ is of type $1$ we assume, as we may, that $\lambda_\alpha > \gamma$. We may check directly that $T = S \delta$ whenever $\gamma$ and $S$ are as given, and that $S$ is of the form given in the corollary. If $T = S \delta$ with $S$ irreducible, then $S$ is of the same type as $T$, so $S$ has a right factor of the form given in the corollary, so is itself of this form. We may then check directly that $S' = S$ and $S$ is of the form given for $\gamma$. The same argument applied to $S$ shows that $S$ is irreducible.

6.2. $R_1$ and $6$ types do not in general have irreducible right factors. We therefore introduce two weaker notions:

**Definition.** We say an $R_1$ type is **reduced** if for every right factor $S$ of $T$ we have $\alpha(S) = \alpha(T)$ and $\lambda(S) = \lambda(T)$. We say that $T$ is (right) **quasi-irreducible** if every right factor $S$ of $T$ is at the same time a right multiple.

If $S$ is a right factor of $T$, $\alpha(S) \leq \alpha(T)$ and $\lambda(S) \leq \lambda(T)$. It follows that $T$ is reduced if it is q.i.

We shall need two other notions:

**Definition.** We define ordinal functions $\tau$ and $\theta$ of $T$ by setting $\lambda(T) = \tau + \omega^\theta$ in normal form (i.e. $\tau = 0$ or $\omega > 0$). Note that for $R_1$, $2, 3, 4$ types $\theta \neq 0$, since $\lambda$ is the second kind.

**Definition.** We say that an increasing sequence $A = (\lambda(n))$ of ordinals is **left step-periodic** if for some step $a$ and period $p \neq 0$ we have $\lambda(n + p) = a + \lambda(n)$ for all sufficiently large $n$. It may be shown that the set of periods is a semi-ideal in the set of positive integers, so that $A$ has a fundamental period. We say $A$ is **step-periodic** if it is both left and right step-periodic (defined in the corresponding manner).

6.3. Suppose now $T = \sum (\lambda_\alpha, \lambda_\beta) + 1$, of $\delta = (\lambda_\alpha, \lambda_\beta, r)$ iff for some $e$
\[ \beta = \omega^\alpha \cdot \omega^\beta + (\lambda_\alpha, \lambda_\beta, N - 1), \quad \delta = \delta_\gamma. \]

**Lemma 1.** The left factors of $\delta$ or $T$ are precisely those $\beta$ such that either $\delta \beta < \delta$ or $\beta$ is an end-segment of $\delta$ or $T$.

**Proof.** If $\delta$ or $T = \beta U$ and $U$ is of the second kind, then $\delta \beta < \delta_\gamma$ and for every such $\beta$ we have $\beta U = \gamma U$, where
\[ U = \sum (-\delta, \lambda_\alpha, \lambda_\beta) \]
the sum being finite or infinite.

If $U$ is of the first kind, and $U = \sum (\lambda_\gamma, m_\delta)$, $\beta = \omega^\alpha \cdot \omega^\beta + 1$, then
\[ \delta \beta = \lambda_\gamma \lambda_\delta, \quad \text{for some } \gamma \geq \lambda_\delta. \]

Since $\delta < \delta + 1$, this is in normal form as it stands, so that $\delta$ satisfies (1) for $e = m_\delta$ and some $N$. Also $\delta \beta = \lambda_\gamma \lambda_\delta$. Conversely, with $\beta$ as in (1), we have $\beta U = T$, where
\[ U = \sum (-\delta, \lambda_\gamma, \lambda_\delta) + 1, \quad \delta = \delta_\gamma. \]

This lemma suggests that $R_1$ types may be regarded as "infinite long" ordinals, and that by continuing to "peel off" right factors we can obtain an "infinite" normal factorization (a.f.) of ordinals (see [3], p. 340, or [11], p. 88).

**Definition.** Let $\Pi = (\tau_1, \tau_2, ..., \tau_N)$ be a sequence of non-zero ordinals and $T$ of type $4$. Then we write $T \sim \Pi$ if every final segment $\gamma$ of $T$ is a final segment of every product $\tau_\gamma \tau_{\gamma'}$ for $\gamma < \gamma'$ some $N = N(\gamma)$.

Given ordinals $\gamma = (a_1, a_2, a_3, r)$ and $\delta = (\lambda_\delta, a_4, s)$, we say $\gamma$ is strictly longer than $\delta$ iff $r > s$ and $a_1 = a_4$ and $\gamma_1 = \delta_1$ for $0 < i < s$.

**Lemma 2.** Given $a, \beta > 0$, $\alpha$ is an end-segment of $a \beta$ iff $\alpha$ is of the first kind, and $a \beta$ is strictly longer than $a \alpha$ iff $\alpha$ is of the second kind.

**Lemma 3.** Given a sequence $\Pi$ there exists a $T \sim \Pi$ if
a) The $\alpha_n$ are not ultimately all finite;
b) only finitely many $\alpha_n$ are of the second kind.

If $T$ exists, it is unique.

**Proof.** The first part follows from Lemma 2, and the second from the fact that $T$ is determined by any coinitial sequence of end-segments.
Lemma 4. If $T \sim H$, then every product $\pi(m) = \pi_1 \pi_2 \ldots \pi_n$ is an end-segment of $T$ iff

- $\pi_i (i > 1)$ is of the first kind.

Proof. Let $\sigma$ be an end-segment of $T$ of length greater than length $\pi(m)$. Suppose $\sigma$ is a final segment of $\pi(n)$, where $n > m$. Then $\pi(m)$ is an end-segment of $\pi(n) = \pi(m) \pi_{m+1} \pi_{m+2} \ldots \pi_{n-1} \pi_n$ by an iteration of Lemma 2, hence of $\sigma$, hence of $T$. The converse follows from Lemma 2.

Suppose now $T$ satisfies

- $\pi_i (i > 1)$ is either finite $> 1$ or infinite of the first kind and irreducible;
- $\pi_i$ satisfies $d$ or is of the form $\omega^\alpha (\alpha > 0)$;
- No two adjacent $\pi_i$ are both finite.

Then we say $T$ (or $\sigma$) has the amalgamated normal factorisation (anf) $\pi_1 \pi_2 \ldots \pi_k \pi_{k+1} \ldots \pi_n$ (in $T \sim H$). Note that conditions $a$, $b$, $c$ are satisfied.

Lemma 5. Every $\sigma = (\lambda_1, \lambda_2, \lambda_n)$ has the unique anf

$$\sigma = \omega^{\alpha_0} \xi_1 \xi_2 \ldots \xi_n \xi_{mn},$$

where $\alpha_0 = \omega^{\alpha_0} + 1$, $\xi(i) = -\lambda_{i-1} + \lambda_i$, and we omit $\xi(i)$ if $\lambda_i = 1$, and $\omega^\alpha$ if $\lambda_0 = 0$.

Proof. It is clear that this is an anf. It may be obtained from the n.f. by amalgamating adjacent factors which are both finite or both of the second kind. Suppose now $\sigma = \alpha \pi_i \pi_{i+1} \ldots \pi_n$ is any anf. On refining each $\pi_i$ to its own n.f. we clearly obtain an n.f. for $\sigma$. On amalgamating, we return to $\pi_i \pi_{i+1} \ldots \pi_n$, but also we must obtain (4), since the n.f. is unique. Thus the two anfs are in fact the same.

Theorem 11. Each $\beta \lambda_4$ type $T$ has a unique normal factorisation: that is, there is a unique $H$ such that $T \sim H$ and

- each $\pi_i$ is irreducible;
- no $\pi_i$ of the second kind follows one of the first kind;
- $\pi_1 \ldots \pi_n$ which are both of the second kind, or both finite, occur in non-increasing order.

Proof. Note first that it is legitimate in an expression

$$T \sim \pi(1) \pi(2) \ldots$$

both to refine individual terms—thus $T \sim \Sigma$, where

$$\pi(i) = \sigma(\pi(i-1)) \sigma(\pi(i-2)) \ldots \sigma(\pi(1)),$$

and to amalgamate blocks of adjacent terms—thus $T \sim \Sigma$ where

$$\pi(i) = \pi(1) \pi(n-1) \ldots \pi(n-1).$$

We may obtain one n.f. for $T$ by refining the terms of the anf to their own n.f. Unicity is proved by the reverse of the argument used in Lemma 2.

In terms of the type of factorisation introduced here for $\beta \lambda_4$ types, and also of their normal form, we may regard them as the natural generalization of ordinals, and take the irreducible $\beta \lambda_4$ types to be the irreducible ordinals, and those alone. We now consider ordinary factorisation for $\beta \lambda_4$ types.

6.4. Lemma 6. An $\beta \lambda_4$ type $T$ is reduced (iff $\tau(0) = 0 = \alpha(T)$).

Proof. We may choose $\alpha$ such that $\lambda_0 > 0$. Then

- $\alpha(U) = 0$;
- $\lambda(U) = \lim \lambda_{\beta_0} (\lambda(U)) = -\lambda_0 + \lambda_0 + \lambda = -\lambda_0 + \lambda = \alpha$.

iff $\tau = 0$. If $\alpha(T) = 0$, then $T$ is of the first kind, and so is any right factor $U$ so that $\alpha(U) = 0$. 

Lemma 7. T of type 4 is q.i. iff for some β we have $T \sim \beta \beta \beta \cdots$, and $T$ is irreducible iff we may take β irreducible.

Proof. If $T$ is q.i. it is reducible, so of the first kind. Let $q > 1$ be a left factor. Then $T = qT'$, $T' = \sigma T$, and $T = \beta T$ where $β > 1$ is of the first kind. If $β = n_1n_2 \ldots n_a$ in, and then an easy argument from Theorem 10 (iii) shows that $T$ has anf $n_1n_2 \ldots n_a$, $n_1$, $n_2$, $n_3$, $\ldots$ with $n_a$ and $n_1$ amalgamated if they are both finite; Thus $T \sim \beta \beta \beta \cdots$ Conversely, if $T \sim \beta \beta \beta \cdots$ and $U$ is a right factor, then, by Theorem 10 (ii), $U \sim \tau \beta \beta \beta \cdots$ for some right factor $τ$ of $β$ and then $\tau T = U$ by Theorem 10 (iii), so that $T$ is q.i.

If $T$ is irreducible it follows from Theorem 10 (ii) that every right factor $τ$ of $β$ is also a left factor, Lemmas 1 and 3 applied to the anf for $β$ then show that $β$ is the $\alpha$-th power of an irreducible ordinal.

Knowing the structure of reduced, q.i. and irreducible $\mathcal{B} \mathcal{D} 4$ types, and applying Lemma 1 and Theorem 10 (iii), we have

Theorem 12. If $T = \sum (λ_n, l_n)$ is of type 4, exactly one of (i), (ii), (iii) holds:

(i) The following equivalent conditions are satisfied:
   a) $T$ has a unique irreducible right factor.
   b) $T \sim \beta \beta \beta \cdots$ for some irreducible $β$.
   c) The terms $l_n$ are ultimately 1, and for some $N$ (and hence all sufficiently large $N$) $(-λ_N + κ_{N+1})$ is step-periodic of period 1.
   d) $T = \sum (λ_n + κ_{N+1}, 1 + γ) : γy = δ$.

(ii) The following equivalent conditions are satisfied:
   a) $T$ has a finite number $m > 1$ of q.i. right factors, the right factors of each being precisely the others.
   b) $T \sim \beta \beta \beta \cdots$ for some $β > 1$ a power of an irreducible ordinal.
   c) The terms $l_n$ are ultimately periodic of period $p$, and for some $N$ (and hence all sufficiently large $N$) $(-λ_N + κ_{N+1})$ is left step-periodic of period $q$, say, and for no $m$ does $pq_m = l_1 = 1$ for all $n > M$.
   d) $T = \sum \alpha + κ_{N+1} + γ : γ = δ < α$, and we cannot take $e = 1$.

(iii) The following equivalent conditions are satisfied:
   a) $T$ has a countable infinity of distinct reduced right factors, none of them q.i.
   b) The terms in any $\Pi$ such that $T \sim \Pi$ are not ultimately periodic.
   c) For no $N$ is it true that both $(λ_N + κ_{N+1})$ is periodic and $(-λ_N + κ_{N+1})$ is left step-periodic.

Notes. In cases (i) and (ii) $T$ may have reduced right factors which are not q.i. The left factor corresponding to a given right factor of $T$ is unique iff the right factor is not q.i. In (ii) we may take $β$ to have no $α$-th root ($m > 1$), and in this case the q.i. right factors of $T$ are precisely the $S(τ) \sim \tau \beta \beta \beta \cdots$ one for each right factor $τ$ of $β$. We have $S(τ) = γS(τ')$ iff $y = \gamma\alpha'$, where $\alpha' = \beta$. In (iii) the reduced right factors are precisely those $U$ given by (2) for which $λN \geq \tau(T)$. Further properties of the set of factors of $T$ of type 4 are given in section 7.

6.5. We now consider the $\mathcal{B} \mathcal{D} 6$ types. We assume a standard notation

$T = \sum (λ_n, l_n) + α : \lim \lambda_n = λ$

$U = \sum (μ_n, l_n) + γ : \lim \mu_n = μ$

All assertions and formulas are given modulo an appropriate omission or change in some final segment of $Σ$, and the corresponding renumbering. In particular, if $δ < λ$ we may always assume $δ < λ_n$ for all $n$.

Lemma 8. If $T$ is of type 6, then $T = βU$ iff $δ = λ$ and

$U = \sum (-\beta \alpha + λ_n, l_n) + (-\beta \beta + α_1, α_1, τ)$.

Proof. If $U = Y + λ$, $βU = Y' + β$ is in normal form, so that, for some $i$, $βi = α_1 \geq λ \geq β = λ = γ + μ$, a contradiction, since $μ$ is a limit ordinal. Thus $U$ is of the second kind, and $βU = \sum (β \beta + α_1, μ_n) + (β \beta + γ_1, α_1, τ)$.

The result follows at once.

Lemma 9. $T$ of type 6 is reduced iff $τ = 0$.

Proof. Let $τ = 0$ and $T = βU$. Then $βU < β = α'$, $βU < δ$, so that

$μ = \lim (-\beta \alpha + λ_n) - (-\beta \beta + λ) + α = α' + λ$.

Next, $α \geq α'$ so that $α \geq δ$. Thus $β < βU < β + α$, and $β < βU < β + α'$. Since $βU < δ < γ_1$, it is not in normal form, but then $α_1 = β + γ_1 = γ_1$, and $γ = τ$ by Lemma 8.

Suppose conversely $τ = 0$. Then $T = αS$ where

$S = S(T) = \sum (-τ + λ_n, l_n) + (-τ + α_1, α_1, τ)$

and $lim(-τ + λ_n) = -τ + λ = α' + λ$, so that $T$ is not reduced.

Lemma 10. Each $T$ of type 6 has at least one reduced right factor $8$.

If $T = βU$ for some reduced $U$, then $τ \leq βU < λ$, and $τ = α'$.

Proof. We have exhibited one such $S$ in (7). If conversely $T = βU$ for some $β$ and reduced $U$, we have $μ = α '$, say, by Lemma 9. Then $θ + μ = α$, $θ(θ + μ) = φ = 0$. But $φ(θ + μ) = φ = α$, so that $α = 0$ and $θ + μ = τ + α = τ + α'$. Since the right side is in normal form, we must have $θ = τ$ with $θ < δ$. So that $δ \leq τ < τ + α' = τ$. We now assign $T$ to type 6a or 6b according as $δ$ is of the second or first kind respectively.
Lemma 11. If \( U \) is a right factor of \( T \), then \( U \) is of type 6a (6b) iff \( T \) is of type 6a (6b).

6.6. Lemma 12. Suppose now \( T \) is of type 6a. Then the \( S \) of Lemma 10 is unique, and \( \beta S = T \) for every \( \beta \) such that \( \tau < \phi \beta < \lambda \).

Proof. Suppose \( T = \beta U \) for some \( \beta \) and reduced \( U \). By Lemma 10, \( \phi \beta = \tau + \pi \cdot \pi < \omega^\alpha \), so that \( \phi \pi < \theta \). Since \( \theta \) is of the second kind, we may choose \( \phi \theta \) so that \( \phi \pi < \theta < \phi \theta \). Then \( \mu = \phi \theta \), so that \( \mu \mu \geq \mu \) for all \( \mu \).

Thus \( \lambda = \phi \beta + \mu \mu = \tau + \pi + \mu \mu = \tau + \mu \).

Next, since \( U \) is of type 6, \( \phi \gamma U > \theta \), so that \( \phi \gamma U = \tau + \pi + \gamma \). Together with Lemma 8 this shows that \( U = S = \beta S(T) \), given by (7).

Conversely if \( \tau < \phi \beta < \lambda \) and \( \beta = \tau + \pi \) with \( \phi \pi < \theta \), then we may write \( \lambda = \tau + \pi + \delta \) with \( \phi \delta \geq \phi \mu \), so that if \( \beta = U \), then

\[
\mu = \tau + \pi + (\delta - \lambda) = \tau + \pi + \delta = \tau + \delta = \lambda,
\]

and \( \gamma = \tau + \pi + (\delta - \mu) = \mu \) similarly. Together with Lemma 8 this shows \( U = T \).

Theorem 13. An \( R \) \( 6a \) type \( T \) has a unique irreducible right factor \( S \), given by (7). We have \( T = \beta S \) iff \( \tau < \phi \beta < \lambda \).

Proof. By Lemmas 10 and 12 there exists a \( \beta \) and a unique reduced \( S \) such that \( T = \beta S \). If \( S \) is a right factor of \( T \), it is reduced. Since \( S \) is also a right factor of \( T \), \( S = S(T) \) by Lemma 13. Thus \( S \) is irreducible.

The second part follows from Lemmas 10 and 12.

Corollary. The following conditions on \( T \) of type 6a are equivalent:
(i) \( T \) is irreducible.
(ii) \( T \) is reduced.
(iii) \( \tau(T) = 0 \).
(iv) \( T = \Sigma(\lambda_\alpha, \alpha) + \omega^\alpha \delta_\alpha \) where \( \sigma \geq 1 \), \( \delta \geq 1 \), and \( \lim \lambda_\alpha = \omega^\sigma \).

The simplest example is

\[
T = \sum \omega^\alpha \omega^\alpha \omega^\alpha + \omega^\omega + \omega + \omega^\omega.
\]

6.7. Suppose now \( S = \Sigma(\lambda_\alpha, \alpha) \) is reduced of type 6b. Set \( \phi = \xi + 1 \). Then \( \lambda_\alpha = \omega^\alpha \delta_\alpha + \delta \) in normal form, where \( \{ \omega^\alpha \} \) is non-decreasing and \( \lim \omega^\alpha = \omega \). We write \( S = (\xi, \omega, \delta_\alpha, \lambda_\alpha, \gamma) \) as standard notation. By Lemma 8 we have \( S = \beta U \) for some \( \beta \) and \( U \) iff \( \phi \beta \leq \xi \) and

\[
U = S(u) = (\xi, -\xi, \omega, \delta, \lambda, \gamma)
\]

where \( \phi \beta = \omega^u \delta (u \geq 0) \) in normal form.

It follows that the set of all right factors and reduced right multiples of \( S \) is precisely the set

\[
E(S) = \{ S(u) : -\infty < u < \infty \}.
\]

If no two \( S(u) \) are equal, then \( E(S) \) is infinite of type \( \omega^\alpha + \omega \), each member a right factor of all its predecessors, and \( S \) is not \( q.i. \). If for some \( v \neq u \) we have \( S(v) = S(U) \), then

\[
\beta S(v + \beta) = S(u) = S(v) = \beta S(v + \beta),
\]

where \( \phi \beta = \omega^\alpha \delta \). Since we may left-cancel \( \beta \), we easily see that \( E(S) \) is infinite. If its cardinal is \( \iota \), then \( S(u) = \beta S(\gamma) \) whenever \( \phi \beta = \omega^\alpha \delta + \delta \), where \( \alpha \gamma = \alpha - \beta \) (mod \( \iota \)), so that \( S \) is \( q.i. \). In this situation \( \iota \) is the smallest integer \( m \) such that for some \( d \geq 0 \) we have

\[
u(u + d) = \nu(n + m), \quad \delta(u + d) = \delta(n), \quad l(u + d) = l(n)
\]

for all \( u \) (where we write \( u(u) \) for \( u(n) \), etc.). Thus \( u(n) \) is step-periodic, and \( (\nu_\alpha) \) and \( (\delta_\alpha) \) are periodic, and if their fundamental periods are \( p, q, r \), and \( m \) is the fundamental step of \( u(n) \), then

\[
t = \min(p, q, r).
\]

We may take it that

\[
u(u^{d+1} - 1) \leq \nu(u) \leq 4d + 1,
\]

where \( \nu(u + s) = \delta(n + s) = l(n), \quad n \leq d \leq d + 1; \quad m \geq 0 \).

On setting \( u(0) = f, \quad u(0) = u(0), \) we find

Lemma 13. \( S \) of type 6b is \( q.i. \) iff \( \lambda_\alpha \) is periodic and \( (\lambda_\alpha) \) is left-step-periodic. In this case we may write

\[
S = S(t - 1) + \sum \omega^\alpha \delta + \gamma,
\]

where \( \pi(n) = \omega^\pi(n), \quad \pi \gamma > \xi > \phi \beta \).

In terms of the discussion above, we have

\[
\delta = \omega^\pi \delta + \delta(0), \quad d \geq 1.
\]

The right factors of \( S \) are just the sets \( S(k), \quad 0 \leq k < t \).

In particular, \( S \) is irreducible iff \( t = 1 \), so that \( m = 1 \) and \( p = d \).

Then \( u(d) = u(0) + 1 < 2d \), by (9). But the \( u(n) \) are non-decreasing, so that
0 = u(0) = ... = u(d−1) < u(d) = 1. Thus u(su + s) = 0 (0 ≤ s ≤ d−1).

In this case we may simplify and renumber in (9) to obtain

\[ S = \sum_{n > 0} \omega^{n\alpha} + \gamma, \quad \varphi < \gamma < \varphi. \]

Knowing the structure of reduced, q. i., and irreducible \( E_{6b} \) types, and applying Lemma 10, we have

**Theorem 14.** If \( T \) is of type 6b we may write \( \lambda = \tau + \omega^{n\alpha} + \delta \), where \( (\alpha, \omega) \) is non-decreasing with limit \( \alpha \). Subject to appropriate changes and eliminations of a finite number of the \( \lambda \)'s, and the appropriate renumbering, exactly one of the following holds:

(i) \( T \) has a unique irreducible right factor \( S \), given by (7). Equivalently, \( \lambda = \tau + \omega^{n\alpha} + \delta \) is left step-periodic of fundamental period \( p \), and a corresponding step of the form \( \alpha \), and \( \lambda \) is periodic of period dividing \( p \). We have \( T = \beta S \) if \( \tau + \varphi < \lambda \).

(ii) \( T \) has a finite number of \( q. i. \) right factors, the factors of which being precisely the others. Equivalently \( \lambda = \tau + \omega^{n\alpha} + \delta \) is left step-periodic and \( \lambda \) is periodic, and if \( \alpha \) is their least common period, then \( \lambda \) has a step \( \omega^{n\alpha} \) corresponding to the period \( \alpha \). The \( q. i. \) right factors are \( S(\alpha) \) (0 ≤ \( j < \alpha \)) given by (10), where \( \gamma = -\tau + \omega^{n\alpha} + \delta \), and \( T = \beta S(\alpha) \) if \( \varphi = \alpha \omega^{n\alpha} + \gamma \) is normal form. Also \( S(\alpha) = \beta S(\alpha) \) if \( \varphi = \alpha \omega^{n\alpha} + \gamma \) is normal form.

(iii) \( T \) has an infinite chain \( \{ S(u): u > 0 \} \) of reduced right factors, not \( q. i. \), the right factors of each being precisely its successors. \( S(u) \) is given by (8), where \( \gamma = -\tau + \omega^{n\alpha} + \delta \). Equivalently, \( \lambda \) is not periodic, or \( \lambda = \tau + \omega^{n\alpha} + \delta \) is not step-periodic for any \( \lambda = \beta S(u) \) if \( \varphi = \alpha \omega^{n\alpha} + \gamma \) is normal form.

**Summary.** \( E_{1, 2, 3, 5, 6a} \) types have a unique irreducible right factor. The corresponding left factor is unique for types 2 and 5. \( E_{6b} \) and 6b types have either

(i) a unique irreducible right factor, or

(ii) a finite number of \( q. i. \) right factors, the right factors of each being just the others, or

(iii) a countable infinity of reduced right factors, none of them \( q. i. \). The corresponding left factor is unique only for cases (ii) of type 4.

Condition (iii) holds in particular if \( \varphi \omega^{n\alpha} \) is not ultimately constant. We may think of such "bad" order types as a countable sum of rapidly increasing ordinals, just as the "bad" Liouville transcendentals are an infinite sum of rapidly decreasing rationals.

**7. Arithmetic of \( E_{7} \) Types.** We first extend the definition of \( \varphi, \psi \) to arbitrary \( E_{7} \) types by setting

\[ \varphi T = \max \{ \lambda T, \varphi_{T} \}, \quad \psi T = \min \{ \lambda T, \psi_{T} \}. \]

This agrees with the definition of \( E_{7} \) 0 types. We refer to types 4j and 6f (j = (i), (ii), (iii)) in accordance with Theorems 12 and 14.

**Theorem 15.** The following relations hold whenever the expressions make sense (i.e., \( C, D \) are any \( E_{7} \) types; \( \gamma, \delta \) are of type 0):

1. \( \varphi(C + \gamma) = \max \{ \varphi C, \varphi \gamma \} \)
2. \( \psi(C + \gamma) = \psi \gamma + \psi C \)
3. \( \varphi(C + \gamma) = \psi \gamma (\gamma \neq 0) \)
4. \( \varphi(C + \gamma) = \varphi \gamma + \varphi C \) (of the first kind)
5. \( \psi(C + \gamma) = \psi \gamma + \psi C \) (of the second kind)

Cancellations:

1. \( C + \gamma = D + \gamma \) iff \( C \gamma = D \gamma \) for some \( \varepsilon \) such that \( \varepsilon < \gamma \).
2. \( C + \gamma = C + \delta \) iff \( C \gamma \gamma \) is of type 1 and \( \delta = \omega \gamma + \alpha(C + \gamma) \gamma \), where \( m > 0 \) if \( \alpha(C + \gamma) \neq \mu(C) \).

3. \( \gamma = \gamma D \), \( D \neq D \) iff \( \gamma = 0 \).
4. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = 0 \).
5. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = \gamma (\gamma \neq 0) \).
6. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = \gamma (\gamma \neq 0) \).
7. \( \gamma = \gamma D \), \( D \neq D \) iff \( \gamma = 0 \).
8. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = \gamma (\gamma \neq 0) \).
9. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = \gamma (\gamma \neq 0) \).
10. \( \gamma = \alpha C \), \( C \neq \gamma \) iff \( \gamma = \gamma (\gamma \neq 0) \).

**7.2.** The sets \( L(T) \) and \( R(T) \) of left and right factors of \( E_{7} \) set \( T \) have a well-defined structure which we now consider. For given ordered sets \( A, B \), let us write \( A < B \) iff \( A \) is a left divisor of \( B \). Then \( < \) is a transitive relation, but need not be a partial order. Let us say a class \( \mathcal{A} \) of order types is almost linearly ordered by \( < \) iff \( \mathcal{A} \) is a partial order on \( \mathcal{A} \) such that whenever \( A, B \in \mathcal{A} \) are incomparable, then for some \( m, n > 0 \) and \( C \in \mathcal{A} \) we have \( A = C m, B = C n \). We make the corresponding definitions for \( > \).
4. $L_2(T)$ is almost linearly ordered by $<_j$.
5. Every member of $L_2(T)$ has the left factor $\omega^\varepsilon$.

Corollary. $(L_2(T), <_j)$ is a distributive lattice, and $(L(T), <_j)$ is a lattice. It is distributive iff $\forall T \leq 1$.

The proof is by consideration of cases, using property 4. If $\forall T \geq 2$, the lattice is not even modular, since then $a = 2$, $b = \omega$, $c = \omega + 1$ are left factors of $T$ with $a <_j b$, and $a \lor (c \land b) = 2$ while $(a \lor c) \land b = \omega$.

Compare [1], p. 81.

Theorem 16B. If $T$ is Rj we have $R(T) = R_0(T) \lor R_2(T)$, where 0. $R_0(T)$ is the set of right factors of $T$ corresponding to the left factors in $L_2(T)$ ($i = 1, 2$).
1. $R_0(T) <_j R_2(T)$.
2. $R_0(T)$ is empty iff $T$ is of the first kind, and otherwise is a finite set of Rj sets of the same type and kind as $T$, and is linearly ordered by $<_j$.
3. $R_2(T)$ is a set of Rj sets which is countably infinite iff $T$ is of type 4(ii) or 0b(iii), and otherwise finite.
4. $[R_2(T), <_j]$ is
   a) a linearly ordered set if $T$ is of type 3, 6,
   b) an (almost) linearly ordered set followed by a finite loop iff $T$ is of type 4(ii) or 0b(ii),
   c) an almost linearly ordered set in all other cases.

Corollary. $(L_0(T), <_j)$ is a lattice iff $T$ is not of type 4 (ii) or 0b (ii), and is then moreover distributive.

The "almost" in both parts of Theorem 16 can be omitted iff one of the following holds:
(i) $T$ is of type 0(4) and every $a_i(\omega_0)$ is a prime power (where the $a_i$ are the coefficients of $\omega = a(T)$);
(ii) $T$ is of type 2 or 5 and $a_i$ is a prime power for every $i$ such that $a_i < \omega$.

7.3. Goldbach's hypothesis may be stated in the form: any $J$-set is expressible as a sum of $\leq 3$ irreducible sets. Here a $J$-set is one whose only cuts are of type $J$. It is of course either finite or one of $\omega^\omega$, $\omega^\omega + \omega$.

For Rj sets we have

Theorem 17. Suppose $T$ is Rj not of type 1 or 4, and $a = a(T) = (a_0, a_1, r)$. Then $T$ is expressible as a finite sum of irreducible sets iff one of the following relations holds:
(i) $T = \sum (a_0, 1) + \omega^\varepsilon$,
(ii) $a_0 = \omega^\varepsilon$ but not $T = \sum (\lambda_0, \lambda_0) + \omega^\varepsilon$: $\delta = \omega^\varepsilon$, unless (i).
(iii) $a_0 = 0$: $a_0 \neq 2$.
(iv) $a_0 = 0$: $a_0 = 2$ and either $T = 2$ or $a_0 = \omega^\varepsilon (i \geq 0)$.

References

Reçu par la Rédaction le 21.1.1962