Infinite Boolean polynomials I*

by

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Introduction. This work treats Boolean polynomials (on a fixed set of variables) in which the operations of join, $\bigvee$, and meet, $\bigwedge$, may apply to sets of arbitrary high powers. Thus if X is any set of such polynomials then $\bigvee X$ and $\bigwedge X$ are Boolean polynomials as well. If the variables of such a polynomial are given values in a complete Boolean algebra, then the polynomial has a value in this Boolean algebra, which is defined in a natural way, and two polynomials which always yield the same values for the same values of the variables are identified. The main result of part I is that the Boolean polynomials on $\aleph_0$ variables do not constitute a set.

In Sec. 0 we outline how these Boolean polynomials and the infinite operations on them can be defined, using the usual tools of set theory, and avoiding the axiom of choice. In Sec. 1 and Sec. 2 certain properties of these Boolean polynomials are established, and in Sec. 3 and Sec. 4 a construction is carried out, which, using the results of Sec. 1 and Sec. 2, implies that the Boolean polynomials on $\aleph_0$ variables do not form a set. From this it follows that there is no completely free Boolean algebra on $\aleph_0$ variables. Such an algebra, if it existed, would be a complete Boolean algebra, generated (in the general sense) by $\aleph_0$ generators, and free for all infinite Boolean operations (i.e., every mapping of the free generators into any other complete Boolean algebra could be extended to a complete homomorphism). The result gives a negative answer to a question posed by Rieger [7]. Moreover, the same is shown to hold for Boolean polynomials with δ variables, where δ is an infinite regular cardinal, even if we add the (δ, ∞) distributive law (i.e., $\bigwedge_{i\in I}\bigvee_{j\in J_i} b_j = \bigvee_{f\in \Pi J_i}\bigwedge_{i\in I} b_{f(i)}$ whenever I is of power $<\delta$). These results were also obtained, independently and about the same time, using quite a different approach, by A. Hales [5].

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Using the results we get also that the α-free Boolean algebra (by which we mean the Boolean algebra which is free for joins and meets of less than α elements) on β generators, where α and β are infinite, is at least of power $\alpha+\beta$. An upper bound can be obtained by direct calculation, and, under the assumption of the general continuum hypothesis, the power of this Boolean algebra turns out to be α if $\alpha > \beta$ and α is regular.

In the second part further results, based on those of the first part and concerning Boolean polynomials, will be presented. In particular, the construction used in the first part will be employed to prove a general hierarchy theorem for Boolean polynomials. The Lebesgue theorem concerning the hierarchy of Borel sets turns out to be a special case of this hierarchy theorem for Boolean polynomials.

Preliminaries. Boolean algebras will be denoted by $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, etc. "B. a." stands for "Boolean algebra".

B.a.'s are conceived here as algebras of the form $\langle B, \vee, \wedge, {}^- \rangle$, where $\wedge$, $\vee$, and ${}^-$, are the join, meet, and complement operations, respectively. "0" and "1" will denote the minimal and maximal elements of the B.a., respectively. The partial order of the B.a. is denoted by "$\leqslant$", i.e., $a \leqslant b$ if $a \vee b = b$ and $a < b$ if $a \leqslant b$ and $a \neq b$. "$a-b$" stands for "$a \wedge \bar{b}$". If $\mathfrak{B} = \langle B, \vee, \wedge, {}^- \rangle$ then we put $|\mathfrak{B}| = B$.

If $X \subseteq |\mathfrak{B}|$ and the least upper bound of the elements of X exists, then it will be denoted by "$\bigvee X$" and referred to as the join of X. Similarly, the greatest lower bound of X, if it exists, is denoted by "$\bigwedge X$" and is referred to as the meet of X. Self-explanatory notations, such as $\bigvee_{i \in I} a_i$, $\bigvee_{i=1}^{\infty} a_i$, etc., will be used as well.

Ordinals will be denoted by "α", "β", "γ", "δ", "λ", "μ", "ν" with or without subscripts. Cardinals are identified with their initial ordinals and natural numbers with finite ordinals. The power of K is denoted by $\overline{\overline{K}}$.

"Iff" means if and only if.

By α-completeness of a B.a. we mean the existence of $\bigvee X$ and $\bigwedge X$ whenever $\overline{\overline{X}} < \alpha$. Thus a B.a. is complete iff it is α-complete for all α. An α-complete B.a. is referred to as an α-B.a. A B.a. is (α, β)-distributive, where α and β are infinite cardinals, if the equality $\bigwedge_{i \in I} \bigvee_{j \in J_i} a_j = \bigvee_{f \in \Pi J_i} \bigwedge_{i \in I} a_{f(i)}$ holds whenever $\overline{\overline{I}} < \alpha$, $\overline{\overline{J_i}} < \beta$ for all $i \in I$, and all the joins and meets on both sides exist. (ΠJ_i is the cartesian product of the J_i's as $i \in I$.) A B.a. is (α, ∞)-*distributive* if it is (α, β) distributive for all β.

Similarly we use the other well-known α-concepts to mean that the property in question holds for all joins and meets of less than α elements. Thus an α-*homomorphism* between two α-B.a.'s is a homomorphism which preserves joins and meets of less than α elements. An α *subalgebra* of an α-B.a. $\mathfrak{B}$ is a subalgebra $\mathfrak{B}'$ such that for all $X \subseteq |\mathfrak{B}'|$, $\overline{\overline{X}} < \alpha$ implies that $\bigvee X, \bigwedge X \in |\mathfrak{B}'|$, where $\bigvee X$ and $\bigwedge X$ are the join and meet of X in $\mathfrak{B}$. An α-B.a. is α-*generated* by a set X of its elements if every α-subalgebra containing all members of X coincides with it. The notion of the α-free B.a. on α generators is defined accordingly. A complete B.a. is ∞-generated by X if for some α it is α-generated by X.

We assume all the well-known properties of these concepts (see [1] and [8]), as well as the existence and well-known properties of the normal completion (or completion by cuts) of a B.a. (see [1], p. 161). In particular, we mention the fact that the normal completion of an (α, ∞)-distributive B.a. is also (α, ∞)-distributive. This, however, is not generally true for (α, β)-distributive B.a.'s (see [6]).

§ 0. For our purposes it is convenient to use a set theory which has proper classes, e.g., Bernays set theory. There is no loss of generality in this, since with obvious modifications the whole work can be carried out in Zermelo-Fraenkel set theory without affecting the results.

One can imagine a general theory of complete B.a.'s in which the operations are $\bigvee$, $\bigwedge$, and ${}^-$; $\bigvee$ and $\bigwedge$ operate on arbitrary sets of terms, yielding new terms, and ${}^-$ operates on single terms. In this way, starting from a fixed set of variables $x_0, \ldots, x_\lambda, \ldots, \lambda < \delta$, we get a class of Boolean terms, BT, which has the following properties (these are all the properties which we need):

(0.1) $x_\lambda \in \mathrm{BT}$ for all $\lambda < \delta$.

(0.2) If $X \subseteq \mathrm{BT}$ and X is a set, then $\bigvee X, \bigwedge X \in \mathrm{BT}$, and if $f \in \mathrm{BT}$ then $\bar{f} \in \mathrm{BT}$ (X may be empty).

(0.3) BT is the smallest class satisfying (0.1) and (0.2).

(0.4) For every member f of BT exactly one of the following possibilities holds: (i) For some unique $\lambda < \delta$ $f = x_\lambda$; (ii) For some unique set X, $f = \bigvee X$; (iii) For some unique set X, $f = \bigwedge X$; (iv) For some unique term g, $f = \bar{g}$.

A concrete definition of BT which will imply (0.1)-(0.4) can be easily given. For instance, put $x_\lambda = \lambda+1$, and define $\bigvee X$ to be the ordered pair $\langle X, 0\rangle$, $\bigwedge X$ to be $\langle X, 1\rangle$ and $\bar{f}$ to be $\langle f, 2\rangle$. Then define BT to be the smallest class satisfying (0.1) and (0.2). It is easily seen that all the required properties hold.

DEFINITION 0.1. $\mathrm{BT}_0 = \{x_0, \ldots, x_\lambda, \ldots\}_{\lambda<\delta}$. BT_α, where $\alpha > 0$, is the set of all Boolean terms which are of one of the forms: $f, \bar{f}, \bigvee X, \bigwedge X$, where $f \in \bigcup_{\beta<\alpha} \mathrm{BT}_\beta$ and $X \subseteq \bigcup_{\beta<\alpha} \mathrm{BT}_\beta$.

It follows that every BT_α is a set and $\mathrm{BT} = \bigcup_\alpha \mathrm{BT}_\alpha$, where α varies over all ordinals. Roughly speaking, BT_α is the set of all Boolean terms in the construction of which no more than α iterations of the operations are used.

DEFINITION 0.2. Let $\mathfrak{B}$ be complete B.a. and put $b = \langle b_0, \ldots, b_\lambda, \ldots\rangle_{\lambda<\delta}$, where $b_\lambda \in |\mathfrak{B}|$ for $\lambda < \delta$. Let $\bigvee$, $\bigwedge$ and $^-$ be the Boolean operations in $\mathfrak{B}$. Define $f(b; \mathfrak{B})$, for $f \in \mathrm{BT}$, as follows:

(i) $x_\lambda(b; \mathfrak{B}) = b_\lambda$ for $\lambda < \delta$.

(ii) $\bigvee X(b; \mathfrak{B}) = \bigvee \{f(b; \mathfrak{B}) | f \in X\}$.

(iii) $\bigwedge X(b; \mathfrak{B}) = \bigwedge \{f(b; \mathfrak{B}) | f \in X\}$.

(iv) $\bar{g}(b; \mathfrak{B}) = \overline{g(b; \mathfrak{B})}$.

This is a definition by induction on the smallest α such that $f \in \mathrm{BT}_\alpha$. It is legitimate because of (0.4). For every $f \in \mathrm{BT}$, $f(b; \mathfrak{B}) \in |\mathfrak{B}|$. (Note that if $x = \emptyset$ then $\bigvee X(b; \mathfrak{B}) = 0$ and $\bigwedge X(b; \mathfrak{B}) = 1$.)

DEFINITION 0.3. If $f, g \in \mathrm{BT}$ then $f \equiv g$ if for every complete B.a. $\mathfrak{B}$ and every sequence b, of length δ, of members of $\mathfrak{B}$, $f(b; \mathfrak{B}) = g(b; \mathfrak{B})$.

Obviously $\equiv$ is an equivalence relation. It is also a congruence relation in the following sense: if $f \equiv g$ then $\bar{f} \equiv \bar{g}$, and if X and Y are subsets of BT such that for every f in X there is a g in Y for which $f \equiv g$ and vice versa, then $\bigvee X \equiv \bigvee Y$ and $\bigwedge X \equiv \bigwedge Y$.

DEFINITION 0.4. If $f \in \mathrm{BT}$ then $[f] = \bigvee \{g | g \in \mathrm{BT}_{\alpha(f)}$ and $g \equiv f\}$, where $\alpha(f)$ is the smallest α for which there is a g in BT_α such that $g \equiv f$.

DEFINITION 0.5. φ is a Boolean polynomial if, for some f in BT, $\varphi = [f]$.

Thus a Boolean polynomial is, by this definition, a special kind of Boolean term. Obviously, if $f, g \in \mathrm{BT}$ then $f \equiv g$ iff $[f] = [g]$.

"φ", "ψ", "σ", "τ", with or without subscripts, will denote Boolean polynomials. "B.p." will stand for "Boolean polynomial".

DEFINITION 0.6. $\mathrm{BP} = \{\varphi | \varphi$ is a B.p.$\}$.

We pose the following problem: For α an infinite cardinal let BT^α be the set of all Boolean terms in which $\bigvee$ and $\bigwedge$ operate only on sets of power $<\alpha$ (i.e. BT^α is the smallest subclass of BT satisfying: $x_\lambda \in \mathrm{BT}^\alpha$, for $\lambda < \delta$, if $f \in \mathrm{BT}^\alpha$ then $\bar{f} \in \mathrm{BT}^\alpha$, and if $X \subseteq \mathrm{BT}^\alpha$ and $\overline{\overline{X}} < \alpha$ then $\bigvee X$, $\bigwedge X \in \mathrm{BT}^\alpha$). Assume that $f_1 \in \mathrm{BT}^\alpha$, $f_2 \in \mathrm{BT}_\beta$, and $[f_1] = [f_2]$. Does there always exist an f_3 in $\mathrm{BT}^\alpha \cap \mathrm{BT}_\beta$ such that $[f_1] = [f_2] = [f_3]$? If this is not always so, then there exists a B.p. which cannot be represented by a Boolean term which involves operations of power $<\alpha$ and not more than β iterations, but which can be represented in two ways: one involving operations of power $<\alpha$ but more than β iterations, and the other involving not more than β iterations but operations of power $\geqslant \alpha$. Thus a negative answer means that the number of iterations can be essentially reduced at the cost of increase in the powers of the sets to which $\bigvee$ and $\bigwedge$ apply.

DEFINITION 0.7. If X is a set of B.p.'s and f is a B.p. then:

(i) $\bigvee^* X = [\bigvee X]$,

(ii) $\bigwedge^* X = [\bigwedge X]$,

(iii) $\bar{f}^* = [\bar{f}]$.

$\bigvee^*$, $\bigvee^*$, and $^{-*}$, will be referred to as the join, meet, and complement operations for B.p.'s, respectively. Since from now on we deal mostly with B.p.'s, we will omit "*" when treating B.p.'s, and instead of "$\bigvee^*$", "$\bigwedge^*$" and "$^{-*}$" we will use "$\bigvee$", "$\bigwedge$" and "$^-$". For the same reasons "x_λ" will stand for "$[x_\lambda]$". We define $\varphi \vee \psi$ as $\bigvee \{\varphi, \psi\}$ and $\varphi \wedge \psi$ as $\bigwedge \{\varphi, \psi\}$. All our conventions regarding the operations $\bigvee$, $\bigwedge$, $^-$, will be adopted for $\bigvee$, $\bigwedge$, $^-$. Thus $\varphi - \psi = \varphi \wedge \bar{\psi}$, $\bigvee_{\nu<\lambda} \varphi_\nu = \bigvee \{\varphi_\nu | \nu < \lambda\}$, etc. $\mathbf{0}$ is the B.p. $x_0 \wedge \bar{x}_0$ and $\mathbf{1}$ the B.p. $x_0 \vee \bar{x}_0$. Since B.p.'s are Boolean terms, $\varphi(b; \mathfrak{B})$ is defined for every B.p. φ by Definition 0.2. The following properties follow:

(0.5) For all $\lambda < \delta$ $x_\lambda \in \mathrm{BP}$.

(0.6) If X is a subset of BP and $\varphi \in \mathrm{BP}$ then $\bigvee X$, $\bigwedge X$, and $\bar{\varphi}$ belong to BP.

(0.7) BP is the smallest class which satisfies (0.5) and (0.6).

(0.8) If $\mathfrak{B}$ is a complete B.a., $b = \langle b_0, \ldots, b_\lambda, \ldots\rangle_{\lambda<\delta}$ where $b_\lambda \in |\mathfrak{B}|$ for $\lambda < \delta$, then, for every B.p. φ, $\varphi(b; \mathfrak{B}) \in |\mathfrak{B}|$ and we have:

(i) $x_\lambda(b; \mathfrak{B}) = b_\lambda$ for $\lambda < \delta$;

(ii) $\bigvee X(b; \mathfrak{B}) = \bigvee \{\varphi(b; \mathfrak{B}) | \varphi \in X\}$;

(iii) $\bigwedge X(b; \mathfrak{B}) = \bigwedge \{\varphi(b; \mathfrak{B}) | \varphi \in X\}$;

(iv) $\bar{\varphi}(b; \mathfrak{B}) = \overline{\varphi(b; \mathfrak{B})}$,

where $\bigvee$, $\bigwedge$, and $^-$ are the operation in $\mathfrak{B}$.

(0.9) If $\varphi(b; \mathfrak{B}) = \psi(b; \mathfrak{B})$ for all $\mathfrak{B}$ and b then $\varphi = \psi$.

(0.5)-(0.9) are all the properties of B.p.'s which will be required.

Note that if $X = \emptyset$, then $\bigvee X = \mathbf{0}$ and $\bigwedge X = \mathbf{1}$.

The procedure of constructing B.p.'s, as outlined here, follows the main ideas concerning the notion of a polynomial for a class of algebras with some fixed similarity type. The usual procedure is to divide the set of all terms by the congruence relation $\equiv$. Here, however, all the terms form a class and each equivalence class is a proper class; hence if we use this way and wish to treat classes or sets of B.p.'s, we must use a set theory in which there are classes of proper classes. A well-known device is to define the polynomial $[f]$, which is represented by the term f, as the set of all terms of the least rank which are equivalent to f. In our case this would mean defining $[f]$ as $\{g | g \in \mathrm{BT}_{\alpha(f)}$ and $g \equiv f\}$. If we do this and want to define the join of infinitely many B.p.'s, we are forced to use the axiom of choice in order to choose for every B.p. a representative term. The procedure used here uses the fact that in a B.a. $\bigvee$ is an idempotent operation and avoids the axiom of choice. Thus we can define $[f]$ as a term equivalent to f and such that $f \equiv g$ iff $[f] = [g]$; this can be done whenever the algebras in question have an idempotent operation.

We conclude this section with some remarks and problems concerning Boolean terms. If $f \in \mathrm{BT}^{\alpha}$ (i.e. f is a Boolean term involving operations of powers $< \alpha$), then $f(b;\, \mathfrak{B})$ can be defined as in Definition 0.2 not only when $\mathfrak{B}$ is complete, but also when $\mathfrak{B}$ is α-complete. Hence, one can identify two terms f and g, from BT^{α}, by making the apparently stronger requirement that $f(b;\, \mathfrak{B}) = g(b;\, \mathfrak{B})$ for all α-complete B.a.'s; this, however, yields the same B.p.'s, since, if for some α-complete B.a. $\mathfrak{B}$, $f(b;\, \mathfrak{B}) \neq g(b;\, \mathfrak{B})$, then $f(b;\, \mathfrak{B}') \neq g(b;\, \mathfrak{B}')$ where $\mathfrak{B}'$ is the normal completion of $\mathfrak{B}$. This is so because the joins and meets of sets of elements from $\mathfrak{B}$, if they exist in $\mathfrak{B}$, are the same as in the normal completion of $\mathfrak{B}$, from which follows that $f(b;\, \mathfrak{B}) = f(b;\, \mathfrak{B}')$. The situation is, however, changed if we want to define (β, γ)-distributive B.p.'s. If we restrict ourselves only to BT^{α} then one can either identify terms which yield the same values in all complete (β, γ)-distributive B.a.'s, or one can identify only those terms which yield the same values in all α-complete (β, γ)-distributive B.a.'s. Since the normal completion of a (β, γ)-distributive B.a. need not be (β, γ)-distributive (see [6]), it is not clear whether two terms in BT^{α} which yield the same values in all complete (β, γ)-distributive B.a.'s yield also the same values in all α-complete (β, γ)-distributive B.a.'s. In case of (β, ∞)-distributive B.p.'s there is no problem since the normal completion of a (β, ∞)-distributive B.a. is (β, ∞)-distributive.

Sometimes $f(b;\, \mathfrak{B})$ has a meaning even when $\mathfrak{B}$ is some arbitrary B.a. This happens when all the joins and meets which are involved in the definition of $f(b;\, \mathfrak{B})$ exist. If this happens we will say that $f(b;\, \mathfrak{B})$ is *strongly defined*. A precise definition of this notion will be: $x_{\lambda}(b;\, \mathfrak{B})$ is strongly defined and $x_{\lambda}(b;\, \mathfrak{B}) = b_{\lambda}$, if $g(b;\, \mathfrak{B})$ is strongly defined so is $\bar{g}(b;\, \mathfrak{B})$ and $\bar{g}(b;\, \mathfrak{B}) = \overline{g(b;\, \mathfrak{B})}$, and if $g(b;\, \mathfrak{B})$ is strongly defined for all $g \in X$ and the join (meet) of $\{g(b;\, \mathfrak{B}) \mid g \in X\}$ exists in $\mathfrak{B}$, then $\bigvee X(b;\, \mathfrak{B})$ $\big(\bigwedge X(b;\, \mathfrak{B})\big)$ is strongly defined and is equal to $\bigvee \{g(b;\, \mathfrak{B}) \mid g \in X\}$ $(\bigwedge \{g(b;\, \mathfrak{B}) \mid g \in X\})$. Another way of making $f(b;\, \mathfrak{B})$ meaningful is to define it to be $f(b;\, \mathfrak{B}')$ where $\mathfrak{B}'$ is the normal completion of $\mathfrak{B}$, provided that $f(b;\, \mathfrak{B}') \in |\mathfrak{B}|$. Thus we say that $f(b;\, \mathfrak{B})$ is weakly defined if $f(b;\, \mathfrak{B}') \in |\mathfrak{B}|$. Obviously if $f(b;\, \mathfrak{B})$ is strongly defined then it is also weakly defined. The other implication does not hold in general. The problem which we pose is: if $g(b;\, \mathfrak{B})$ is weakly defined is there always a term f such that $f \equiv g$ and $f(b;\, \mathfrak{B})$ is strongly defined?

§ 1. Definition 1.1. $\varphi \leqslant \psi$ if for every complete B.a. $\mathfrak{B}$ and every sequence b of length δ of its elements $\varphi(b;\, \mathfrak{B}) \leqslant \psi(b;\, \mathfrak{B})$. $\varphi < \psi$ if $\varphi \leqslant \psi$ but $\varphi \neq \psi$.

Obviously, $\varphi \leqslant \psi$ iff $\varphi \vee \psi = \psi$ and $\varphi = \psi$ iff $\varphi \leqslant \psi$ and $\psi \geqslant \varphi$.

Definition 1.2. A class G of B.p.'s is *closed* if, for every $\varphi, \psi \in G$, $\varphi \vee \psi \in G$, $\varphi \wedge \psi \in G$, and $\bar{\varphi} \in G$.

Properties (0.5)-(0.9) imply that all equalities and inequalities involving $\bigvee$, $\bigwedge$, and $^{-}$, which hold in all complete B.a.'s hold also for B.p.'s where $\bigvee$, $\bigwedge$, and $^{-}$ are replaced by $\bigvee$, $\bigwedge$, and $^{-}$. All these well-known properties will be assumed in the sequel. It follows that a closed set of B.p.'s together with the operations $\vee$, $\wedge$, and $^{-}$ form a B.a.

Definition 1.3. If G is a closed set of B.p.'s then $B(G)$ is the B.a. $\langle G, \vee, \wedge, ^{-}\rangle$.

It follows that if G is a closed set of B.p.'s and $\varphi, \psi \in G$ then φ is less than or equal to ψ, as members of $B(G)$, iff $\varphi \leqslant \psi$. Consequently, we get

Lemma 1.1. *If G is a closed set $\{\varphi_i\}_{i \in I} \subseteq G$ and $\bigvee_{i \in I} \varphi_i \in G$ $(\bigwedge_{i \in I} \varphi_i \in G)$ then the join (meet) of $\{\varphi_i\}_{i \in I}$ in $B(G)$ exists and is equal to $\bigvee_{i \in I} \varphi_i$ $(\bigwedge_{i \in I} \varphi_i)$. Also the complement in $B(G)$ of φ, where $\varphi \in G$, is $\bar{\varphi}$.*

An intersection of closed classes of B.p.'s is a closed class of B.p.'s. Hence there exists a smallest closed class containing a given class of B.p.'s.

Definition 1.4. Let G be any class of B.p.'s; then: $\mathrm{Cl}(G)$ is the smallest closed class containing G. $J(G) = \{\bigvee Y \mid Y \subseteq G$ and Y is a set$\}$.

$$M(G) = \{\bigvee Y \mid Y \subseteq G \text{ and } Y \text{ is a set}\}. \qquad \mathrm{Com}(G) = \{\bar{\varphi} \mid \varphi \in G\}.$$

$$S(G) = J(G) \cup M(J) \cup \mathrm{Com}(G).$$

Obviously $J(G)$, $M(G) \supseteq G$. If G is a set so are $S(G)$ and $\mathrm{Cl}(G)$; for $S(G)$ this is obvious and for $\mathrm{Cl}(G)$ it follows from the following lemma:

Lemma 1.2. $\mathrm{Cl}(G)$ *is the class of all B.p.'s which are finite joins of finite meets of members of $G \cup \mathrm{Com}(G)$.*

Proof. By the distributivity laws and De Morgan's laws, it follows that the class of all finite joins of finite meets of members of $G \cup \mathrm{Com}(G)$ is closed. On the other hand every member of this class must be in $\mathrm{Cl}(G)$.

Definition 1.5. If G is a class of B.p.'s then $S_0(G) = G$ and $S_{\alpha}(G) = S\big(\bigcup_{\beta<\alpha} S_{\beta}(G)\big)$. $\mathrm{BP}_0 = \{x_0, x_1, \ldots, x_{\nu}, \ldots\}_{\lambda<\delta}$ and $\mathrm{BP}_{\alpha} = S_{\alpha}(\mathrm{BP}_0)$.

Obviously $S_{\alpha}(G) \supseteq S_{\beta}(G)$ if $\alpha \geqslant \beta$ and $\mathrm{BP} = \bigcup_{\alpha} \mathrm{BP}_{\alpha}$ where α ranges over all ordinals. BP_{α} is the set of all B.p.'s which can be represented by terms from BT_{α}, that is, involving at most α iterations of the Boolean operations.

Definition 1.6. A class G of B.p.'s is *self-generated* if there are sets of B.p.'s $G_0, G_1, \ldots, G_{\alpha}, \ldots$, defined for all ordinals α, such that $G_0 = G \cap \mathrm{BP}_0$, $G_{\alpha} \subseteq S(\bigcup_{\alpha<\beta} G_{\beta})$ for all $\alpha > 0$, and $G = \bigcup_{\alpha} G_{\alpha}$.

Roughly speaking G is self-generated if starting from those fundamental B.p.'s $x_0, \ldots, x_{\lambda}, \ldots$ which are in G one can construct all B.p.'s of G by successive use of $\bigvee$, $\bigwedge$ and $^{-}$ using in the process only B.p.'s of G.

A self-generated class may be a set, in which case the G's mentioned in the definition will be equal from a certain point on. Thus the BP_a's which are sets are self-generated. BP is self-generated as well.

We mention here the following problem concerning self-generated classes:

Is every self-generated class G of the form $\bigcup_a G_a$ where $G_a \subseteq S(\bigcup_{\beta<a} G_\beta)$ for $a>0$ and $G_a = G \cap \mathrm{BP}_a$ for $a \geqslant 0$?

This amounts to asking whether every self-generated class can be generated through the "natural hierarchy" of B.p.'s.

LEMMA 1.3. *If G is self-generated and $H \subseteq S(G)$ then $G \cup H$ is self-generated.*

Proof. $G = \bigcup_a G_a$ where the G_a's satisfy the conditions of Definition 1.6. Put $H_a = H \cap S(\bigcup_{\beta<a} G_\beta)$ and $G'_a = G_a \cup H_a$. Then $G \cup H = \bigcup_a G'_a$ and the required conditions hold for $G'_0, \ldots, G'_a, \ldots$

LEMMA 1.4. *If G is self-generated so is* $\mathrm{Cl}(G)$.

Proof. By Lemma 1.3 $G \cup \mathrm{Com}(G)$ is self-generated. $\mathrm{Cl}(G)$ is obtained by taking all finite joins of $G \cup \mathrm{Com}(G)$ and then all finite meets of such joins. Consequently, applying Lemma 1.3 twice more we find that $\mathrm{Cl}(G)$ is self-generated.

It is clear that a union of self-generated classes is self-generated.

LEMMA 1.5. *Let G be a self-generated class and G' a subclass of G such that the following hold*: (i) $G' \subseteq G \cap BP_0$, (ii) if $X \subseteq G'$ *and* $\bigvee X \in G$ *then* $\bigvee X \in G'$, (iii) *if* $X \subseteq G'$ *and* $\bigwedge X \in G$ *then* $\bigwedge X \in G'$, (iv) *if* $\varphi \in G'$ *and* $\bar{\varphi} \in G$ *then* $\bar{\varphi} \in G'$. *Under these assumptions* $G' = G$.

Proof. Put $G = \bigcup_a G_a$ where $G_0 = G \cap \mathrm{BP}_0$ and $G_a \subseteq S(\bigcup_{\beta<a} G_\beta)$. It follows by induction on a that every member of G_a is in G'.

THEOREM 1.1. *Let G be a closed self-generated set of B.p.'s and let $b = \langle b_0, \ldots, b_\lambda, \ldots\rangle_{\lambda<\delta}$ be defined as follows: $b_\lambda = x_\lambda$ if $x_\lambda \in G$ and $b_\lambda = \mathbf{0}$ if $x_\lambda \notin G$. Let $\mathfrak{B}$ be the normal completion of $B(G)$. Then, for every $\varphi \in G$, $\varphi(b; \mathfrak{B}) = \varphi$.*

Proof. If $\varphi \in \mathrm{BP}_0 \cap G$ then $\varphi = x_\lambda$, $\lambda < \delta$, and $b_\lambda = x_\lambda$. Hence $\varphi(b; \mathfrak{B}) = x_\lambda(b; \mathfrak{B}) = b_\lambda = x_\lambda = \varphi$. Now assume that $\varphi = \bigvee X$, $X \subseteq G$, and $\psi(b; \mathfrak{B}) = \psi$ for all $\psi \in X$. From Lemma 1.1 it follows that the join of X in $B(G)$ exists and is equal to $\bigvee X$, hence the same holds for the normal completion of $B(G)$. Consequently $\varphi(\mathfrak{B}; b) = \bigvee\{\psi(\mathfrak{B}; b) | \psi \in X\} = \bigvee X = \bigvee X$. A similar argument applies if $\varphi = \bigwedge X$. Finally if $\varphi = \bar{\psi}$ and $\psi(b; \mathfrak{B}) = \psi$, then, since the complement of ψ in $B(G)$ is $\bar{\psi}$, the same is true for $\mathfrak{B}$ and $\bar{\psi}(b; \mathfrak{B}) = \bar{\psi}$. The theorem follows now from Lemma 1.5.

If $\mathfrak{B}$ is a complete B.a. and $b = \langle b_0, \ldots, b_\lambda, \ldots\rangle_{\lambda<\delta}$ then it is easily seen that $\mathfrak{B}$ is ∞-generated by $\{b_0, \ldots, b_\lambda \ldots\}_{\lambda<\delta}$ iff every member of $\mathfrak{B}$ is of the form $\varphi(b; \mathfrak{B})$ where $\varphi \in \mathrm{BP}$. If $\mathfrak{B}$ is the normal completion of $\mathfrak{B}'$

then every member of $\mathfrak{B}$ is a join (in $\mathfrak{B}$) of members of $\mathfrak{B}'$; hence if every member of $\mathfrak{B}'$ is of the form $\varphi(b; \mathfrak{B})$ the same is true for every member of $\mathfrak{B}$. Therefore, Theorem 1.1 yields the following

COROLLARY. *If $\mathfrak{B}$ is the normal completion of $B(G)$, where G is a closed self-generated set of B.p.'s, then $\mathfrak{B}$ is ∞-generated by $\{b_0, \ldots, b_\lambda \ldots\}_{\lambda<\delta}$, where $b_\lambda = x_\lambda$ if $x_\lambda \in G$ and $b_\lambda = \mathbf{0}$ if $x_\lambda \notin G$.*

If BP is a proper class then the powers of BP_a increase beyond any bound as a increases; hence the normal completions of $B(\mathrm{Cl}(\mathrm{BP}_a))$ would be complete B.a.'s of arbitrary high powers, ∞-generated by $\{x_0, \ldots, x_\lambda \ldots\}_{\lambda<\delta}$. On the other hand if BP is a set then it can be easily established that $B(\mathrm{BP})$ is a complete B.a. ∞-generated by $\{x_0, \ldots, x_\lambda \ldots\}_{\lambda<\delta}$, and this would be the completely free B.a. on δ generators. The power of any other complete B.a. ∞-generated by δ generators would be $\leqslant \overline{\overline{\mathrm{BP}}}$.

§ **2.** THEOREM 2.1. *Let $\{T_i\}_{i\in I}$ be a set of closed self-generated sets of B.p.'s directed under inclusion (i.e., for every i, j in I there is a k in I such that $T_k \supseteq T_i \cup T_j$). Let R be a set of B.p.'s satisfying the following conditions*:

(i) $R \subseteq \bigcup_{i\in I} M(J(T_i))$;

(ii) *If* $\varphi, \psi \in R$ *then* $\varphi \wedge \psi \in R$;

(iii) $\mathbf{0} \notin R$;

(iv) *For every i in I there is a B.p. φ in R depending on i, $\varphi = \varphi(i)$, such that for every ψ in T_i and every σ in R, $\psi \wedge \sigma = \mathbf{0}$ iff $\psi \wedge \varphi(i) = \mathbf{0}$.*

Under these assumptions $\bigwedge R \neq \mathbf{0}$.

Remark. (ii) and (iii) imply that every finite meet of members of R is $\neq \mathbf{0}$, but this finite meet property is easily seen to be insufficient to guarantee that $\bigwedge R \neq \mathbf{0}$.

Proof. Put $G = \bigcup_{i\in I} T_i$. G is self-generated. Since the T_i's are directed under inclusion and closed, G is also closed.

Put $\mathfrak{J} = \{\psi | \psi \in G$ and, for some σ in R, $\psi \wedge \sigma = \mathbf{0}\}$. If $\psi_1, \psi_2 \in \mathfrak{J}$ then $\psi_1 \wedge \sigma_1 = \mathbf{0}$, $\psi_2 \wedge \sigma_2 = \mathbf{0}$ for some $\sigma_1, \sigma_2 \in R$, $\sigma_1 \wedge \sigma_2 \in R$ and $(\psi_1 \vee \psi_2) \wedge (\sigma_1 \wedge \sigma_2) = \mathbf{0}$ hence $\psi_1 \vee \psi_2 \in \mathfrak{J}$. Consequently $\mathfrak{J}$ is an ideal in $B(G)$. Since $\mathbf{0} \notin R$, we have $\mathbf{1} \wedge \sigma \neq \mathbf{0}$ for all $\sigma \in R$, hence $\mathbf{1} \notin \mathfrak{J}$, and therefore $\mathfrak{J}$ is a proper ideal of $B(G)$.

Let $\mathfrak{B}$ be the normal completion of $B(G)/\mathfrak{J}$. Put $b_\lambda = x_\lambda/\mathfrak{J}$ if $x_\lambda \in G$ and $b_\lambda = 0$ (the zero of $\mathfrak{B}$) otherwise. We claim that for all $i \in I$ the following is true:

($*$) *If $\psi \in T_i$ then $\psi(b; \mathfrak{B}) = \psi/\mathfrak{J}$.*

Obviously ($*$) holds if $\psi = x_\lambda$. If ($*$) holds for ψ then $\bar{\psi}(b; \mathfrak{B}) = \overline{\psi(b; \mathfrak{B})} = \overline{\psi/\mathfrak{J}} = \bar{\psi}/\mathfrak{J}$.

Assume that $\bigwedge_{l\in L} \psi_l \in T_i$ where $\{\psi_l\}_{l\in L} \subseteq T_i$ and ($*$) holds for all ψ_l. Put $\psi = \bigwedge_{l\in L} \psi_l$. Then $\psi(b; \mathfrak{B}) = \bigwedge_{l\in L} \psi_l(b; \mathfrak{B}) = \bigwedge_{l\in L} \psi_l/\mathfrak{J}$. Since $\psi_l \geqslant \psi$, for all $l \in L$, we have $\psi_l/\mathfrak{I} \geqslant \psi/\mathfrak{I}$ and $\bigwedge_{l\in L}\psi_l/\mathfrak{I} \geqslant \psi/\mathfrak{I}$. Since $\mathfrak{B}$ is the normal completion of $B(G)/\mathfrak{I}$, in order to prove equality it suffices to show that for all $\sigma \in B(G)$ if $\bigwedge_{l\in L}\psi_l/\mathfrak{I} - \psi/\mathfrak{I} \geqslant \sigma/\mathfrak{I}$ then $\sigma \in \mathfrak{I}$. Let σ be such a B.p.; then, for some j, $\sigma \in T_j$. Let k be such that $T_k \supseteq T_i \cup T_j$. For every $l \in L$, $\sigma/\mathfrak{I} - \psi_l/\mathfrak{I} = 0$ hence $\sigma - \psi_l \in \mathfrak{I}$. Therefore for every $l \in L$ there is a B.p. τ in R such that $(\sigma - \psi_l) \wedge \tau = \mathbf{0}$. Since $\sigma - \psi_l \in T_k$ for all $l \in L$, it follows from (iv) that $(\sigma - \psi_l) \wedge \varphi(k) = \mathbf{0}$ for all $l \in L$. Consequently $\sigma \wedge \varphi(k) \leqslant \psi_l \wedge \varphi(k)$ for all $l \in L$, hence $\sigma \wedge \varphi(k) \leqslant \bigwedge_{l\in L}\psi_l \wedge \varphi(k) = \psi \wedge \wedge \varphi(k)$. On the other hand, $\sigma/\mathfrak{I} \wedge \psi/\mathfrak{I} = 0$ hence $\sigma \wedge \psi \in \mathfrak{I}$ and since $\sigma \wedge \psi \in T_k$, $\sigma \wedge \psi \wedge \varphi(k) = \mathbf{0}$. Therefore $\sigma \wedge \varphi(k) \leqslant \sigma \wedge \psi \wedge \varphi(k) = \mathbf{0}$, hence $\sigma \in \mathfrak{I}$.

The case of $\bigvee_{l\in L}\psi_l$ follows from this by taking complements and using De Morgan's law.

Since every T_i is self-generated, $(*)$ holds for every T_i by Lemma 1.5.

We claim also that for all $i \in I$ the following holds:

$(**)$ *If* $\{\psi_l\}_{l\in L} \subseteq T_i$ *and* $\bigwedge_{l\in L}\psi_l \wedge \tau = \mathbf{0}$ *for some* $\tau \in R$ *then* $(\bigwedge_{l\in L}\psi_l)(b; \mathfrak{B}) = 0$.

By $(*)$ $\bigwedge_{l\in L}\psi_l(b; \mathfrak{B}) = \bigwedge_{l\in L}\psi_l/\mathfrak{I}$. Let $\sigma \in G$ be such that $\bigwedge_{l\in L}\psi_l/\mathfrak{I} \geqslant \sigma/\mathfrak{I}$. Then $\psi_l/\mathfrak{I} \geqslant \sigma/\mathfrak{I}$ for all $l \in L$, hence $\sigma - \psi_l \in \mathfrak{I}$ for all $l \in L$. For some j, $\sigma \in T_j$. Let k be such that $T_k \supseteq T_i \cup T_j$ then $\sigma - \psi_l \in T_k$ for all $l \in L$, hence $(\sigma - \psi_l) \wedge \varphi(k) = \mathbf{0}$. Therefore $\sigma \wedge \varphi(k) \leqslant \bigwedge_{l\in L}\psi_l \wedge \varphi(k)$ and $\sigma \wedge \varphi(k) \wedge \tau \leqslant \bigwedge_{l\in L}\psi_l \wedge \varphi(k) \wedge \tau = \mathbf{0}$. Since $\varphi(k) \wedge \tau \in R$, this implies that $\sigma \in \mathfrak{I}$, which proves $(**)$.

Now let σ be any member of R. By (i) there is a T_i such that $\sigma \in M(J(T_i))$. Therefore $\overline{\sigma} \in J(M(T_i))$, i.e., $\overline{\sigma} = \bigvee_{j\in J}\psi_j$ where $\psi_j \in M(T_i)$ for all $j \in J$. $\psi_j \wedge \overline{\sigma} = \mathbf{0}$ for all $j \in J$ therefore, by $(**)$, $\psi_j(b; \mathfrak{B}) = 0$ for all $j \in J$, hence $\overline{\sigma}(b; \mathfrak{B}) = \bigwedge_{j\in J}\psi_j(b; \mathfrak{B}) = 0$. Consequently $\sigma(b; \mathfrak{B}) = 1$ for all $\sigma \in R$, which implies $\bigwedge\{\sigma(b; \mathfrak{B}) \mid \sigma \in R\} = 1 \neq 0$. Hence $\bigwedge R \neq \mathbf{0}$, q.e.d.

DEFINITION 2.1. If T is a class of B.p.'s then φ is *essential* in ψ with respect to T, or, simply, T-*essential* in ψ, if either $\psi = \mathbf{0}$ or $\psi \neq \mathbf{0}$ and, for some σ in T, $\varphi \geqslant \psi \wedge \sigma$ and $\psi \wedge \sigma \neq \mathbf{0}$. φ is *inessential* in ψ with respect to T, or, simply, T-*inessential* in ψ, if it is not T-essential in ψ.

LEMMA 2.1. *If* T *is closed under finite meets,* $\sigma \in T$, *and* $\sigma \wedge \psi \neq \mathbf{0}$ *then every B.p. which is* T-*essential in* $\sigma \wedge \psi$ *is* T-*essential in* ψ.

Proof. If φ is T-essential in $\sigma \wedge \psi$ then $\varphi \geqslant \sigma' \wedge \sigma \wedge \psi \neq \mathbf{0}$ for some $\sigma' \in T$. $\sigma' \wedge \sigma \in T$ hence φ is T-essential in ψ.

LEMMA 2.2. *Let* T *be a closed class of B.p.'s. If* φ *is* T-*essential in* ψ, φ' T-*inessential in* ψ *and* $\varphi' \in M(T)$, *then* $\varphi - \varphi'$ *is* T-*essential in* ψ.

Proof. If $\psi \neq \mathbf{0}$ then, for some $\sigma \in T$, $\varphi \geqslant \psi \wedge \sigma \neq \mathbf{0}$. Put $\varphi' = \bigwedge_{i\in I}\varphi_i$ where $\{\varphi_i\}_{i\in I} \subseteq T$. Since φ' is T-inessential in ψ, φ' non $\geqslant \sigma \wedge \psi$. Hence, for some $i \in I$, φ_i non $\geqslant \sigma \wedge \psi$ which means that $(\sigma - \varphi_i) \wedge \psi \neq \mathbf{0}$. $\sigma - \varphi_i \in T$, because T is closed, and we have $\varphi - \varphi' \geqslant (\sigma - \varphi_i) \wedge \psi \neq \mathbf{0}$.

THEOREM 2.2. *Let* T *be a closed self-generated set of B.p.'s. Let* φ *be a non-zero B.p. of* $M(J(T))$ *and let* S *be a subset of* $M(T)$ *all of whose members are* T-*inessential in* φ. *Then* $\varphi - \bigvee S \neq \mathbf{0}$.

Proof. Assume that $\psi \in T$ and $\psi \wedge \varphi \neq \mathbf{0}$, then by Lemma 1.1 every member of S is T-inessential in $\psi \wedge \varphi$. $\mathbf{1} \wedge \psi \wedge \varphi = \psi \wedge \varphi \neq \mathbf{0}$ and $\mathbf{1} \in T$ hence $\psi \wedge \varphi$ is T-essential in itself. By Lemma 2.2, $\psi \wedge \varphi - \varphi_1$ is T-essential in $\psi \wedge \varphi$ whenever $\varphi_1 \in S$; again by the same Lemma, if $\varphi_2 \in S$, $(\psi \wedge \varphi - \varphi_1) - \varphi_2$ is T-essential in $\psi \wedge \varphi$ and so on. Hence $\psi \wedge \varphi - \bigvee S' \neq \mathbf{0}$ whenever S' is a finite subset of S.

Now let R be the set of all B.p.'s of the form $\varphi - \bigvee S'$ where S' is a finite subset of S. Obviously $R \subseteq M(J(T))$ and by putting $S' = \emptyset$ we get $\varphi = \varphi - \bigvee S' \in R$. Also if $\psi \in T$ and $\psi \wedge \varphi \neq \mathbf{0}$ then $\psi \wedge \sigma \neq \mathbf{0}$ for all $\sigma \in R$. By Theorem 2.1, putting $\{T_i\}_{i\in I} = \{T\}$, we get $\bigwedge R \neq \mathbf{0}$, but $\bigwedge R = \varphi - \bigvee S$, q.e.d.

Remark. If we change the assumption of Theorem 1.1 by assuming $S \subseteq J(M(T))$ instead of $S \subseteq M(T)$ we still get $\varphi - \bigvee S \neq \mathbf{0}$. This is so since in that case every member of S is a join of B.p.'s from $M(T)$ which must be T-inessential in φ. Subtracting $\bigvee S$ is the same as subtracting all these B.p.'s of $M(T)$.

THEOREM 2.3. *Let* $T_0, T_1, \ldots, T_\mu, \ldots$, $\mu < \alpha$, *be a sequence of closed self-generated sets of B.p.'s, where* α *is some limit ordinal. Assume that* $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_\mu \subseteq T_{\mu+1} \subseteq \ldots$ *Let* $\sigma_0, \ldots, \sigma_\mu, \ldots$, $\mu < \alpha$, *be a sequence of B.p.'s such that for all* $\mu < \alpha$:

(i) $\sigma_\mu \in M(J(T_\mu))$ *and* $\sigma_\mu \neq \mathbf{0}$;

(ii) $\sigma_{\mu+1} = \sigma_\mu - \bigvee R_\mu$ *where* $R_\mu \subseteq M(T_\mu)$ *and every member of* R_μ *is* T_μ-*inessential in* σ_μ;

(iii) $\sigma_\mu = \bigwedge_{\nu<\mu}\sigma_\nu$ *if* μ *is a limit ordinal* >0.

Under these assumptions if $\psi \in T_\beta$, $\beta < \alpha$, *and* $\psi \wedge \sigma_\beta \neq \mathbf{0}$ *then* $\psi \wedge \bigwedge_{\mu<\alpha}\sigma_\mu \neq \mathbf{0}$. *In particular,* $\bigwedge_{\mu<\alpha}\sigma_\mu \neq \mathbf{0}$ (*since* $\mathbf{1} \wedge \sigma_0 = \sigma_0 \neq \mathbf{0}$).

Proof. Put $\sigma_\alpha = \bigwedge_{\mu<\alpha}\sigma_\mu$. We have to show that if $\psi \in T_\beta$ and $\psi \wedge \sigma_\beta \neq \mathbf{0}$ then $\psi \wedge \sigma_\alpha \neq \mathbf{0}$. Since $\sigma_\gamma \geqslant \sigma_\beta$ whenever $\gamma \leqslant \beta$, this amounts to showing that if $\psi \wedge \sigma_\beta \neq 0$, where $\psi \in T_\beta$, then also $\psi \wedge \sigma_{\beta+\mu} \neq \mathbf{0}$ whenever μ is such that $\beta + \mu \leqslant \alpha$. This is done by transfinite induction on μ.

It is true for $\mu = 0$. Assume it to be true for μ and let $\psi \wedge \sigma_\beta \neq \mathbf{0}$ where $\psi \in T_\beta$. By our assumption $\psi \wedge \sigma_{\beta+\mu} \neq \mathbf{0}$. $\psi \wedge \sigma_{\beta+\mu+1} = \psi \wedge \sigma_{\beta+\mu} - \bigvee R_{\beta+\mu}$. Every member of $R_{\beta+\mu}$ is $T_{\beta+\mu}$-inessential in $\sigma_{\beta+\mu}$. Hence by Lemma 2.1 (since $\psi \in T_\beta \subseteq T_{\beta+\mu}$) every member of $R_{\beta+\mu}$ is $T_{\beta+\mu}$-inessential in $\psi \wedge \sigma_{\beta+\mu}$. Therefore by Theorem 2.2 $\psi \wedge \sigma_{\beta+\mu+1} \neq \mathbf{0}$.

Assume it to be true for all $\nu < \mu$ where $\mu = \bigcup\mu > 0$. Let $\psi \wedge \sigma_\beta \neq 0$ where $\psi \in T_\beta$ and let $\beta + \mu \leqslant \alpha$. Consider $\{T_{\beta+\nu}\}_{\nu<\mu}$, and the set $\{\psi \wedge \sigma_{\beta+\nu}\}_{\nu<\mu}$

of B.p.'s. $\psi\wedge\sigma_{\beta+\nu}\in M\big(J(T_{\beta+\nu})\big)$ and, by our induction hypothesis, $\psi\wedge\sigma_{\beta+\nu}\neq\mathbf{0}$ for $\nu<\mu$. Moreover if $\psi'\in T_{\beta+\nu'}$, $\nu'<\mu$ and $\psi'\wedge\psi\wedge\sigma_{\beta+\nu'}\neq\mathbf{0}$ then, since $\psi'\wedge\psi\in T_{\beta+\nu'}$, our induction hypothesis implies that $\psi'\wedge\psi\wedge\sigma_{\beta+\nu}\neq\mathbf{0}$ for all $\nu<\mu$. By Theorem 2.1 (putting $\{T_i\}_{i\in I}=\{T_{\beta+\nu}\}_{\nu<\mu}$ and $R=\{\psi\wedge\sigma_{\beta+\mu}\}_{\nu<\mu}$) we get $\psi\wedge\sigma_{\beta+\mu}=\psi\wedge\bigwedge_{\nu<\mu}\sigma_{\beta+\nu}\neq\mathbf{0}$.

Note that the requirement $R_\mu\subseteq M(T_\mu)$ can be replaced by $R_\mu\subseteq J\big(M(T_\mu)\big)$ (see remark to Theorem 2.1).

We remark here that in a certain sense the converse of Theorem 2.3 is also true. Consider the sequence $\mathrm{BP}_0,\mathrm{BP}_1,\ldots,\mathrm{BP}_\mu,\ldots$ (see Definition 1.5) and let σ be any non-zero B.p. of BP_α, $\alpha>0$. Then there are B.p.'s $\sigma_0,\ldots,\sigma_\mu,\ldots$, $\mu\leqslant\alpha$, such that $\sigma=\sigma_\alpha$, and sets of B.p.'s $R_0,R_1,\ldots,R_\mu,\ldots$, $\mu\leqslant\alpha$, such that the following hold:

(i) for all $\mu<\alpha$ $\sigma_\mu\in M(\bigcup_{\nu<\mu}\mathrm{BP}_\nu)$, $R_\mu\subseteq M(\bigcup_{\nu<\mu}\mathrm{BP}_\nu)$ and every member of R_μ is $(\bigcup_{\nu<\mu}\mathrm{BP}_\nu)$-inessential in σ_μ;

(ii) $\sigma_{\mu+1}=\sigma_\mu-\bigvee R_\mu$ and if $\mu=\bigcup\mu\leqslant\alpha$ then $\sigma_\mu=\bigwedge_{\nu<\mu}\sigma_\nu$.

Moreover, it follows by Theorem 2.3 that the B.p.'s σ_μ are uniquely determined and depend only on σ, and not on the particular α (provided only that $\sigma\in\mathrm{BP}_\alpha$). The R_μ's can be given by: $R_\mu=\{\psi|\psi\in M(\bigcup_{\nu<\mu}\mathrm{BP}_\nu)$ and $\psi\wedge\sigma=\mathbf{0}\}$, and the σ_μ's are defined by: $\sigma_\mu=\bigwedge\{\overline{\psi}|\psi\in\bigcup_{\nu<\mu}R_\nu\}$. This will be discussed in detail in Part II.

§ 3. Let δ be a fixed infinite cardinal. Let $\{N_{\lambda\mu}\}_{\lambda,\mu<\delta}$ be a family of sets of ordinals defined for all $\lambda,\mu<\delta$ such that:

(3.1) $N_{\lambda\mu}\subseteq\delta$ and $\overline{\overline{N}}_{\lambda\mu}=\delta$;

(3.2) If $\langle\lambda,\mu\rangle\neq\langle\lambda',\mu'\rangle$ then $N_{\lambda\mu}\cap N_{\lambda'\mu'}=\emptyset$.

It is easy to establish the existence of such a family.

Define by recursion B.p.'s $\varphi(\alpha,\lambda)$, where $\lambda<\delta$ and α is any ordinal, as follows:

Definition 3.1. $\varphi(0,\lambda)=x_\lambda$ for all $\lambda<\delta$.

$\varphi(\alpha+1,\lambda)=\varphi(\alpha,\lambda)\wedge\bigwedge_{\mu>\delta}\bigvee_{\nu\in N_{\lambda\mu}}\varphi(\alpha,\nu)$.

$\varphi(\alpha,\lambda)=\bigwedge_{\beta<\alpha}\varphi(\beta,\lambda)$ if $\alpha=\bigcup\alpha>0$.

Roughly speaking $\varphi(\alpha+1,\lambda)$ is the meet of $\varphi(\alpha,\lambda)$ with the B.p. obtained by "applying" $\bigwedge\bigvee$ to the $\delta\times\delta$ matrix whose μ-th row consists of the B.p.'s $\varphi(\alpha,\nu)$ as ν ranges over $N_{\lambda\mu}$.

Obviously $\varphi(0,\lambda)\geqslant\varphi(1,\lambda)\geqslant\ldots\geqslant\varphi(\alpha,\lambda)\geqslant\varphi(\alpha+1,\lambda)\geqslant\ldots$ It is our aim to show that, for all $\lambda<\delta$, $\varphi(0,\lambda)>\varphi(1,\lambda)>\ldots>\varphi(\alpha,\lambda)>\varphi(\alpha+1,\lambda)>\ldots$, and that if δ is regular the same continues to hold even if we add the (δ,∞)-distributive rule.

We have to show that $\varphi(\alpha,\lambda)-\varphi(\alpha+1,\lambda)\neq\mathbf{0}$. To do this we will construct a sequence of closed self-generated sets of B.p.'s which will have certain properties with respect to the B.p.'s $\varphi(\alpha,\lambda)$ and we will apply our previously established theorems.

In order to facilitate the notation we put:

$$\varphi(\alpha,\lambda,\mu)=\bigvee\nolimits_{\nu\in N_{\lambda\mu}}\varphi(\alpha,\nu)\,. \tag{3.3}$$

With this we get

$$\varphi(\alpha+1,\lambda)=\varphi(\alpha,\lambda)\wedge\bigwedge\nolimits_{\mu<\delta}\varphi(\alpha,\lambda,\mu) \tag{3.4}$$

and by transfinite induction it is easily seen that:

$$\varphi(\alpha,\lambda)=\varphi(0,\lambda)\wedge\bigwedge\nolimits_{\beta<\alpha}\bigwedge\nolimits_{\mu<\delta}\varphi(\beta,\lambda,\mu)\,, \tag{3.5}$$

$$\overline{\varphi}(\alpha,\lambda)=\overline{\varphi}(0,\lambda)\vee\bigvee\nolimits_{\beta<\alpha}\bigvee\nolimits_{\mu<\delta}\overline{\varphi}(\beta,\lambda,\mu)\,. \tag{3.5'}$$

Definition 3.2. Let T_α be the closure of the set of all B.p.'s which fall under one of the following:

(i) Meets of power $<\delta$ of B.p.'s of the form $\varphi(\beta,\lambda)$ where $2\beta\leqslant\alpha$;

(ii) Joins of power $<\delta$ of B.p.'s of the form $\overline{\varphi}(\beta,\lambda)$ where $2\beta<\alpha$;

(iii) Joins of power $<\delta$ of B.p.'s of the form $\varphi(\beta,\lambda,\mu)$ where $2\beta<\alpha$.

Obviously $T_0\subseteq T_1\subseteq\ldots\subseteq T_\alpha\subseteq T_{\alpha+1}$ and we have:

$$\varphi(\alpha,\lambda),\ \overline{\varphi}(\alpha,\lambda)\in T_{2\alpha},\qquad \varphi(\alpha,\lambda,\mu),\ \overline{\varphi}(\alpha,\lambda,\mu)\in T_{2\alpha+1}\,. \tag{3.6}$$

From (3.5) and (3.3) we get:

$$\varphi(\alpha,\lambda)\in M(\bigcup\nolimits_{\beta<2\alpha}T_\beta)\ ,\qquad \varphi(\alpha,\lambda,\mu)\in J(T_{2\alpha})\,. \tag{3.7}$$

It is easily seen that:

$$T_0=\mathrm{Cl}\big(\{\bigwedge\nolimits_{\lambda\in L}\varphi(0,\lambda)|\overline{\overline{L}}<\delta\}\big)\,, \tag{3.8}$$

$$T_{2\alpha+1}=\mathrm{Cl}\big(T_{2\alpha}\cup\{\bigvee\nolimits_{\lambda\in L}\overline{\varphi}(\alpha,\lambda)|\overline{\overline{L}}<\delta\}\cup\{\bigvee\nolimits_{\langle\lambda\mu\rangle\in M}\varphi(\alpha,\lambda,\mu)|\overline{\overline{M}}<\delta\}\big)\,, \tag{3.9}$$

$$T_{2\alpha}=\mathrm{Cl}\big(\bigcup\nolimits_{\beta<2\alpha}T_\beta\cup\{\bigwedge\nolimits_{\lambda\in L}\varphi(\alpha,\lambda)|\overline{\overline{L}}<\delta\}\big)\,. \tag{3.10}$$

Lemma 3.1. *T_α is closed and self-generated, for all α.*

Proof. Since $x_\lambda\in T_0$ for all $\lambda<\delta$ and T_0 is the closure of a set of meets of x_λ's, it follows (by Lemma 1.4) that T_0 is self-generated. Furthermore, $T_{2\alpha+1}$ is obtained from $T_{2\alpha}$ by adding to it joins of $\varphi(\alpha,\lambda)$'s and $\varphi(\alpha,\lambda,\mu)$'s and taking the closure. Since $\varphi(\alpha,\lambda)\in T_{2\alpha}$ and $\varphi(\alpha,\lambda,\mu)\in J(T_{2\alpha})$, it follows (by Lemma 1.3 and 1.4) that if $T_{2\alpha}$ is self-generated, so is $T_{2\alpha+1}$. Finally, if T_β is self-generated for all $\beta<2\alpha$, then so is $\bigcup_{\beta<2\alpha}T_\beta$ and a similar argument, using (3.7) and (3.10), shows that $T_{2\alpha}$ is self-generated.

Lemma 3.2. *Every B.p. from T_α is a join of B.p.'s (also from T_α) of the form $\bigwedge_{\langle\beta\lambda\rangle\in K}\varphi(\beta,\lambda)\wedge\bigwedge_{\langle\beta\lambda\rangle\in L}\overline{\varphi}(\beta,\lambda)\wedge\bigwedge_{\langle\beta\lambda\mu\rangle\in M}\overline{\varphi}(\beta,\lambda,\mu)$, where K, L, M are sets of power $<\delta$ satisfying the following: $2\beta\leqslant\alpha$ if $\langle\beta\lambda\rangle\in K$, $2\beta<\alpha$ or $\beta=0$ if $\langle\beta\lambda\rangle\in L$, and $2\beta<\alpha$ if $\langle\beta\lambda\mu\rangle\in M$.*

Proof. Every B.p. of T_α is a finite join of finite meets in which every B.p. is of one of the forms (i), (ii), (iii) (in Definition 3.2), or a complement of a B.p. of such a form (see Lemma 1.2). Given such a finite meet use first De Morgan's laws and replace every complement of $\bigwedge_{\langle\beta\lambda\rangle\in N}\varphi(\beta,\lambda)$ by $\bigvee_{\langle\beta\lambda\rangle\in N}\overline{\varphi}(\beta,\lambda)$, every complement of $\bigvee_{\langle\beta\lambda\rangle\in N}\varphi(\beta,\lambda)$ by $\bigwedge_{\langle\beta\lambda\rangle\in N}\overline{\varphi}(\beta,\lambda)$, and every complement of $\bigvee_{\langle\beta\lambda\mu\rangle\in N}\varphi(\beta,\lambda,\mu)$ by $\bigwedge_{\langle\beta\lambda\mu\rangle\in N}\overline{\varphi}(\beta,\lambda,\mu)$. This will yield a finite meet $\sigma_1\wedge\ldots\wedge\sigma_k$ in which every σ_i is of one of the following forms:

(1) $\bigwedge_{\langle\beta\lambda\rangle\in N}\varphi(\beta,\lambda)$, $\overline{\overline{N}}<\delta$ and $2\beta\leqslant\alpha$ if $\langle\beta\lambda\rangle\in N$,

(1′) $\bigvee_{\langle\beta\lambda\rangle\in N}\overline{\varphi}(\beta,\lambda)$, $\overline{\overline{N}}<\delta$ and $2\beta\leqslant\alpha$ if $\langle\beta\lambda\rangle\in N$,

(2) $\bigvee_{\langle\beta\lambda\rangle\in N}\varphi(\beta,\lambda)$, $\overline{\overline{N}}<\delta$ and $2\beta<\alpha$ if $\langle\beta\lambda\rangle\in N$,

(2′) $\bigwedge_{\langle\beta\lambda\rangle\in N}\overline{\varphi}(\beta,\lambda)$, $\overline{\overline{N}}<\delta$ and $2\beta<\alpha$ if $\langle\beta\lambda\rangle\in N$,

(3) $\bigvee_{\langle\beta\lambda\mu\rangle\in N}\varphi(\beta,\lambda,\mu)$, $\overline{\overline{N}}<\delta$ and $2\beta<\alpha$ if $\langle\beta\lambda\mu\rangle\in N$,

(3′) $\bigwedge_{\langle\beta\lambda\mu\rangle\in N}\overline{\varphi}(\beta,\lambda,\mu)$, $\overline{\overline{N}}<\delta$ and $2\beta<\alpha$ if $\langle\beta\lambda\mu\rangle\in N$.

A finite meet of B.p.'s of form (1) is again of this form and the same is true for (2′) and (3′). Since $\overline{\varphi}(\beta,\lambda)=\overline{\varphi}(0,\lambda)\vee\bigvee_{\gamma<\beta}\bigvee_{\mu<\delta}\overline{\varphi}(\gamma,\lambda,\mu)$, we get that every B.p. of form (1′) is of the form $\bigvee_{\langle\beta\lambda\rangle\in N}\big(\overline{\varphi}(0,\lambda)\vee\bigvee_{\gamma<\beta}\bigvee_{\mu<\delta}\overline{\varphi}(\gamma,\lambda,\mu)\big)$ where $2\beta\leqslant\alpha$ if $\langle\beta\lambda\rangle\in N$, that is, of the form $\bigvee_{\lambda\in L}\overline{\varphi}(0,\lambda)\vee\vee\bigvee_{\langle\gamma\lambda\mu\rangle\in M}\overline{\varphi}(\gamma,\lambda,\mu)$ where $2\gamma<\alpha$ if $\langle\gamma\lambda\mu\rangle\in M$. Also, since $\varphi(\beta,\lambda,\mu)=\bigvee_{\nu\in N_{\lambda\mu}}\varphi(\beta,\nu)$, every B.p. of form (2) is of the form $\bigvee_{\langle\beta\nu\rangle\in M}\varphi(\beta,\nu)$ where $2\beta<\alpha$ if $\langle\beta\gamma\rangle\in M$. If we distribute now over the finite meet we get a join of B.p.'s satisfying the statement of the lemma, q.e.d.

From Definition 3.1 it follows that $\bigwedge_{i\in I}\varphi(\alpha_i,\lambda)=\varphi(\alpha,\lambda)$ where $\alpha=\bigcup_{i\in I}\alpha_i$. Consider $\bigwedge_{\langle\alpha\lambda\rangle\in K}\varphi(\alpha,\lambda)$, if for every λ for which there is an α such that $\langle\alpha\lambda\rangle\in K$ we put $\alpha_\lambda=\bigcup\{\alpha|\langle\alpha\lambda\rangle\in K\}$, then it follows that $\bigwedge_{\langle\alpha\lambda\rangle\in K}\varphi(\alpha,\lambda)=\bigwedge_{\langle\alpha\lambda\rangle\in K'}\varphi(\alpha,\lambda)$, where K' is the set of all pairs $\langle\lambda\alpha_\lambda\rangle$. In particular we get that every B.p. of the form $\bigwedge_{\langle\alpha\lambda\rangle\in K}\varphi(\alpha,\lambda)$ is also of the form $\bigwedge_{\langle\alpha\lambda\rangle\in L}\varphi(\alpha,\lambda)$, where L is of power $\leqslant\overline{\overline{K}}$ and satisfies the following property:

(3.11) If $\langle\beta\lambda\rangle\in L$ then there is a maximal μ for which $\langle\mu\lambda\rangle\in L$.

Hence in Lemma 3.2 we can also add that K satisfies (3.11).

Definition 3.3. F_α^1 is the set of all sets K of pairs of ordinals such that $\overline{\overline{K}}<\delta$, for all $\langle\beta\lambda\rangle$ in K $2\beta\leqslant\alpha$ and $\lambda<\delta$, and K satisfies (3.11).

F_α^2 is the set of all sets K of pairs of ordinals such that $\overline{\overline{K}}<\delta$ and, for all $\langle\beta\lambda\rangle$ in K, $2\beta<\mathrm{Max}(\alpha,1)$ and $\lambda<\delta$.

F_α^3 is the set of all sets K of triples of ordinals such that $\overline{\overline{K}}<\delta$ and, for all $\langle\beta\lambda\mu\rangle$ in K, $2\beta<\alpha$ and $\lambda,\mu<\delta$.

$$F_\alpha=F_\alpha^1\times F_\alpha^2\times F_\alpha^3.$$

As a result we get:

Lemma 3.3. *Every B.p. of T_α is a join of B.p.'s of the form*

$\bigwedge_K\varphi(\beta,\lambda)\wedge\bigwedge_L\overline{\varphi}(\beta,\lambda)\wedge\bigwedge_M\overline{\varphi}(\beta,\lambda,\mu)$ *where* $\langle KLM\rangle\in F_\alpha$.

("$\bigwedge_K\varphi(\beta,\lambda)$" stands for "$\bigwedge_{\langle\beta\lambda\rangle\in K}\varphi(\beta,\lambda)$" and similar conventions are assumed for "$\bigwedge_L\overline{\varphi}(\beta,\lambda)$" and "$\bigwedge_M\overline{\varphi}(\beta,\lambda,\mu)$").

The reason why the T_α's are defined so that $\varphi(\alpha,\lambda)\in T_{2\alpha}$, and not $\varphi(\alpha,\lambda)\in T_\alpha$, is that we want to obtain an increasing sequence of self-generated sets, in which the members of every set are finite combinations of joins and meets of members of the preceding sets. $\varphi(\alpha+1,\lambda)$ is obtained by applying $\bigwedge\bigvee$ to $\varphi(\alpha,\mu)$'s, which amounts to a double step. This causes the appearance of the factor 2 in the calculations. On the other hand the definition could be simplified by replacing "joins of power $<\delta$" and "meets of power $<\delta$" by "finite joins" and "finite meets". The proof that $\varphi(\alpha,\lambda)>\varphi(\alpha+1,\lambda)$ would carry through. The definition in its present form is, however, necessary for proving that the inequality holds also if we add the (δ,∞)-distributive law.

Lemma 3.4. *If* $\bigwedge_K\varphi(\beta,\lambda)\wedge\bigwedge_L\overline{\varphi}(\beta,\lambda)\wedge\bigwedge_M\overline{\varphi}(\beta,\lambda,\mu)\neq\mathbf{0}$ *then* K, L, M *satisfy the following 3 conditions:*

(3.12) *If* $\langle\beta\lambda\rangle\in K$ *and* $\langle\beta'\lambda\rangle\in L$ *then* $\beta'>\beta$.

(3.13) *If* $\langle\beta\lambda\rangle\in K$, $\langle\beta'\nu\mu\rangle\in M$ *and* $\lambda\in N_{\nu\mu}$ *then* $\beta'>\beta$.

(3.14) *If* $\langle\beta\lambda\rangle\in K$ *and* $\langle\beta'\lambda\mu\rangle\in M$ *then* $\beta'\geqslant\beta$.

Proof. Since for $\beta'\leqslant\beta$ $\varphi(\beta',\lambda)\geqslant\varphi(\beta,\lambda)$ we get $\varphi(\beta,\lambda)\wedge\overline{\varphi}(\beta',\lambda)=\mathbf{0}$ if $\beta'\leqslant\beta$, hence (3.12). For the same reason $\varphi(\beta,\lambda)\wedge\overline{\varphi}(\beta',\nu,\mu)=\varphi(\beta,\lambda)\wedge\bigwedge_{\lambda\in N_{\nu\mu}}\overline{\varphi}(\beta'\lambda)=\mathbf{0}$ in case $\lambda\in N_{\nu\mu}$ and $\beta'\leqslant\beta$, hence (3.13). Finally, $\varphi(\beta'+1,\lambda)=\varphi(\beta',\lambda)\wedge\bigwedge_{\mu<\delta}\varphi(\beta',\lambda,\mu)$, hence $\varphi(\beta',\lambda,\mu)\geqslant\varphi(\beta'+1,\lambda)$ for all $\mu<\delta$. Therefore, if $\beta'<\beta$, $\beta'+1\leqslant\beta$, and consequently $\varphi(\beta,\lambda)\wedge\overline{\varphi}(\beta',\lambda,\mu)=\mathbf{0}$. Hence (3.14).

It is our aim to show that for every α if $\langle KLM\rangle\in F_\alpha$ the converse of the last lemma holds. That is, if (3.12)-(3.14) are satisfied then $\bigwedge_K\varphi(\beta,\lambda)\wedge\bigwedge_L\overline{\varphi}(\beta,\lambda)\wedge\bigwedge_M\overline{\varphi}(\beta,\lambda,\mu)\neq\mathbf{0}$. This will imply our main result, namely: putting $K=\{\langle\beta\lambda\rangle\}$, $L=\{\langle\beta+1,\lambda\rangle\}$ and $M=\emptyset$ we will get $\varphi(\beta,\lambda)\wedge\overline{\varphi}(\beta+1,\lambda)\neq\mathbf{0}$, i.e., $\varphi(\beta,\lambda)>\varphi(\beta+1,\lambda)$. This is done in the next section.

§ 4. Definition 4.1. $c(K,L,M)$ means that (3.12), (3.13), and (3.14) hold for K,L,M.

$\mathrm{pr}(\alpha)$ means that for every $\langle KLM\rangle$ in F_α if $c(K,L,M)$ then

$$\bigwedge_K\varphi(\beta,\lambda)\wedge\bigwedge_L\overline{\varphi}(\beta,\lambda)\wedge\bigwedge_M\overline{\varphi}(\beta,\lambda,\mu)\neq\mathbf{0}.$$

Since $F_\alpha\subseteq F_\beta$ for $\alpha\leqslant\beta$, $\mathrm{pr}(\beta)$ implies $\mathrm{pr}(\alpha)$ if $\alpha\leqslant\beta$.

It is our aim to prove that for all α $\mathrm{pr}(\alpha)$. The outline of the proof is as follows. First in Lemma 4.1 $\mathrm{pr}(1)$ is proved by direct verification. Next we assume that $\mathrm{pr}(\beta)$ for all $\beta<\alpha$ where $\alpha>1$. For $\beta\leqslant\alpha$ let $\beta=2\beta_0$ if β is even and $\beta=2\beta_0+1$ if β is odd. Given $\langle KLM\rangle$ in F_α put $K_\beta=\{\langle\gamma\lambda\rangle|\langle\gamma\lambda\rangle\in K$ and $\gamma\leqslant\beta_0\}\cup\{\langle\beta_0\lambda\rangle|\langle\gamma\lambda\rangle\in K$ for some $\gamma>\beta_0\}$,

$L_\beta = \{\langle\gamma\lambda\rangle | \langle\gamma\lambda\rangle \in L$ and $2\gamma < \mathrm{Max}(\beta, 1)\}$, $M_\beta = \{\langle\gamma\lambda\mu\rangle | \langle\gamma\lambda\mu\rangle \in L$ and $2\gamma < \beta\}$. Obviously $\langle KLM\rangle = \langle K_\alpha L_\alpha M_\alpha\rangle$. Put $\sigma_\beta = \bigwedge_{K_\beta}\varphi(\gamma, \lambda) \wedge \wedge \bigwedge_{L_\beta}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M_\beta}\overline{\varphi}(\gamma, \lambda, \mu)$. If $c(K, L, M)$ then it is easily seen that $c(K_\beta, L_\beta, M_\beta)$ for $\beta \leqslant \alpha$, hence $\sigma_\beta \neq \mathbf{0}$ for all $\beta < \alpha$. If $0 < \beta = \bigcup\beta$ then $\sigma_\beta = \bigwedge_{\gamma<\beta}\sigma_\gamma$. Moreover, it is shown that for all $1 \leqslant \beta < \alpha$ the B.p. $\sigma_{\beta+1}$ is obtained by subtracting from σ_β (which is a member of $M(\bigcup_{\gamma<\beta}T_\gamma)$) B.p.'s, which are in $M(\bigcup_{\gamma<\beta}T_\gamma)$ and which are inessential in σ_β with respect to $\bigcup_{\gamma<\beta}T_\gamma$. This is done in Lemma 4.2 for β even and in Lemma 4.3 for β odd. It follows then that if $\alpha = (\alpha-1)+1$ then $\sigma_\alpha \neq \mathbf{0}$ by Theorem 2.2 and if $\alpha = \bigcup\alpha$ then $\sigma_\alpha \neq \mathbf{0}$ by Theorem 2.3. Hence for all α $\mathrm{pr}(\alpha)$ by induction on α.

LEMMA 4.1. $\mathrm{pr}(1)$.

Proof. If $\langle KLM\rangle \in F_1$ then K and L are sets of pairs of the form $\langle 0\lambda\rangle$ and M is a set of triples of the form $\langle 0\lambda\mu\rangle$. Hence

$$\bigwedge_K\varphi(\beta, \lambda) \wedge \bigwedge_L\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M\overline{\varphi}(\beta, \lambda, \mu)$$
$$= \bigwedge\{x_\lambda | \langle 0\lambda\rangle \in K\} \wedge \bigwedge\{\overline{x}_\lambda | \langle 0\lambda\rangle \in L\} \wedge \bigwedge\{\overline{x}_\lambda | \lambda \in N_{\nu\mu} \text{ and } \langle 0\nu\mu\rangle \in M\}.$$

If $c(K, L, M)$ then no x_λ appears in this meet together with its complement. This is so because if $\langle 0\lambda\rangle \in K$ then by (3.12) $\langle 0\lambda\rangle \notin L$, and by (3.13) if $\lambda \in N_{\nu\mu}$ then $\langle 0\nu\mu\rangle \notin M$. Consequently the meet is not zero.

LEMMA 4.2. *Let $\alpha = 2\alpha_0 > 0$. Let $\langle KLM\rangle \in F_\alpha$ and let λ_0 be such that $\langle\alpha_0\lambda_0\rangle \notin K$. Then if $c(K, L, M)$ and if we assume $\mathrm{pr}(\alpha)$ it follows that $\varphi(\alpha_0, \lambda_0)$ is inessential in $\bigwedge_K\varphi(\beta, \lambda) \wedge \bigwedge_L\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M\overline{\varphi}(\beta, \lambda, \mu)$ with respect to $\bigcup_{\nu<\alpha}T_\nu$.*

Proof. Put $\sigma = \bigwedge_K\varphi(\beta, \lambda) \wedge \bigwedge_L\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M\overline{\varphi}(\beta, \lambda, \mu)$. Since $\mathrm{pr}(\alpha)$, $\sigma \neq \mathbf{0}$. If a B.p. ψ is essential in σ with respect to $\bigcup_{\nu<\alpha}T_\nu$ then $\psi \geqslant \psi' \wedge \wedge \sigma > \mathbf{0}$ where $\psi' \in \bigcup_{\nu<\alpha}T_\nu$. Since by Lemma 3.3 every member of T_β is a join of B.p.'s of the form $\bigwedge_{K'}\varphi(\gamma, \lambda) \wedge \bigwedge_{L'}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M'}\overline{\varphi}(\gamma, \lambda, \mu)$ where $\langle K'L'M'\rangle \in F_\beta$, it follows that $\psi \geqslant (\bigwedge_{K'}\varphi(\gamma, \lambda) \wedge \bigwedge_{L'}\overline{\varphi}(\gamma, \lambda) \wedge \wedge \bigwedge_{M'}\overline{\varphi}(\gamma, \lambda, \mu)) \wedge \sigma > \mathbf{0}$ for some $\langle K'L'M'\rangle$ in F_β, where $\beta < \alpha$.

Therefore, in order to prove the lemma it suffices to show that if $\sigma' \wedge \sigma \neq \mathbf{0}$, where $\sigma' = \bigwedge_{K'}\varphi(\gamma, \lambda) \wedge \bigwedge_{L'}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M'}\overline{\varphi}(\gamma, \lambda, \mu)$ and $\langle K'L'M'\rangle \in F_\beta$, $\beta < \alpha$, then $\varphi(\alpha_0, \lambda_0)$ non $\geqslant \sigma' \wedge \sigma$. Put $\sigma'' = \sigma' \wedge \sigma$, then $\sigma'' = \bigwedge_{K''}\varphi(\gamma, \lambda) \wedge \bigwedge_{L''}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M''}\overline{\varphi}(\gamma, \lambda, \mu)$ where $K'' = K' \cup K$, $L'' = L' \cup L$, and $M'' = M' \cup M$. Since (3.11) holds for K and K', it is easily seen to hold for K''. Also $\overline{\overline{K''}}, \overline{\overline{L''}}, \overline{\overline{M''}} < \delta$. Since $\beta < \alpha$, it is easily seen that $\langle K'', L'', M''\rangle \in F_\alpha$. If $\langle\gamma\lambda\rangle \in K'$ then $2\gamma \leqslant \beta < \alpha$ hence $\gamma < \alpha_0$. Therefore $\langle\alpha_0\lambda_0\rangle \notin K'$ and consequently $\langle\alpha_0\lambda_0\rangle \notin K''$. Thus all we need to prove is the following: if $\sigma'' = \bigwedge_{K''}\varphi(\gamma, \lambda) \wedge \bigwedge_{L''}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M''}\overline{\varphi}(\gamma, \lambda, \mu) > \mathbf{0}$ where $\langle K''L''M''\rangle \in F_\alpha$ and $\langle\alpha_0\lambda_0\rangle \notin K''$ then $\varphi(\alpha_0, \lambda_0)$ non $\geqslant \sigma''$, i.e., $\sigma'' \wedge \overline{\varphi}(\alpha_0, \lambda_0) \neq \mathbf{0}$. Let γ_0 be the greatest γ such that $\langle\gamma\lambda_0\rangle \in K''$ (if for no γ $\langle\gamma\lambda_0\rangle \in K''$ put $\gamma_0 = 0$), then $\gamma_0 < \alpha_0$. Consider $\{\lambda |$ for some γ, $\langle\gamma\lambda\rangle \in K''\}$, since $\overline{\overline{K}} < \delta$ this set is of power $< \delta$, therefore there is a μ such that no member of this set belongs to $N_{\lambda_0\mu}$ (see (3.1) and (3.2)). Let μ_0 be such a μ. We claim that $\sigma'' \wedge \overline{\varphi}(\gamma_0, \lambda_0, \mu_0) \neq \mathbf{0}$. This is so since $\sigma'' \wedge \overline{\varphi}(\gamma_0, \lambda_0, \mu_0) = \bigwedge_{K''}\varphi(\gamma, \lambda) \wedge \bigwedge_{L''}\overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{M'''}\overline{\varphi}(\gamma, \lambda, \mu)$ where $M''' = M'' \cup \{\langle\gamma_0\lambda_0\mu_0\rangle\}$. Obviously $\langle K''L''M'''\rangle \in F_\alpha$. Since $c(K'', L'', M'')$, (3.12)-(3.14) hold as far as members of K'', L'', and M'' are concerned. Since $\lambda \notin N_{\lambda_0\mu_0}$ whenever $\langle\gamma\lambda\rangle \in K''$, (3.13) holds for K'' and M''', and since $\gamma \leqslant \gamma_0$ whenever $\langle\gamma\lambda_0\rangle \in K''$, (3.14) holds as well. Hence $c(K'', L'', M''')$ and consequently $\sigma'' \wedge \overline{\varphi}(\gamma_0, \lambda_0, \mu_0) \neq \mathbf{0}$. $\overline{\varphi}(\alpha_0, \lambda_0) \geqslant \overline{\varphi}(\gamma, \lambda_0, \mu)$ whenever $\gamma < \alpha_0$ (by (3.5')), hence $\sigma'' \wedge \overline{\varphi}(\alpha_0, \lambda_0) \geqslant \sigma'' \wedge \overline{\varphi}(\gamma_0, \lambda_0, \mu_0) > \mathbf{0}$, q.e.d.

LEMMA 4.3. *Let $\alpha = 2\alpha_0+1$ and let $\langle KLM\rangle \in F_\alpha$. If $\langle\alpha_0\lambda_0\rangle \in K$, $\langle\alpha_0\lambda_0\mu_0\rangle \notin M$ and $c(K, L, M)$, then, by assuming $\mathrm{pr}(\alpha)$, it follows that $\overline{\varphi}(\alpha_0, \lambda_0, \mu_0)$ is inessential in $\bigwedge_K\varphi(\beta, \lambda) \wedge \bigwedge_L\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M\overline{\varphi}(\beta, \lambda, \mu)$ with respect to $T_{\alpha-1}$.*

Proof. Again as in the proof of Lemma 4.2 it suffices to prove that if $\langle K'L'M'\rangle \in F_{\alpha-1}$, $K'' = K \cup K'$, $L'' = L \cup L'$, $M'' = M \cup M'$, and $\sigma = \bigwedge_{K''}\varphi(\beta, \lambda) \wedge \bigwedge_{L''}\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_{M''}\overline{\varphi}(\beta, \lambda, \mu)$ then $\overline{\varphi}(\alpha_0, \lambda_0, \mu_0)$ non $\geqslant \sigma$. That is, $\sigma \wedge \varphi(\alpha_0, \lambda_0, \mu_0) \neq \mathbf{0}$. $\overline{\overline{N}}_{\lambda_0\mu_0} = \delta$ and $\overline{\overline{L''}}, \overline{\overline{M''}} < \delta$, therefore there is an ordinal ν_0 in $N_{\lambda_0\mu_0}$ such that, for all β, $\langle\beta\nu_0\rangle \notin L''$ and, for all β and μ, $\langle\beta\nu\mu\rangle \notin M''$. Put $K''' = K'' \cup \{\langle\alpha_0\nu_0\rangle\}$. Since $\sigma \neq \mathbf{0}$, (3.12)-(3.14) hold as far as members of K'', L'', and M'' are concerned (Lemma 3.4). Since, for all β, $\langle\beta\nu_0\rangle \notin L''$ (3.12) holds for K''' and L'', and since, for all β and μ, $\langle\beta\nu_0\mu\rangle \notin M''$, (3.14) holds for K''' and M''. $\langle\alpha_0\lambda_0\rangle \in K''$, since $c(K'', L'', M'')$, it follows (by (3.14)) that, for all $\beta < \alpha_0$ and all μ, $\langle\beta\lambda_0\mu\rangle \notin M''$. $\langle K'L'M'\rangle \in F_{\alpha-1}$ hence $\langle\alpha_0\lambda_0\mu_0\rangle \notin M'$, and, since $\langle\alpha_0\lambda_0\mu_0\rangle \notin M$, it follows that, for all β, $\langle\beta\lambda_0\mu_0\rangle \notin M''$. Now if $\nu_0 \in N_{\lambda\mu}$ then $\langle\lambda\mu\rangle = \langle\lambda_0\mu_0\rangle$ ($N_{\lambda\mu} \cap N_{\lambda'\mu'} = \emptyset$ if $\langle\lambda\mu\rangle \neq \langle\lambda'\mu'\rangle$) therefore there is no $\langle\lambda\mu\rangle$ such that $\nu_0 \in N_{\lambda\mu}$ and $\langle\beta\lambda\mu\rangle \in M''$. This shows that (3.13) holds for K''' and M''. Thus we have $c(K''', L'', M'')$ and assuming $\mathrm{pr}(\alpha)$ we get (since $\langle K'''L''M''\rangle \in F_\alpha$) $\sigma \wedge \varphi(\alpha_0, \nu_0) = \bigwedge_{K'''}\varphi(\beta, \lambda) \wedge \bigwedge_{L''}\overline{\varphi}(\beta, \lambda) \wedge \bigwedge_{M''}\overline{\varphi}(\beta, \lambda, \mu) \neq \mathbf{0}$. Since $\varphi(\alpha_0, \lambda_0, \mu_0) \geqslant \varphi(\alpha_0, \nu_0)$, this proves the lemma.

LEMMA 4.4. *For all α $\mathrm{pr}(\alpha)$.*

Proof. As indicated at the beginning of this section, the proof is by induction on α. From Lemma 4.1 we have $\mathrm{pr}(1)$. Assume $\mathrm{pr}(\beta)$ for all $\beta < \alpha$, where $\alpha > 1$. Let $\langle KLM\rangle \in F_\alpha$. For all β such that $1 \leqslant \beta \leqslant \alpha$ put $\beta = 2\beta_0$ if β is even and $\beta = 2\beta_0+1$ if β is odd, and define $\langle K_\beta L_\beta M_\beta\rangle$ as follows:

$$K_\beta = \{\langle\gamma\lambda\rangle | \langle\gamma\lambda\rangle \in K \text{ and } 2\gamma \leqslant \beta\} \cup \{\langle\beta_0\lambda\rangle | \text{ for some } \gamma > \beta_0,\ \langle\gamma\lambda\rangle \in K\},$$
$$L_\beta = \{\langle\gamma\lambda\rangle | \langle\gamma\lambda\rangle \in L \text{ and } 2\gamma < \beta\},$$
$$M_\beta = \{\langle\gamma\lambda\mu\rangle | \langle\gamma\lambda\mu\rangle \in M \text{ and } 2\gamma < \beta\}.$$

Put $\sigma_\beta = \bigwedge_{K_\beta}\varphi(\gamma,\lambda)\wedge\bigwedge_{L_\beta}\overline{\varphi}(\gamma,\lambda)\wedge\bigwedge_{M_\beta}\overline{\varphi}(\gamma,\lambda,\mu)$. It is easily seen that $\langle K_\beta L_\beta M_\beta\rangle \in F_\beta$ and $\langle K_\alpha L_\alpha M_\alpha\rangle = \langle KLM\rangle$. Moreover, if $c(K, L, M)$ then also $c(K_\beta, L_\beta, M_\beta)$ for all $1 \leqslant \beta \leqslant \alpha$. Hence we get $\sigma_\beta \neq \mathbf{0}$ for all $\beta < \alpha$. Note that from (3.7) it follows that $\sigma_\beta \in M(\bigcup_{\gamma<\beta}T_\gamma)$ for all $1 \leqslant \beta \leqslant \alpha$.

If $\beta = 2\beta_0 > 0$, then $K_{\beta+1} = K_\beta$, $L_{\beta+1} = L_\beta \cup \{\langle\beta_0\lambda\rangle | \langle\beta_0\lambda\rangle \in L\}$ and $M_{\beta+1} = M \cup \{\langle\beta_0\lambda\mu\rangle | \langle\beta_0\lambda\mu\rangle \in M\}$. Hence $\sigma_{\beta+1} = \sigma_\beta\wedge\bigwedge\{\overline{\varphi}(\beta_0,\lambda) | \langle\beta_0\lambda\rangle \in L\}\wedge\bigwedge\{\overline{\varphi}(\beta_0,\lambda,\mu) | \langle\beta_0\lambda\mu\rangle \in M\} = \sigma_\beta - \bigvee(\{\varphi(\beta_0,\lambda) | \langle\beta_0\lambda\rangle \in L\} \cup \{\varphi(\beta_0,\lambda) | \lambda \in N_{\nu\mu}$ and $\langle\beta_0\nu\mu\rangle \in M\})$. If $\langle\beta_0\lambda\rangle \in L$ then, because $c(K, L, M)$, we have (by (3.12)) $\langle\beta_0\lambda\rangle \notin K$, hence $\langle\beta_0\lambda\rangle \notin K_\beta$. If $\lambda \in N_{\nu\mu}$ and $\langle\beta_0\nu\mu\rangle \in M$ then (by (3.13)) $\langle\beta_0\lambda\rangle \notin K$, hence $\langle\beta_0\lambda\rangle \notin K_\beta$. Consequently by Lemma 4.2 the B.p.'s subtracted from σ_β to get $\sigma_{\beta+1}$ are inessential in it with respect to $\bigcup_{\gamma<\beta}T_\gamma$. Note also that $\varphi(\beta_0,\lambda) \in M(\bigcup_{\gamma<\beta}T_\gamma)$.

If $\beta = 2\beta_0+1$ then $K_{\beta+1} = (K_\beta - \{\langle\beta_0\lambda\rangle |$ for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K\}) \cup \{\langle\beta_0+1, \lambda\rangle |$ for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K\}$, $L_{\beta+1} = L_\beta$ and $M_{\beta+1} = M_\beta$. Since $\varphi(\beta_0+1,\lambda) \leqslant \varphi(\beta_0,\lambda)$, we get $\bigwedge_{K_{\beta+1}}\varphi(\gamma,\lambda) = \bigwedge_{K_\beta}\varphi(\gamma,\lambda)\wedge\bigwedge\{\varphi(\beta_0+1,\lambda) |$ for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K\}$. If, for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K$ then $\langle\beta_0\lambda\rangle \in K_\beta$, therefore, since $\varphi(\beta_0+1,\lambda) = \varphi(\beta_0,\lambda)\wedge\bigwedge_{\mu<\delta}\varphi(\beta_0,\lambda,\mu)$, we get $\sigma_{\beta+1} = \sigma_\beta\wedge\bigwedge\{\varphi(\beta_0,\lambda,\mu) |$ for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K\} = \sigma_\beta - \bigvee\{\overline{\varphi}(\beta_0,\lambda,\mu) |$ for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K\}$. If, for some $\gamma > \beta_0+1$, $\langle\gamma\lambda\rangle \in K$ then, by (3.14), $\langle\beta_0\lambda\mu\rangle \notin M$ hence $\langle\beta_0\lambda\mu\rangle \notin M_\beta$; as we noted also, we have in this case $\langle\beta_0\lambda\rangle \in K_\beta$. Consequently, by Lemma 4.3, $\sigma_{\beta+1}$ is obtained by subtracting from σ_β B.p.'s which are in $M(T_{\beta-1})$ and are $T_{\beta-1}$-inessential in σ_β.

Finally it is easy to verify that for all β such that $0 < \beta = \bigcup\beta \leqslant \alpha$, $\sigma_\beta = \bigwedge_{\gamma<\beta}\sigma_\gamma$. Therefore, if $\alpha = (\alpha-1)+1$, $\sigma_\alpha \neq \mathbf{0}$ by Theorem 2.2 and, if $\alpha = \bigcup\alpha$, $\sigma_\alpha \neq \mathbf{0}$ by Theorem 2.3. This proves $\mathrm{pr}(\alpha)$, q.e.d.

Putting $K = \{\langle\alpha\lambda\rangle\}$, $L = \{\langle\alpha+1, \lambda\rangle\}$ and $M = \emptyset$ we get $\varphi(\alpha,\lambda)\wedge\overline{\varphi}(\alpha+1,\lambda) \neq \mathbf{0}$. Thus we get:

THEOREM. *The B.p.'s defined in Definition* 3.1 *satisfy the property that, for all $\lambda < \delta$ and all ordinals α, $\varphi(\alpha,\lambda) > \varphi(\alpha+1,\lambda)$. Consequently there is no set consisting of all B.p.'s on δ variables.*

LEMMA 4.5. *Let α be any ordinal and let $\lambda < \delta$. If $\psi \in \bigcup_{\beta<2\alpha}T_\beta$ and $\psi \neq \mathbf{0}$ then $\varphi(\alpha,\lambda)$* non $\geqslant \psi$. *(Hence $\varphi(\alpha,\lambda) \notin \bigcup_{\beta<2\alpha}T_\beta$.)*

Proof. In view of Lemma 3.3 we may assume that $\psi = \bigwedge_K\varphi(\gamma,\nu)\wedge\bigwedge_L\overline{\varphi}(\gamma,\nu)\wedge\bigwedge_M\overline{\varphi}(\gamma,\nu,\mu)$ where $\langle KLM\rangle \in F_\beta$ and $\beta < 2\alpha$. We have $c(K, L, M)$ (because $\psi \neq \mathbf{0}$), and, for all α', if $\langle\alpha'\lambda\rangle \in K$ then $\alpha' < \alpha$ (because $2\alpha' \leqslant \beta < 2\alpha$). Consequently if $L' = L \cup \{\langle\alpha\lambda\rangle\}$ then $c(K, L', M)$, and, by Lemma 4.4, $\overline{\varphi}(\alpha,\lambda)\wedge\psi = \bigwedge_K\varphi(\gamma,\nu)\wedge\bigwedge_{L'}\overline{\varphi}(\gamma,\nu)\wedge\bigwedge_M\overline{\varphi}(\gamma,\nu,\mu) \neq \mathbf{0}$, q.e.d.

Thus the first β for which $\varphi(\alpha,\lambda) \in T_\beta$ is 2α. It can also be shown that the first β such that $\varphi(\alpha,\lambda,\mu) \in T_\beta$ is $2\alpha+1$. In Part II we will show that this implies also that $\varphi(\alpha,\lambda) \notin \mathrm{Cl}(\mathrm{BP}_\beta)$ whenever $\beta < 2\alpha$, and that

$\varphi(\alpha,\lambda,\mu) \notin \mathrm{Cl}(\mathrm{BP}_\beta)$ whenever $\beta < 2\alpha+1$. It will be shown that this is a generalization of Lebesgue's hierarchy theorem concerning Borel sets. Lebesgue's theorem follows from this by putting $\delta = \omega_0$ and considering the B.p.'s $\varphi(\alpha,\lambda)$ where $\lambda < \delta$ and $\alpha < \omega_1$. It leads also to a very easy construction of Borel sets at any given place of the hierarchy.

§ 5. DEFINITION 5.1. $\varphi/(\alpha,\infty) \geqslant \psi/(\alpha,\infty)$ if for every complete (α,∞)-distributive B.a. $\mathfrak{B}$ and every sequence of length δ of its elements, b, $\varphi(b;\mathfrak{B}) \geqslant \psi(b;\mathfrak{B})$. $\varphi/(\alpha,\infty) = \psi/(\alpha,\infty)$ if $\varphi/(\alpha,\infty) \geqslant \psi/(\alpha,\infty)$ and $\psi/(\alpha,\infty) \geqslant \varphi/(\alpha,\infty)$. $\varphi/(\alpha,\infty) > \psi/(\alpha,\infty)$ if $\varphi/(\alpha,\infty) \geqslant \psi/(\alpha,\infty)$ and $\varphi/(\alpha,\infty) \neq \psi/(\alpha,\infty)$.

It is our aim to show that if δ is regular our results of § 4 continue to hold if we add the (δ,∞) distributive law. This means that for every α if $\psi \in T_\alpha$ and $\psi \neq \mathbf{0}$ then also $\psi/(\delta,\infty) \neq \mathbf{0}/(\delta,\infty)$. In particular, this implies that $\big(\varphi(\alpha,\lambda)\wedge\overline{\varphi}(\alpha+1,\lambda)\big)/(\delta,\infty) \neq \mathbf{0}/(\delta,\infty)$, which means that $\varphi(\alpha,\lambda)/(\delta,\infty) > \varphi(\alpha+1,\lambda)/(\delta,\infty)$ (obviously $\varphi(\alpha,\lambda)/(\delta,\infty) \geqslant \varphi(\alpha+1,\lambda)/(\delta,\infty)$). In order to show that $\psi/(\delta,\infty) \neq \mathbf{0}/(\delta,\infty)$ we have to establish the existence of a complete (δ,∞)-distributive B.a. $\mathfrak{B}$ and a sequence b of its elements for which $\psi(b;\mathfrak{B}) \neq 0$. The B.a.'s which we use for our purposes are the normal completions of the B.a.'s $B(T_\alpha)$. If $\mathfrak{B}$ is the normal completion of $B(T_\alpha)$ and $\psi \in T_\alpha$ then (since T_α is self-generated and $x_\lambda \in T_\alpha$ for all $\lambda < \delta$) it follows from Theorem 1.1 that $\psi(b;\mathfrak{B}) = \psi$, where $b = \langle x_0, \ldots, x_\lambda \ldots\rangle_{\lambda<\delta}$. Hence to prove that $\psi/(\delta,\infty) \neq \mathbf{0}/(\delta,\infty)$ it suffices to show that the normal completion of $B(T_\alpha)$ is (δ,∞)-distributive.

LEMMA 5.1. *A sufficient condition for the (δ,∞)-distributivity of a complete B.a. $\mathfrak{B}$ is the existence of a set B of elements of $\mathfrak{B}$ having the following properties:*

(i) *B is dense in $\mathfrak{B}$ (i.e., for every $b \in |\mathfrak{B}|$ if $b > 0$ then, for some $b' \in B$, $b \geqslant b' > 0$).*

(ii) *If $B' \subseteq B$, $\overline{\overline{B'}} < \delta$, and every finite meet of elements of B' is non zero then $\bigwedge B' \neq 0$.*

Proof. It is well known that $\mathfrak{B}$ is (δ,∞)-distributive if, whenever $\overline{\overline{J}} < \delta$, $\bigwedge_{j\in J}\bigvee_{i\in I_j}b_i > 0$ implies the existence of an f in $\Pi_{j\in J}I_j$ such that $\bigwedge_{j\in J}b_{f(j)} > 0$.

Assume $\overline{\overline{J}} < \delta$ and $\bigwedge_{j\in J}\bigvee_{i\in I_j}b_i > 0$. We may assume that $J = \alpha$ where α is some ordinal $< \delta$. Since $\bigwedge_{\lambda<\alpha}\bigvee_{i\in I_\lambda}b_i > 0$, there is a b_0 in B such that $\bigwedge_{\lambda<\alpha}\bigvee_{i\in I_\lambda}b_i \geqslant b_0 > 0$. Since $\bigvee_{i\in I_0}b_i \geqslant b_0$, there is an i_0 in I_0 such that $b_{i_0}\wedge b_0 > 0$. Let b_1 be a member of B for which $b_{i_0}\wedge b_0 \geqslant b_1 > 0$. Since $\bigvee_{i\in I_1}b_i \geqslant b_1$, there is an i_1 in I_1 such that $b_{i_1}\wedge b_1 > 0$. Let b_2 be a member of B for which $b_{i_1}\wedge b_1 \geqslant b_2 > 0$; and so on.

In general if b_λ and b_{i_λ} are defined for all $\lambda < \beta$, where $0 < \beta < \alpha$, so that $b_0 \geqslant b_1 \geqslant \ldots \geqslant b_\lambda \geqslant b_{\lambda+1} \ldots$, $b_\lambda > 0$ for all $\lambda < \beta$, and $b_{i_\lambda}\wedge b_\lambda \geqslant b_{\lambda+1}$, then from the properties of B we get $\bigwedge_{\lambda<\beta} b_\lambda > 0$. Consequently $\bigwedge_{\lambda<\beta} b_{i_\lambda} > 0$. $\bigvee_{i\in I_\beta} b_i \geqslant \bigwedge_{\lambda<\beta} b_i > 0$, therefore we can choose from B a b_β and from I_β an i_β so that $\bigwedge_{\lambda<\beta} b_\lambda \wedge b_{i_\beta} \geqslant b_\beta > 0$.

Carrying the construction through we finally get $\bigwedge_{\lambda<\beta} b_{i_\lambda} \geqslant \bigwedge_{\lambda<\beta} b^\lambda > 0$, q.e.d.

LEMMA 5.2. *If δ is regular then for every a the normal completion of $B(T_a)$ is (δ, ∞)-distributive.*

Proof. Let a be given. Let R be the set of all B.p.'s which are of the form $\bigwedge_K \varphi(\beta, \lambda) \wedge \bigwedge_L \overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M \overline{\varphi}(\beta, \lambda, \mu)$, where $\langle KLM \rangle$ is a member of F_a such that whenever $\langle \beta\lambda \rangle \in K$ and $2(\beta+1) < a$ then $\langle \beta+1, \lambda \rangle \in L$. Obviously $R \subseteq T_a$. We claim that:

(a) If $\psi \in T_a$ and $\psi > \mathbf{0}$ then $\psi \geqslant \psi' > \mathbf{0}$ for some $\psi' \in R$.

(b) If $R' \subseteq R$, $\overline{\overline{R'}} < \delta$ and $\psi \wedge \psi' \neq \mathbf{0}$ for all $\psi, \psi' \in R'$ then $\bigwedge R' \neq \mathbf{0}$.

Proof of (a). By Lemma 3.3 every B.p. of T_a is a join of B.p.'s of the form $\bigwedge_K \varphi(\beta, \lambda) \wedge \bigwedge_L \overline{\varphi}(\beta, \lambda) \wedge \bigwedge \overline{\varphi}(\beta, \lambda, \mu)$ where $\langle KLM \rangle \in F_a$, hence it suffices to prove (a) assuming that $\psi = \bigwedge_K \varphi(\beta, \lambda) \wedge \bigwedge_L \overline{\varphi}(\beta, \lambda) \wedge \wedge \bigwedge \overline{\varphi}(\beta, \lambda, \mu)$, where $\langle KLM \rangle \in F_a$. Since $\psi \neq \mathbf{0}$ we have (by Lemma 3.4) $c(K, L, M)$. For every λ such that $\langle \beta\lambda \rangle \in K$, for some β, there is a maximal γ such that $\langle \gamma\lambda \rangle \in K$. Put $K' = \{\langle \gamma\lambda \rangle | \langle \gamma\lambda \rangle \in K$ and, for all $\gamma' > \gamma$, $\langle \gamma'\lambda \rangle \notin K\}$. Since $\varphi(\beta, \lambda) \geqslant \varphi(\gamma, \lambda)$ if $\beta \leqslant \gamma$, it follows that $\bigwedge_K \varphi(\beta, \lambda) = \bigwedge_{K'} \varphi(\beta, \lambda)$.

Put $L' = L \cup \{\langle \beta+1, \lambda \rangle | \langle \beta\lambda \rangle \in K'$ and $2(\beta+1) < a\}$, and $\psi' = \bigwedge_{K'} \varphi(\beta, \lambda) \wedge \bigwedge_{L'} \overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M \overline{\varphi}(\beta, \lambda, \mu)$. Obviously $\psi' \in R$ and $\psi \geqslant \psi'$. Since $K' \subseteq K$ and (3.13), (3.14) hold for K and M, they hold also for K' and M. If $\langle \beta\lambda \rangle \in K'$ and $\beta' > \beta$ then $\langle \beta'\lambda \rangle \notin K'$, therefore (3.12) holds for K' and L'. Thus $c(K', L', M)$. By our main result of § 4 (Lemma 4.4), $c(K'L'M)$ implies $\psi' \neq \mathbf{0}$. This proves (a).

Proof of (b). Put $R' = \{\psi_i | i \in I\}$ where $\overline{\overline{I}} < \delta$, $\psi_i = \bigwedge_{K_i} \varphi(\beta, \lambda) \wedge \wedge \bigwedge_{L_i} \overline{\varphi}(\beta, \lambda) \wedge \bigwedge_{M_i} \overline{\varphi}(\beta, \lambda, \mu)$ and K_i, L_i, and M_i satisfy the conditions defining the members of R. Assume that $\psi_i \wedge \psi_j \neq \mathbf{0}$ for all $i, j \in I$.

$\bigwedge_{i\in I} \psi_i = \bigwedge_K \varphi(\beta, \lambda) \wedge \bigwedge_L \overline{\varphi}(\beta, \lambda) \wedge \bigwedge_M \overline{\varphi}(\beta, \lambda, \mu)$ where $K = \bigcup_{i\in I} K_i$, $L = \bigcup_{i\in I} L_i$, $M = \bigcup_{i\in I} M_i$. It is easily seen that if one of the conditions (3.12), (3.13), and (3.14) holds for all $K_i \cup K_j$, $L_i \cup L_j$, $M_i \cup M_j$, where $i, j \in I$, then this same condition holds also for the unions $\bigcup_{i\in I} K_i$, $\bigcup_{i\in I} L_i$, $\bigcup_{i\in I} M_i$. Since $\psi_i \wedge \psi_j \neq \mathbf{0}$ for all $i, j \in I$, we have $c(K_i \cup K_j, L_i \cup L_j, M_i \cup M_j)$ for all $i \in I$, hence $c(K, L, M)$.

The regularity of δ and the fact that $\overline{\overline{K}}_i, \overline{\overline{L}}_i, \overline{\overline{M}}_i < \delta$ for all $i \in I$ imply that $\overline{\overline{K}}, \overline{\overline{L}}, \overline{\overline{M}} < \delta$. (This is the only place that the regularity of δ is used). Thus in order to show that $\langle KLM \rangle \in F_a$ it remains to prove that for any β if $\langle \beta\lambda \rangle \in K$ then, for some maximal γ, $\langle \gamma\lambda \rangle \in K$ (the rest of the conditions defining member of F_a are easily seen to hold).

If $\langle \beta\lambda \rangle \in K$ and there is no maximal γ for which $\langle \gamma\lambda \rangle \in K$ then there are β' and β'' such that $\beta < \beta' < \beta''$ and $\langle \beta'\lambda \rangle$, $\langle \beta''\lambda \rangle \in K$. Hence there are i and j such that $\langle \beta\lambda \rangle \in K_i$, $\langle \beta''\lambda \rangle \in K_j$, and $\beta'' \geqslant \beta+2$. Since $2\beta'' \leqslant a$, we have $2(\beta+1) < a$ and therefore $\langle \beta+1, \lambda \rangle \in L_i$. This yields:

$$\psi_j \wedge \psi_i \leqslant \bigwedge_{K_j} \overline{\varphi}(\gamma, \lambda) \wedge \bigwedge_{L_i} \overline{\varphi}(\gamma, \lambda) \leqslant \varphi(\beta'', \lambda) \wedge \overline{\varphi}(\beta+1, \lambda) = \mathbf{0}.$$

Therefore $\langle KLM \rangle \in F_a$, and using Lemma 4.4 we infer that $\bigwedge_{i\in I} \psi_i \neq \mathbf{0}$. This proves (b).

Consider R as a set of members of $B(T_a)$. It is dense in $B(T_a)$, hence it is dense in the normal completion of $B(T_a)$. By Lemma 5.1 the normal completion of $B(T_\beta)$ is (δ, ∞)-distributive, q.e.d.

This proves the following

THEOREM. *If δ is regular and ψ is a non-zero B.p. of some T_β then $\psi/(\delta, \infty) \neq \mathbf{0}/(\delta, \infty)$. In particular $\varphi(a, \lambda)/(\delta, \infty) > \varphi(a+1, \lambda)/(\delta, \infty)$ for all $\lambda < \delta$ and all a.*

§ 6. The power of the α-free Boolean algebra. As is well known, if a and δ are infinite cardinals then the B.a. of all a-B.p.'s on δ variables, is the a-free B.a. on δ generators. By an a-B.p. we mean a B.p. that can be represented by a Boolean term in which the joins and the meets are only over sets of power $< a$. The set BP^a of all a-B.p.'s is defined by

DEFINITION 6.1. BP^a is the smallest set of B.p.'s satisfying:

(i) $x_\lambda \in \mathrm{BP}^a$ for all $\lambda < \delta$.

(ii) If $\varphi \in \mathrm{BP}^a$ then $\overline{\varphi} \in \mathrm{BP}^a$ and if $R \subseteq \mathrm{BP}^a$ and $\overline{\overline{R}} < a$ then $\bigvee R \in \mathrm{BP}^a$ and $\bigwedge R \in \mathrm{BP}^a$.

(It can be easily established that $\varphi \in \mathrm{BP}^a$ iff $\varphi = [f]$ for some f in BT^a, where BT^a is the set of all a-terms defined in § 0).

BP^a is used by Rieger to establish the existence of the a-free B.a., [7]. This, however, is a special case of the general method to prove the existence of free algebras (see Birkhoff [2]).

Note that in $\varphi(\beta, \lambda)$ the joins and meets are applied only to sets whose power is at most $\mathrm{Max}(\overline{\overline{\beta}}, \delta)$. Consequently if a is a cardinal and $\delta < a$ the B.p.'s $\varphi(0, 0), \varphi(1, 0), \ldots, \varphi(\beta, 0), \ldots$, $\beta < a$, are all a-B.p.'s. Since by our results they are all different, we get $\overline{\overline{\mathrm{BP}^a}} \geqslant a$. This proves the following

THEOREM. *If a and δ are infinite cardinals then the power of the a-free B.a. on δ generators is at least $\mathrm{Max}(a, \delta)$.*

An upper bound for the power of the a-free B.a. on δ generators can be obtained by the following calculation.

Put $\mathrm{BP}^a_0 = \{x_0, \ldots, x_\lambda \ldots\}_{\lambda<\delta}$ and define by induction BP^a_β to be the set of all B.p.'s which are of one of the forms $\overline{\varphi}$, $\bigvee R$, $\bigwedge R$, where $\varphi \in \bigcup_{\gamma<\beta} \mathrm{BP}^a_\gamma$ and R is a subset of $\bigcup_{\gamma<\beta} \mathrm{BP}^a_\gamma$ of power $< a$. Assume that a is regular, then it is clear that $\mathrm{BP}^{\alpha} = \bigcup_{\beta<\alpha}\mathrm{BP}^{\alpha}_{\beta}$. (The case of a singular α need no discussion since in that case the α-free B.a. is also the α^{+}-free B.a. where α^{+} is the next cardinal after α.) If $\beta > 0$ then the power of $\mathrm{BP}^{\alpha}_{\beta}$ is at most that of the family of all subsets of $\bigcup_{\gamma<\beta}\mathrm{BP}^{\alpha}_{\gamma}$ which are of power $<\alpha$. This, in case $\overline{\overline{\bigcup_{\gamma<\beta}\mathrm{BP}^{\alpha}_{\gamma}}} \geqslant \alpha$, is $\sum_{\lambda<\alpha}(\overline{\overline{\bigcup_{\gamma<\beta}\mathrm{BP}^{\alpha}_{\gamma}}})^{\lambda}$, where the summation is over all cardinals $<\alpha$ (cardinal summation and exponentiation), and in case $\alpha > \overline{\overline{\bigcup_{\gamma<\beta}\mathrm{PB}^{\alpha}_{\gamma}}}$ it is $2^{\bigcup_{\gamma<\beta}\mathrm{BP}^{\alpha}_{\gamma}}$. We distinguish two cases:

(i) $\delta > \alpha$. Then $\overline{\overline{\mathrm{BP}^{\alpha}_{0}}} = \delta$, $(\sum_{\beta<\alpha}\delta^{\beta})^{\beta} = \sum_{\beta<\alpha}\delta^{\beta}$ whenever $\beta < \alpha$, hence $\sum_{\beta<\alpha}(\sum_{\beta<\alpha}\delta^{\beta})^{\beta} = \sum_{\beta<\alpha}\delta^{\beta}$. Consequently $\overline{\overline{\mathrm{BP}^{\alpha}_{\beta}}} \leqslant \sum_{\beta<\alpha}\delta^{\beta}$ for all $\beta < \alpha$ and therefore $\overline{\overline{\mathrm{BP}^{\alpha}}} \leqslant \sum_{\beta<\alpha}\delta^{\beta}$. If we assume the general continuum hypothesis this sum would be either δ or δ^{+}. (It will be δ if δ is not of the form $\bigcup_{\lambda<\gamma}\beta_{\lambda}$ where $\gamma < \alpha$ and $\beta_{\lambda} < \delta$ for all $\lambda < \gamma$.)

(ii) $\alpha > \delta$. Here if we assume the general continuum hypothesis it is easily seen that $\overline{\overline{\mathrm{BP}^{\alpha}_{\beta}}} \leqslant \alpha$ for all $\beta < \alpha$ hence $\overline{\overline{\mathrm{BP}^{\alpha}}} \leqslant \alpha$. Without the general continuum hypothesis the estimate is more complicated: Let $f(\delta, \beta)$ be defined by $f(\delta, 0) = \delta$, $f(\delta, \beta+1) = 2^{f(\delta,\beta)}$ and $f(\delta, \beta) = \sum_{\gamma<\beta} f(\delta, \gamma)$ if $\beta = \bigcup\beta > 0$. If, for no $\beta < \alpha$, $f(\delta, \beta) \geqslant \alpha$, then $\overline{\overline{\mathrm{BP}^{\alpha}}} \leqslant \alpha$. Otherwise, if β_0 is the first $\beta < \alpha$ for which $f(\delta, \beta) \geqslant \alpha$, then $\overline{\overline{\mathrm{BP}^{\alpha}}} \leqslant \sum_{\gamma<\alpha} f(\delta, \beta_0)^{\gamma}$.

We get the following

THEOREM. *If α and δ are infinite cardinals and α is regular, then the α-free B.a. on δ generators is at least of power* $\mathrm{Max}(\alpha, \delta)$. *Assuming the general continuum hypothesis it is of power α if $\alpha > \delta$ and either of power δ or δ^{+} if $\alpha \leqslant \delta$.*

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Ein eindimensionales Kompaktum im E^3, das sich nicht lagetreu in die Mengersche Universalkurve einbetten läßt

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1. Vorbemerkungen. E^3 sei der dreidimensionale euklidische Raum, den wir auf ein festes kartesisches Koordinatensystem bezogen denken. Der *Einheitswürfel* W_e ist die Menge aller Punkte $p = (\xi_1, \xi_2, \xi_3)$, für die $0 \leqslant \xi_i \leqslant 1$ gilt $(i = 1, 2, 3)$. Um die Mengersche Universalkurve U zu definieren, bezeichnen wir mit D_i die Vereinigung aller offenen Intervalle $(k/3^i, (k+1)/3^i)$, wo k alle ganzen Zahlen zwischen 1 und 3^i durchläuft, die bei Division durch 3 den Rest 1 lassen und mit R_i die Menge aller Punkte aus W_e, von deren Koordinaten wenigstens zwei zu D_i gehören. Es ist dann $U = W_e \setminus \bigcup_{i=1} R_i$. Jeder eindimensionale metrische Raum läßt sich homöomorph in U einbetten ([2], Kap. XII).

Es soll hier ein in E^3 enthaltenes eindimensionalen Kompaktum X konstruiert werden, zu dem es keinen Homöomorphismus h von E^3 auf sich mit der Eigenschaft $h(X) \subseteq U$ gibt.

2. Eine Eigenschaft von U. *Ist C eine zahme einfach geschlossene Kurve in E^3 und $\varepsilon > 0$, so gibt es einen Homöomorphismus h von E^3 auf sich, der jeden Punkt von C um weniger als ε verrückt mit der Eigenschaft $h(C) \cap U = 0$.*

Beweis. Es sei δ eine beliebige positive Zahl. Da C zahm ist, gibt es einen Homöomorphismus h_1 von E^3 auf sich, der jeden Punkt von C um weniger als δ verrückt und für den $C_1 = h_1(C)$ ein einfach geschlossenes Polygon ist (Approximation von Homöomorphismen in den E^3 durch semilineare Abbildungen; siehe [3]). Man findet leicht einen zweiten Homöomorphismus h_2 von E^3 auf sich, der jeden Punkt von C_1 um weniger als δ verrückt und für den $C_2 = h_2(C_1)$ ein Polygon ist, dessen sämtliche Strecken zu Koordinatenachsen parallel sind. Mit R_i^* wollen wir die Menge aller der Punkte bezeichnen, deren sämtliche Koordinaten in $D_i \cup (E^1 \setminus [0, 1])$ liegen. Die Vereinigung $R^* = \bigcup_{i=1}^{\infty} R_i^*$ ist dann eine offene dichte Teilmenge von E^3. Der Homöomorphismus h_3 sei schließlich eine