consequently with suitable $g_1, \ldots, g_n$ we have $a_j = g_j(b_1, \ldots, b_n)$, $j = 1, \ldots, n$. Since the system $(g_1, \ldots, g_n)$ is evidently independent, it follows from the independence of the systems $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ that $(g_1, \ldots, g_n) = f^{-1}$, which ends the proof.

(ii) *If $\mathfrak{A}$ is a v^{**}-algebra with a finite basis in which every independent set can be extended to a basis, then $\mathfrak{A}$ is a v^*-algebra.*

Proof. Let $\mathfrak{A}$ possess a basis of n elements. Then every independent n-tuple forms a basis (because it can be extended to a basis and from theorem I it follows that every basis has n elements) and it remains to apply the foregoing statement.

(iii) *If A is a v^{**}-algebra with a finite basis and $\mathfrak{A}$ does not contain a subalgebra different from $\mathfrak{A}$ but with $\mathfrak{A}$ isomorphic, then $\mathfrak{A}$ is a v^*-algebra.*

This statement is an immediate consequence of (i), since it follows from the assumption that every independent n-tuple (where n is the power of the basis) is a basis.

The last statement shows that if we introduce the notion of dimension for v^{**}-algebra with a finite basis as the cardinal number of the basis (which is well-defined in view of theorem I), then the dimension of a subalgebra can be equal to the dimension of the algebra and that this peculiarity does not occur only for v^*-algebras.

Statement (iii) is not true for algebras with an infinite basis, because $[a_1, a_2, \ldots] \approx [a_2, \ldots]$. We do not know whether (ii) is false for algebras with an infinite basis.

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Reçu par la Rédaction le 7. 12. 1962

Convergence functions and their related topologies *

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Introduction

A *convergence function* is a correspondence between the filters on a given set S and the subsets of S which specifies which filters converge to which points of S. This concept is defined to include types of convergence which are more general than that defined by specifying a topology on S. Thus a convergence function may be regarded as a generalization of a topology.

Various generalizations of the latter concept have been made in the past with the help of convergence criteria; structures of this type have been identified with such names as "limitierung", "pseudo-topologie", and "pretopologie". These latter structures may be regarded as special cases of convergence functions, more topology-like than the basic structures which we investigate.

The method used here to study the convergence function is to place it in the ordered environment of a complete lattice $\mathcal{C}(S)$, whose elements are all the convergence functions on an arbitrary set S. Letting q be an arbitrary convergence function on S, we associate with q various topologies which are related to q in a more or less natural way. To associate topologies with q systematically, the concept of *linkage function* is introduced. A linkage function may be regarded as a method for obtaining a topology from a convergence function which is valid for any convergence function in $\mathcal{C}(S)$.

We investigate and compare four fundamental linkage functions. The first section introduces some relevant definitions, gives certain structural properties of $\mathcal{C}(S)$, and defines what is perhaps the simplest and most natural of linkage functions. A different linkage function is investigated in each of the remaining three sections.

* Research supported by the National Science Foundation, Grant NSF-G-21219. The work reported here was performed in partial fulfillment of the requirements for the Ph. D. Degree at the University of New Mexico.

I. The concepts convergence function and linkage function

We consider S to be an arbitrary set unless otherwise specified. Two sets related to S are defined as follows:

(1) $\mathfrak{P}(S)$ denotes the set of all subsets of S, partially ordered [1] by set inclusion.

(2) $F(S)$ designates the set of all filters on S, partially ordered by set inclusion.

The symbol $\mathfrak{F}$ will be used to denote a filter in $F(S)$. For all $x \in S$, $\mathfrak{F}_x$ represents the ultrafilter generated by the set $\{x\}$.

Most of the sets with which we deal, such as $\mathfrak{P}(S)$, $F(S)$, and $\mathfrak{C}(S)$, will be partially ordered; the symbol $\leqslant$ will always be used to describe the order relation. In case the poset consists of filters or topologies, $x \leqslant y$ may be read "y is finer than x". If $x \in S$ and $A \subset S$, $x \leqslant A$ means $x \leqslant y$ for all $y \in A$. An order preserving function from one poset to another is said to be *isotone*. The symbols $<$, $\geqslant$, $>$ will also be employed with their usual meanings.

Definition 1. An isotone function q mapping $F(S)$ into $\mathfrak{P}(S)$ is called a *convergence function* if and only if $x \in q(\mathfrak{F}_x)$ for all $x \in S$.

If $x \in q(\mathfrak{F})$ for some arbitrary filter $\mathfrak{F} \in F(S)$, $\mathfrak{F}$ is said to *q-converge to x*. We see immediately that convergence relative to any topology satisfies the conditions of Definition 1, and thus defines a convergence function.

For any convergence function q, let $\mathfrak{V}_q(x)$ designate the intersection of the set of all ultrafilters which q-converge to x. $\mathfrak{V}_q(x)$ is called the *q-neighborhood filter at x*.

Definition 2. A convergence function q is *pretopological* if and only if $\mathfrak{V}_q(x)$ q-converges to x for all $x \in S$.

Definition 3. A convergence function q is *topological* if and only if q is pretopological, and for each $x \in S$, the filter $\mathfrak{V}_q(x)$ has a filter base $\mathfrak{G}_q(x) \subset \mathfrak{V}_q(x)$ with the following property: $y \in G(x) \in \mathfrak{G}_q(x)$ implies $G(x) \in \mathfrak{G}_q(y)$.

If q is a topological convergence function, then the members of $\mathfrak{G}_q(x)$ form a base for the family of open neighborhoods at x under some topology; thus q uniquely defines a topology on S. Conversely, given a topology on S, the open sets which contain x generate a filter $\mathfrak{V}_q(x)$ satisfying the requirements of Definition 3. Consequently, we shall use the terms "topological convergence function" and "topology" interchangeably. A pretopological convergence function will usually be called a "pretopology".

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[1] A partial ordering is a reflexive, anti-symmetric, and transitive relation. (We abbreviate "partially ordered set" by "poset".)

Structures similar to convergence functions have been studied by Fischer [2] and Choquet [1]. The "convergence function" considered here is more general than the "limitierung" of Fischer and "pseudo-topologie" of Choquet. For a convergence function q to be a limitierung, it is necessary and sufficient that the following condition be satisfied:

(a) $x \in q(\mathfrak{F}_1)$ and $x \in q(\mathfrak{F}_2)$ implies $x \in q(\mathfrak{F}_1 \cap \mathfrak{F}_2)$.

For q to be a pseudo-topology, the following additional condition is necessary and sufficient:

(b) $x \in q(\mathfrak{F})$ if $x \in q(\mathfrak{F}')$ for all ultrafilters $\mathfrak{F}'$ finer than $\mathfrak{F}$.

A pretopological convergence function is a "pseudo-topologie" in the sense of Choquet. These structures may be listed in order of increasing generality as follows: topology, pretopology, pseudo-topology, limitierung, convergence function.

Let $\mathfrak{C}(S)$ be the set of all convergence functions on S, partially ordered as follows: $q_1 \leqslant q_2$ iff $q_1(\mathfrak{F}) \supset q_2(\mathfrak{F})$, for all $\mathfrak{F} \in F(S)$. For any $q \in \mathfrak{C}(S)$ we define the following related convergence functions:

(1) $\tilde{q}$: $x \in \tilde{q}(\mathfrak{F})$ if and only if there are filters $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ q-converging to x such that $\mathfrak{F} = \bigcap_{i=1}^{n} \mathfrak{F}_i$.

(2) q^*: $x \in q^*(\mathfrak{F})$ if and only if $x \in q(\mathfrak{F}')$ for each ultrafilter $\mathfrak{F}'$ finer than $\mathfrak{F}$.

(3) $\hat{q}$: $x \in \hat{q}(\mathfrak{F})$ if and only if $\mathfrak{F} \geqslant \mathfrak{V}_q(x)$.

(4) $\lambda(q)$: $x \in \lambda(q)(\mathfrak{F})$ if and only if $\mathfrak{F} \geqslant \mathfrak{U}_q(x)$, where $\mathfrak{U}_q(x)$ is the filter generated by the sets $U \in \mathfrak{V}_q(x)$ which have the property: $y \in U$ implies $U \in \mathfrak{V}_q(y)$.

At least one member of $\mathfrak{V}_q(x)$ has this property, namely S. Equivalently, $\mathfrak{U}_q(x)$ may be defined as the filter generated by those sets $U \in \mathfrak{V}_q(x)$ for which $y \in U$ and $y \in q(\mathfrak{F})$ implies $U \in \mathfrak{F}$.

Theorem 1. (1) *$\tilde{q}$ is the finest limitierung coarser than q.*

(2) *q^* is the finest pseudo-topology coarser than q.*

(3) *$\hat{q}$ is the finest pretopology coarser than q.*

(4) *$\lambda(q)$ is the finest topology coarser than q.*

(5) $\lambda(q) \leqslant \hat{q} \leqslant q^* \leqslant \tilde{q}$.

Proof. Parts (1), (2), (3) and (5) of the Theorem are immediate consequences of the relevant definitions. (4) is proved by Fischer [2] in the case where q is a limitierung; no alteration is necessary in extending the proof to the present situation.

The $\lambda(q)$-open sets are those sets $U \subset S$ such that $y \in U$ implies $U \in \mathfrak{V}_q(y)$.

In general there is no coarsest topology finer than q; this result also extends to the other three specializations of a convergence function which we consider.

The poset $\mathfrak{C}(S)$ is a complete lattice, whose greatest and least elements are the discrete and indiscrete topologies respectively. For any non-void set $Q \subset \mathfrak{C}(S)$, the convergence functions $\sup Q$ and $\inf Q$ always exist and are given by

$$(\sup Q)(\mathfrak{F}) = \inf\{q(\mathfrak{F}):\ q \in Q\},$$
$$(\inf Q)(\mathfrak{F}) = \sup\{q(\mathfrak{F}):\ q \in Q\}.$$

Let $\mathfrak{I}(S)$ designate the subset of $\mathfrak{C}(S)$ consisting of all topological convergence functions on S, ordered by inheritance. Equivalently, we may consider $\mathfrak{I}(S)$ to be the set of all topologies on S ordered by set inclusion. $\mathfrak{I}(S)$ is a complete lattice, but not a sub-complete-lattice of $\mathfrak{C}(S)$, since the infimum of a set of topologies in $\mathfrak{C}(S)$ need not be a topology.

THEOREM 2. *For each set $T \subset \mathfrak{I}(S) \subset \mathfrak{C}(S)$, $\sup T$ is a topology.*

Proof. If $q = \sup T$, then, by Theorem 1, $\lambda(q) \geqslant T$, implying $\lambda(q) \geqslant q$. Since $\lambda(q) \leqslant q$, the result follows.

For any set $T \subset \mathfrak{I}(S)$, there are two distinct infima which we may consider: we denote by $\inf_c T$ the infimum of $T \subset \mathfrak{I}(S)$ with respect to the lattice $\mathfrak{C}(S)$, and by $\inf_t T$ the infimum relative to $\mathfrak{I}(S)$. It is clear that $\inf_c T \geqslant \inf_t T$ in $\mathfrak{C}(S)$, and $\inf_c T = \inf_t T$ iff $\inf_c T \in \mathfrak{I}(S)$.

DEFINITION 4. A *linkage function* is a mapping of $\mathfrak{C}(S)$ into $\mathfrak{I}(S)$ under which the members of $\mathfrak{I}(S)$ are fixed points. An isotone linkage function is called a *linkage homomorphism*.

Let λ designate the function which assigns to each convergence function q the first topology coarser than q.

THEOREM 3. *λ is a linkage homomorphism.*

Proof. If q is a topology, then $\lambda(q) = q$ is obvious. Let $q_1 \geqslant q_2$. Then $q_1 \geqslant q_2 \geqslant \lambda(q_2)$. Since $\lambda(q_1)$ is the finest topology coarser than q_1, $\lambda(q_1) \geqslant \lambda(q_2)$.

As a corollary to Theorem 1, we note that $\lambda(q) = \lambda(\tilde{q}) = \lambda(q^*) = \lambda(\hat{q})$.

II. The linkage function φ

In the theory of partially ordered sets, *order convergence* is defined as follows: $\mathfrak{F}$ order-converges to x if and only if $x = \inf U(\mathfrak{F}) = \sup L(\mathfrak{F})$, where $U(\mathfrak{F}) = \{y:$ there is an $F \in \mathfrak{F}$ such that $y \geqslant F\}$ and $L(\mathfrak{F})$ is defined dually. If P is any poset, then order convergence defines a convergence function on P, as is easily verified. The so-called "order topology" on P is derived from order convergence by a procedure described in References 3 and 4. We shall now generalize this procedure, thereby obtaining the linkage function φ on $\mathfrak{C}(S)$.

Let q be an arbitrary member of $\mathfrak{C}(S)$.

DEFINITION 1. The set function Γ_q: $\mathfrak{P}(S) \to \mathfrak{P}(S)$ is defined on all subsets $A \subset S$ by:

$\Gamma_q(A) = \{x \in S$: there is an ultrafilter $\mathfrak{F}$ q-converging to x with $A \in \mathfrak{F}\}$.

The set operation Γ_q defines what Choquet [1] calls a "pre-adherence structure" on S; this is a closure structure in the topological sense, except in general $\Gamma_q(\Gamma_q(A)) \neq \Gamma_q(A)$; an example in which $\Gamma_q(\Gamma_q(A)) \neq \Gamma_q(A)$ is given by Rennie ([3], Example 5, p. 399) in the case where S is a poset and q is order convergence.

DEFINITION 2. The set function $\overline{\Gamma}_q$: $\mathfrak{P}(s) \to \mathfrak{P}(S)$ is defined for all $A \subset S$ by letting $\overline{\Gamma}_q(A)$ be the intersection of all sets in the range of Γ_q which contain A.

The set function $\overline{\Gamma}_q$ satisfies the topological closure axioms. We denote by $\varphi(q)$ the topology consisting of those sets which are complements of sets in the range of $\overline{\Gamma}_q$. If S is a partially ordered set and q is order convergence, $\varphi(q)$ is the "order topology" on S.

If q is a topology, then Γ_q assigns to each set $A \subset S$ its usual topological closure $\overline{A}$; thus $\varphi(q) = q$. From this we conclude that φ is a linkage function. An example at the end of this section will show that φ is not a linkage homomorphism.

The following lemma is an immediate consequence of Definition 1:

LEMMA 1. *If $q_1 \geqslant q_2$, then $\Gamma_{q_1}(A) \subset \Gamma_{q_2}(A)$ for all $A \subset S$.*

THEOREM 1. *For all $A \subset S$, $\Gamma_{\hat{q}}(A) = \Gamma_q(A)$.*

Proof. Since $\hat{q} \leqslant q$, $\Gamma_q(A) \subset \Gamma_{\hat{q}}(A)$ follows from Lemma 1. Suppose $x \in \Gamma_{\hat{q}}(A)$. Then an ultrafilter $\mathfrak{F}$ $\hat{q}$-converges to x and contains A. If all ultrafilters which q-converge to x fail to contain A, then $\mathrm{Co}\,A \in \mathfrak{V}_q(x)$, contradicting the assumption $\mathfrak{F} \geqslant \mathfrak{V}_q(x)$. Thus at least one ultrafilter which q-converges to x contains A, and $x \in \Gamma_q(A)$.

THEOREM 2. *$\Gamma_q(A) = A$ if and only if A is closed relative to the topology $\lambda(q)$.*

Proof. Assume A is $\lambda(q)$-closed. Then if $x \in \mathrm{Co}\,A$ and $x \in q(\mathfrak{F})$, $\mathrm{Co}\,A \in \mathfrak{F}$. Thus there are no ultrafilters which q-converge to x and contain A; hence $x \notin \Gamma_q(A)$. Conversely, assume $\Gamma_q(A) = A$. Let $x \in \mathrm{Co}\,A$. If an ultrafilter $\mathfrak{F}$ q-converges to x, A cannot be a member of $\mathfrak{F}$, otherwise $\Gamma_q(A) \neq A$. Thus for each ultrafilter $\mathfrak{F}$ which q-converges to x, $\mathrm{Co}\,A \in \mathfrak{F}$; hence $\mathrm{Co}\,A$ is $\lambda(q)$-open.

We denote the neighborhood filter at x with respect to the topology $\varphi(q)$ by $\mathfrak{W}_q(x)$. Each $\varphi(q)$-open set is a union of sets of the form $\mathrm{Co}\,\Gamma_q(A)$; thus sets of this form which contain x generate the filter $\mathfrak{W}_q(x)$.

THEOREM 3. *$\mathrm{Co}\,\Gamma_q(A) \in \mathfrak{W}_q(x)$ iff $\mathrm{Co}\,A \in \mathfrak{V}_q(x)$.*

Proof. (1) $x \in \mathrm{Co}\,\Gamma_q(A)$ implies that each ultrafilter which refines $\mathfrak{V}_q(x)$ contains $\mathrm{Co}\,A$, and hence $\mathrm{Co}\,A \in \mathfrak{V}_q(x)$.

(2) $x \notin \mathrm{Co}\Gamma_q(A)$ implies that some ultrafilter $\mathfrak{F} \geqslant \mathfrak{V}_q(x)$ contains A. From this it follows that $\mathrm{Co}A \notin \mathfrak{V}_q(x)$.

COROLLARY. $\lambda(q) \leqslant \hat{q} \leqslant \varphi(q)$.

THEOREM 4. *The following statements are equivalent:*

(1) $\lambda(q) = \varphi(q)$,

(2) $\hat{q}$ *is a topology,*

(3) $\Gamma_q = \bar{\Gamma}_q$.

Proof. (1) implies (2) by the Corollary to Theorem 3.

(2) implies (3), since $\Gamma_q = \Gamma_{\hat{q}} = \bar{\Gamma}_{\hat{q}} = \bar{\Gamma}_q$.

(3) implies (1): Indeed, since $\Gamma_q(\Gamma_q(A)) = \Gamma_q(A)$, $\Gamma_q(A)$ is $\lambda(q)$-closed by Theorem 2. Thus $\lambda(q) \geqslant \varphi(q)$. (1) now follows from the above Corollary.

To conclude this section, we cite examples to show that φ is not a linkage homomorphism.

EXAMPLE. Let S be an infinite set, partitioned into a family of mutually disjoint sets, each of which consists of a pair $\{x, x'\}$ of elements of S. A pretopology q_1 is defined by specifying the neighborhood filter $\mathfrak{V}_{q_1}(x)$ at $x \in S$ to consist of the sets S and $S-\{x'\}$; similarly, $\mathfrak{V}_{q_1}(x')$ consists of the sets S and $S-\{x\}$. This assignment of neighborhood filters is extended to all pairs $\{x, x'\}$ in the stated partition of S. Since $\Gamma_{q_1}(\{x'\}) = S-\{x\}$, the set $\{x\}$ is $\varphi(q_1)$-open; since this is true for all $x \in S$, $\varphi(q_1)$ is the discrete topology. One easily sees that $\lambda(q_1)$ is the indiscrete topology.

To see that φ is not a linkage homomorphism, we define q_2 to be the topology whose open sets are complements of finite sets. Since $\varphi(q_2) = q_2$, we have $q_2 > q_1$, but $\varphi(q_2) < \varphi(q_1)$.

III. The linkage function σ

If there were a coarsest topology finer than a given convergence function q, then it would be possible to define a linkage function dual to λ in an obvious way. The linkage function σ is perhaps as close as one can come to accomplishing this objective.

DEFINITION 1. For each $q \in \mathfrak{C}(S)$, let $M(q) = \{p\colon p \in \mathfrak{T}(S)$ and $q \leqslant p\}$. Let $\sigma(q) = \inf_t M(q)$.

THEOREM 1. *σ is a linkage homomorphism. For all $q \in \mathfrak{C}(S)$, $\sigma(q) \geqslant \lambda(q)$.*

Proof. (1) If q is topological then $q \in M(q)$, and hence $q = \inf M(q)$.

(2) If $q_1 \leqslant q_2$, then $M(q_2) \subset M(q_1)$ and $\sigma(q_1) \leqslant \sigma(q_2)$.

(3) $\lambda(q) \leqslant q \leqslant M(q)$.

From the definition of the linkage function φ it is clear that $\varphi(q_1) = \varphi(q_2)$ whenever q_1 and q_2 agree on ultrafilters; the same remark holds for λ. In contrast, we have the following result for σ.

THEOREM 2. *If q_1 and q_2 are convergence functions which coincide on non-ultrafilters, then $\sigma(q_1) = \sigma(q_2)$.*

Proof. Let p be a topological convergence function with $p \geqslant q_1$. Then $\mathfrak{V}_p(x)$ q_1-converges to x. If $\mathfrak{V}_p(x)$ is a non-ultrafilter, then $\mathfrak{V}_p(x)$ q_2-converges to x, since q_1 and q_2 agree on non-ultrafilters. If $\mathfrak{V}_p(x) = \mathfrak{F}_x$, then $\mathfrak{V}_p(x)$ q_2-converges to x, since q_2 is a convergence function. Thus $p \geqslant q_2$. This argument is reciprocal, therefore $M(q_1) = M(q_2)$.

LEMMA 1. *Let $q \in \mathfrak{C}(S)$ and let $\mathfrak{F}$ q-converge to x, with $\mathfrak{V}_q(x) \leqslant \mathfrak{F} \leqslant \mathfrak{F}_x$. Then there is $p \in M(q)$ such that $\mathfrak{V}_p(x) = \mathfrak{F}$.*

Proof. (1) To construct p, let $\mathfrak{S}$ denote the class of all sets of the form $C \cup F$, where $C \subset S-\{x\}$ and $F \in \mathfrak{F}$ or $F = \Phi$. One easily verifies that the class $\mathfrak{S}$ is a topology, which we denote by p.

(2) $p \geqslant q$. If $y \neq x$, then $\{y\} \in \mathfrak{S}$ and hence $\mathfrak{V}_p(y) = \mathfrak{F}_y$. If $F \in \mathfrak{F}$, then $F \cup \Phi \in \mathfrak{S}$, $x \in F$, and hence $F \in \mathfrak{V}_p(x)$. Thus $\mathfrak{V}_p(x) \geqslant \mathfrak{F}$. Thus for all $y \in S$, $\mathfrak{V}_p(y)$ q-converges to y.

(3) $\mathfrak{V}_p(x) = \mathfrak{F}$. If $U \in \mathfrak{S}$ and $x \in U$, then $U = C \cup F$, $F \in \mathfrak{F}$. Thus $U \in \mathfrak{F}$.

Let $\bar{q} = \inf M(q)$.

THEOREM 3. *If q is a limitierung, then $q = \bar{q}$.*

Proof. $\bar{q} \geqslant q$ is clear. Suppose $x \in q(\mathfrak{F})$. Then $\mathfrak{F} \cap \mathfrak{F}_x$ q-converges to x, and by Lemma 1 there is $p \in M(q)$ such that $\mathfrak{V}_p(x) = \mathfrak{F} \cap \mathfrak{F}_x$. Since $\mathfrak{V}_p(x)$ $\bar{q}$-converges to x, $x \in \bar{q}(\mathfrak{F})$ and $\bar{q} \leqslant q$.

The set of all topologies which are lower bounds of $M(q)$ coincides with the set of all topologies which are lower bounds of $\bar{q}$. From this we deduce the following corollary:

COROLLARY. *If q is a limitierung, then $\lambda(q) = \lambda(\bar{q}) = \sigma(q)$.*

THEOREM 4. *There is a coarsest limitierung (respectively pseudo-topology, pretopology, topology) finer than q if and only if $\bar{q}$ is a limitierung (respectively pseudo-topology, pretopology, topology).*

Proof. (1) Let q' be the coarsest limitierung finer than q. Then $M(q') \subset M(q)$ and $\inf_c M(q') = q' \geqslant \bar{q}$. On the other hand, if $L(q)$ denotes the set of all limitierungs finer than q, then $L(q') = L(q) \supset M(q)$; hence $\bar{q} \geqslant \inf_c L(q') = q'$.

(2) Conversely, assume $\bar{q}$ is a limitierung. If p is a limitierung finer than q, then $M(p) \subset M(q) = M(\bar{q})$. Hence $\bar{q} \leqslant \inf_c M(p) = p$.

(3) The proof requires no alteration if "limitierung" is replaced by "pseudo-topologie", "pretopology", or "topology".

IV. The linkage function ϱ

DEFINITION 1. A convergence function q is *quasi-topological* if and only if there is a topology p such that p and q coincide on non-ultrafilters.

For each $q \in \mathfrak{C}(S)$, let $K(q)$ be the set of all quasi-topological convergence functions which are coarser than q. For each $r \in K(q)$, let p_r be the topology which coincides with r on non-ultrafilters. Let $K'(q) = \{p_r\colon r \in K(q)\}$.

DEFINITION 2. For each $q \in \mathfrak{C}(S)$, let $\varrho(q) = \sup K'(q)$.

THEOREM 1. (1) *ϱ is a linkage homomorphism.*

(2) $\lambda(q) \leqslant \varrho(q) \leqslant \sigma(q)$.

Proof. (1) If q is topological, then $q \in K'(q)$ and $q = \sup K'(q)$. If $p \geqslant q$, then $K'(p) \supset K'(q)$ and thus $\varrho(p) \geqslant K'(q)$, implying $\varrho(p) \geqslant \varrho(q)$.

(2) If $r \in K'(q)$ and $p \in M(q)$, then since $\mathfrak{V}_r(x)$ q-converges to x for all $x \in S$, it follows that $\mathfrak{V}_p(x) \geqslant \mathfrak{V}_r(x)$ and $p \geqslant r$. Thus $\varrho(q) = \sup K'(q) \leqslant \inf_t M(q) = \sigma(q)$. Furthermore, since $\lambda(q) \in K'(q)$, it is clear that $\lambda(q) \leqslant \varrho(q)$.

THEOREM 2. *$\varrho(q) \geqslant q$ if and only if q is quasi-topological.*

Proof. If q is quasi-topological, then $p_q \geqslant q$. Since $p_q \in K'(q)$, $p_q = \varrho(q) \geqslant q$. Conversely, if q is not quasi-topological, then each member of $K'(q)$ is strictly finer than q on non-ultrafilters, whence $\varrho(q)$ non $> q$. Furthermore, $\varrho(q) \neq q$, since q is all the more non-topological.

The next three statements are all immediate consequences of previous results.

(1) If q_1 and q_2 agree on non-ultrafilters, then $\varrho(q_1) = \varrho(q_2)$.

(2) If q is quasi-topological, then $\varrho(q) = \sigma(q) \geqslant q$.

(3) If q is a limitierung, then $\varrho(q) = \sigma(q) = \lambda(q) \leqslant q$.

To prove that σ and ϱ are distinct convergence functions, it suffices to prove the two conditions "$\sigma(q) > q$" and "q not quasi-topological" may coexist.

EXAMPLE. Let S be the three element set $\{a, b, c\}$. Let q be defined as follows: $q(\mathfrak{F}_a) = \{a, b\}$; $q(\mathfrak{F}_b) = \{b\}$; $q(\mathfrak{F}_c) = \{b, c\}$; $q(\mathfrak{F}_{ac}) = \{b\}$, where $\mathfrak{F}_{ac}$ denotes the filter generated by $\{a, c\}$; $q(\mathfrak{F}) = \Phi$ for all other filters $\mathfrak{F}$ on $\{a, b, c\}$. q is not a quasi-topological convergence function, however, $\sigma(q)$ is the discrete topology.

Concluding remarks

For a given convergence function q, the four topologies $\lambda(q)$, $\varphi(q)$, $\sigma(q)$, and $\varrho(q)$ do not exhaust the list of topologies which are related to q. In the following two paragraphs, we show how additional topologies, in general distinct from the four which we have considered, may be associated with a convergence function q, using techniques which are available to us.

Suppose $\varphi(q) > \lambda(q)$ for a given convergence function q. We can introduce the set function Γ_q^2 defined by $\Gamma_q^2(A) = \Gamma_q(\Gamma_q(A))$ for all $A \in \mathfrak{P}(S)$. In the same way that $\varphi(q)$ is constructed from Γ_q, we construct $\varphi_2(q)$ from Γ_q^2. It can be shown that $\lambda(q) \leqslant \varphi_2(q) < \varphi(q)$. If $\lambda(q) < \varphi_2(q)$, the process may be repeated, yielding $\varphi_3(q)$, with $\lambda(q) \leqslant \varphi_3(q) < \varphi_2(q)$. If $\varphi_j(q) \neq \lambda(q)$, $j = 1, 2, \ldots, n-1$, we obtain the sequence of topologies $\lambda(q) \leqslant \varphi_n(q) < \varphi_{n-1}(q) < \ldots < \varphi_2(q) < \varphi(q)$. If $\varphi_n(q) > \lambda(q)$ for all integers n, one can extend the family to include an infinite chain of topologies between $\lambda(q)$ and $\varphi(q)$. The chain terminates when, for some ordinal number α, $\varphi_\alpha(q) = \lambda(q)$.

A second family of topologies related to q may be generated by exactly the same procedure if $\varphi(\bar{q}) \neq \sigma(q)$, where $\bar{q} = \inf_c M(q)$. In this case, we define $\bar{\varphi}(q) = \varphi(\bar{q})$; we can then define $\bar{\varphi}_2(q), \bar{\varphi}_3(q)$, etc., the process terminating only when, for some ordinal number α, $\bar{\varphi}_\alpha(q) = \sigma(q)$.

Since $\lambda(q) = \sigma(q) = \varrho(q)$ when q is a limitierung, a justification for considering the more abstract concept of convergence function seems appropriate. One argument is that order convergence in a partially ordered set (defined in Section II) defines a convergence function but not a limitierung in general. As an example, let S be the complete lattice composed of the set union of two replicas of the open interval $(0, 1)$ of the real line, with the addition of a greatest and a least element. One easily verifies that the order convergence function in this poset is not a limitierung.

I wish to express my sincere gratitude to Professor J. Mayer-Kalkschmidt for his assistance and encouragement.

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Reçu par la Rédaction le 19. 12. 1962