On uniform sets in a complete separable metric space

by

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1. Introduction. Let R denote a complete separable metric space. For definitions see Halmos [3]. Let T denote a group of homeomorphisms of R onto itself. For each set A contained in R, we let $S(A)$ denote the collection of all those subsets B of R for which a sequence $\tau_1, \tau_2, \ldots$ of elements of T exists with $B \subseteq \bigcup_{i=1}^{\infty} \tau_i(A)$. We call a subset A of R *uniform* (with respect to the group T), if for each open set U for which $U \cap A \neq \emptyset$, we have $A \in S(U \cap A)$. The purpose of this paper is to prove

THEOREM 4. *Let A be a closed uniform subset of R, and let $\mathfrak{A}$ be an uncountable collection of closed subsets of A, such that $A \in S(B)$ for each $B \in \mathfrak{A}$. If either*

(i) *T is an abelian group, or*

(ii) *R is the real line and T the group of isometries of R,*

then there are at least two sets B_1 and B_2 of $\mathfrak{A}$ such that $A \in S(B_1 \cap B_2)$.

This theorem reminds one of the theorem: If $\mathfrak{A}$ is an uncountable collection of Lebesgue measurable subsets of the real line, each with positive Lebesgue measure, then at least two of the sets have an intersection with positive Lebesgue measure. The similarity is even closer when we notice that the real line can be covered, to within a set of measure zero, by a countable number of isometric copies of any set with positive Lebesgue measure.

At the end of this paper, we construct a closed set A in the real line, containing an uncountable collection $\mathfrak{A}$ of disjoint closed sets B, such that $A \in S(B)$ for each $B \in \mathfrak{A}$. Here, the group T consists of the translations of the real line. This set is of interest in another connection, in that it is an example of a closed set for which there exists no Hausdorff measure giving it a non-zero σ-finite measure. See for example [1].

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2. Theorems 1, 2 and 3, of this section are valid without the additional conditions on R and T assumed in Theorem 4.

THEOREM 1. *If A is a closed uniform subset of R, and if $A \in S(\bigcup_{i=1}^{\infty} A_i)$ where each A_i is a closed set, then $A \in S(A_k)$ for some integer k.*

Proof. Let $\tau_1, \tau_2, \ldots$ be a sequence of elements of T, such that $A \subseteq \bigcup_{j=1}^{\infty} \tau_j(\bigcup_{i=1}^{\infty} A_i)$. Now $\bigcup_{j=1}^{\infty} \tau_j(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i,j}^{\infty} \tau_j(A_i)$, and by our hypothesis on T, $\tau_j(A_i)$ is a closed set for each i and j. Therefore, the Baire category theorem applied to the closed set A and the covering $\bigcup_{i,j} \tau_j(A_i)$ of A, provides us with an open set U and integers i_0, j_0, such that $U \cap A \neq \emptyset$ and $U \cap A \subseteq \tau_{j_0}(A_{i_0})$. Since A is uniform, $A \in S(U \cap A)$ and so $A \in S\big(\tau_{j_0}(A_{i_0})\big)$. But then $A \in S(A_{i_0})$, which completes the proof of the theorem.

THEOREM 2. *Let A be a closed uniform set in R, and let $\mathfrak{A}$ be an uncountable set of closed subsets of A with $A \in S(B)$ for each $B \in \mathfrak{A}$. Then, there is a closed set D with $A \in S(D)$, and an uncountable subset $\mathfrak{A}_0 \subseteq \mathfrak{A}$, such that for each $B \in \mathfrak{A}_0$ there is a $\tau \in T$ with $\tau(D) \subseteq B$.*

Proof. We choose first a countable base of open sets $U_1, U_2, \ldots$ for the open sets of R. Such a base exists because R is assumed to be separable.

Let B be a fixed element of $\mathfrak{A}$. Since $A \in S(B)$, there is a sequence $\tau_1, \tau_2, \ldots$ of elements of T such that $A \subseteq \bigcup_{i=1}^{\infty} \tau_i(B)$. B is closed, and so by our original hypothesis on T, $\tau_i(B)$ is closed for each integer i. By an application of the Baire category theorem to the closed set A, there is an open set V with $V \cap A \neq \emptyset$ and $V \cap A \subseteq \tau_i(B)$ for some integer i. Since $U_1, U_2, \ldots$ is a base for the open sets, we can find an integer j, such that $U_j \subseteq V$ and $U_j \cap A \neq \emptyset$. Clearly, $\tau_i^{-1}(U_j \cap A) \subseteq B$. Therefore, for each $B \in \mathfrak{A}$, there is an integer j and a $\tau \in T$, such that $U_j \cap A \neq \emptyset$ and $\tau(U_j \cap A) \subseteq B$. Because $\mathfrak{A}$ is uncountable, uncountably many of these integers j must be equal to some fixed integer, say j_0. Let $\mathfrak{A}_0$ denote the set of those $B \in \mathfrak{A}$ corresponding to j_0. Since $U_{j_0} \cap A \neq \emptyset$, and since A is uniform, $A \in S(U_{j_0} \cap A)$. Since U_{j_0} is open, $U_{j_0} = \bigcup_{i=1}^{\infty} C_i$ where the sets C_i are closed sets. Therefore, $A \in S(\bigcup_{i=1}^{\infty} C_i \cap A)$, and since each $C_i \cap A$ is closed, we have, by theorem 1, that $A \in S(C_{i_0} \cap A)$ for some integer i_0. Let $D = C_{i_0} \cap A$. Then D is closed, $A \in S(D)$, and since $D \subseteq U_{j_0} \cap A$, there is, for each $B \in \mathfrak{A}_0$, a $\tau \in T$ with $\tau(D) \subseteq B$. This proves theorem 2.

THEOREM 3. *Let D be a closed set in R, such that, whenever $D \in S(\bigcup_{i=1}^{\infty} A_i)$ with each A_i closed, it follows that $D \in S(A_i)$ for some integer i. Then D contains a closed uniform subset D_0 with $D \in S(D_0)$.*

Proof. Let $\mathcal{E}$ be the set of all those open sets U in R for which $D \notin S(D \cap U)$, and let U_0 be the union of these open sets. We prove that the closed set $D_0 = D - U_0$ is the required set.

Since R is separable, each $U \in \mathcal{E}$ can be represented as a union of open sets taken from a countable base. It follows easily, that U_0 can be written as a union of a countable number of the open sets of $\mathcal{E}$, say $U_1, U_2, U_3, \ldots$ Furthermore, we can write $U_i = \bigcup_{j=1}^{\infty} C_{ij}$ for $i = 1, 2, 3, \ldots$, where each C_{ij} is closed. Since $D \notin S(U_i \cap D)$, we must have $D \notin S(D \cap C_{ij})$ for all i, j. Therefore, since $D = (D - U_0) \cup (D \cap U_0) = D_0 \cup \bigcup_{i,j} (D \cap C_{ij})$, it follows, from the hypothesis on D, that $D \in S(D_0)$.

It remains to prove that D_0 is uniform. Let U be an open set with $U \cap D_0 \neq \emptyset$. Then $U \notin \mathcal{E}$, and so $D \in S(D \cap U)$. We write $U = \bigcup_{i=1}^{\infty} C_i$ where the sets C_i are closed. Then

$$U \cap D = (U \cap D_0) \cup (U \cap \bigcup_{i,j} C_{ij} \cap D) = \bigcup_{i=1}^{\infty} (C_i \cap D_0) \cup \bigcup_{i,j,k} (C_k \cap C_{ij} \cap D).$$

The components are all closed sets, and for all i, j, k, $D \notin S(C_k \cap C_{ij} \cap D)$. Therefore, $D \in S(C_i \cap D_0)$ for some integer i, implying that $D \in S(U \cap D_0)$. Since $D_0 \subseteq D$, we have also $D_0 \in S(U \cap D_0)$, and this proves that D_0 is uniform.

3. We now proceed to prove theorem 4.

Let $U_1, U_2, \ldots$ be a countable base of open sets for the open sets of R, and let $\tau \in T$. Then, if U is an open set in R, $\tau^{-1}(U)$ is open, and so $\tau^{-1}(U) = \bigcup_{j=1}^{\infty} U_{i_j}$ for some sequence of integers $i_1, i_2, \ldots$ Therefore, $U = \bigcup_{j=1}^{\infty} \tau(U_{i_j})$, and so it follows that $\tau(U_1), \tau(U_2), \ldots$ forms a countable base of open sets also.

Let D be a closed set, let $\mathfrak{A}_0$ be an uncountable subset of $\mathfrak{A}$, and for each $B \in \mathfrak{A}_0$ let $\tau_B \in T$ such that

(1) $$A \in S(D),$$

(2) $$\tau_B(D) \subseteq B.$$

By theorem 2, this is possible. Now, suppose $D \in S(\bigcup_{i=1}^{\infty} A_i)$ where each A_i is closed. Then, since $A \in S(D)$, $A \in S(\bigcup_{i=1}^{\infty} A_i)$, and so by theorem 1,

$A \in S(A_{i_0})$ for some integer i_0. This, together with (2), implies $D \in S(A_{i_0})$, so D satisfies the hypothesis of theorem 3. Therefore, let D_0 be a uniform closed subset of D such that $D \in S(D_0)$. Then, since (1) holds, we have also $A \in S(D_0)$. Let $\sigma_1, \sigma_2, \sigma_3, \ldots$ be a sequence of elements of T such that $A \subseteq \bigcup_{i=1}^{\infty} \sigma_i(D_0)$.

Let $B \in \mathfrak{A}_0$. Then, by (2), and since $D_0 \subseteq D$, $\tau_B(D_0)$ is covered by $\bigcup_{i=1}^{\infty} \sigma_i(D_0)$. By the Baire category theorem, there is an open set U such that $U \cap \tau_B(D_0) \neq \emptyset$ and $U \cap \tau_B(D_0) \subseteq \sigma_i(D_0)$ for some integer i. Now, $\tau_B(U_1), \tau_B(U_2), \ldots$ is a countable base for the open sets of R, and so there is an integer j such that

$$\emptyset \neq \tau_B(U_j) \cap \tau_B(D_0) \subseteq \sigma_i(D_0)\,. \tag{3}$$

Since $\mathfrak{A}_0$ is uncountable, there are integers j_0 and i_0 such that for uncountably many $B \in \mathfrak{A}_0$,

$$\emptyset \neq \tau_B(U_{j_0} \cap D_0) \subseteq \sigma_{i_0}(D_0)\,. \tag{4}$$

Clearly, $U_{j_0} \cap D_0 \neq \emptyset$, and since D_0 is uniform, we have $D_0 \in S(U_{j_0} \cap D_0)$. Then, since $A \in S(D_0)$, $A \in S(U_{j_0} \cap D_0)$. The proof of the theorem is completed when we have shown that, in both of the cases (i) and (ii), B_1 and B_2 can be selected from $\mathfrak{A}_0$ such that $B_1 \cap B_2$ contains the image of $U_{j_0} \cap D_0$ under some $\tau \in T$.

Suppose that (i) holds, that is, T is abelian. Let B_1 and B_2 be any two elements of $\mathfrak{A}_0$ satisfying (4). Then

$$\tau_{B_1}\sigma_{i_0}^{-1}\tau_{B_2}(D_0 \cap U_{j_0}) \subseteq \tau_{B_1}(D_0) \subseteq B_1\,,$$
$$\tau_{B_2}\sigma_{i_0}^{-1}\tau_{B_1}(D_0 \cap U_{j_0}) \subseteq \tau_{B_2}(D_0) \subseteq B_2\,.$$

Therefore, $B_1 \cap B_2$ contains $\tau(D_0 \cap U_{j_0})$ where $\tau = \tau_{B_1}\sigma_{i_0}^{-1}\tau_{B_2} = \tau_{B_2}\sigma_{i_0}^{-1}\tau_{B_1}$.

On the other hand, suppose that (ii) holds. Since each $\tau \in T$, in this case, is either a translation or a reflection, we can choose B_1 and B_2 such that (4) is satisfied, and both τ_{B_1} and τ_{B_2} are translations or both are reflections. By replacing D_0 by $-D_0$ if necessary, we may in fact assume both τ_{B_1} and τ_{B_2} are translations. We have two cases to consider, depending on whether σ_{i_0} is a translation or a reflection. If σ_{i_0} is a translation, then the homeomorphisms under consideration commute, implying, as before, that $B_1 \cap B_2 \supseteq \tau_{B_1}\sigma_{i_0}^{-1}\tau_{B_2}(U_{j_0} \cap D_0)$. If σ_{i_0} is a reflection, then $\sigma_{i_0}^2$ is the identity homeomorphism and $\tau_{B_1}\sigma_{i_0}\tau_{B_1} = \tau_{B_2}\sigma_{i_0}\tau_{B_2} = \sigma_{i_0}$. In this case we have $\sigma_{i_0}(D_0 \cap U_{j_0}) = \tau_{B_k}\sigma_{i_0}\tau_{B_k}(D_0 \cap U_{j_0}) \subseteq \tau_{B_k}\sigma_{i_0}\big(\sigma_{i_0}(D_0)\big) = \tau_{B_k}(D_0) \subseteq B_k$, for $k = 1, 2$, and so $B_1 \cap B_2 \supseteq \sigma_{i_0}(D_0 \cap U_{j_0})$. This completes the proof of the theorem.

4. In Theorem 4, the set A is assumed to be uniform. This condition can be relaxed to the condition on the set D of Theorem 3. One needs only to observe that the closed uniform subset D_0 of D, constructed in the proof, satisfies $D \notin S(D - D_0)$. From this it follows easily that $D_0 \in S(D_0 \cap B)$ for each $B \in \mathfrak{A}$. Then apply Theorem 4 to D_0 and $\{D_0 \cap B \mid B \in \mathfrak{A}\}$.

However, we construct a closed set to show that some conditions are needed. The closed set that we give is an effective example in the sense that the axiom of choice is not needed for its construction.

According to von Neumann [4], the set of numbers a_p, where

$$a_p = \sum_{n=1}^{\infty} \frac{2^{2^{[pn]}}}{2^{2^{n^2}}} \qquad (p > 0)\,,$$

is a set of distinct algebraically independent real numbers. I have chosen to take the sum from 1 rather than 0 in order to have $0 < a_p < 1$ whenever $0 < p < 1$. By restricting p to a perfect subset of real numbers which does not contain rational points, the subset so obtained will be a perfect subset of reall numbers. We let A denote the real numbers a_p obtained by taking $0 < p < 1$.

Let $E_1, E_2, \ldots$ be a sequence of disjoint perfect subsets of $(0, 1)$, each of which does not contain any rational point. Such a sequence is easily constructed by taking E_i to be a subset of $\big(1/(1+1), 1/i\big)$ obtained by excluding small enough open sets about the rational points. We also let f_i be one-to-one functions from the unit interval $(0, 1)$ into E_i, $i = 1, 2, \ldots$ These can be constructed as in Denjoy [2], page 8.

Let $A_i = \{a_p \mid p \in E_i\}$ for $i = 1, 2, \ldots$ Then the sets A_i are disjoint perfect subsets of A. Let $D_n = A_1 + \ldots + A_n = \{x_1 + \ldots + x_n \mid x_i \in A_i, i = 1, \ldots, n\}$, and

$$D = \bigcup_{n=1}^{\infty} \left(\frac{n(n+1)}{2} + D_n\right).$$

Each D_n is a perfect set. Furthermore, since $D_n \subseteq (0, n+1)$, the set $\frac{1}{2}n(n+1) + D_n$ is contained in $\big(\frac{1}{2}n(n+1), \frac{1}{2}(n+1)(n+2)\big)$, for $n = 1, 2, \ldots$ and so D is a perfect set. If $t_1, t_2, \ldots$ is a sequence of real numbers with $t_i \in A_i$, $i = 1, 2, \ldots$, then the set $E = \bigcup_{n=2}^{\infty} \big(\frac{1}{2}n(n+1) + t_n + D_{n-1}\big)$ is a perfect subset of D for which there is a sequence of translations $\sigma_1, \sigma_2, \ldots$ such that $D \subseteq \bigcup_{n=1}^{\infty} \sigma_n(E)$. If $t'_1, t'_2, \ldots$ is another sequence of real numbers with $t'_i \in A_i$ and $t'_i \neq t_i$ for $i = 1, 2, \ldots$ and if E' is the corresponding set, then $E \cap E' = \emptyset$ because for $n = 2, 3, \ldots$, $(t_n + D_{n-1}) \cap (t + D_{n-1}) = \emptyset$. For each $t \in (0, 1)$, let E_t be the set obtained by taking the sequence $a_{f_1(t)}, a_{f_2(t)}, \ldots$ Then the sets E_t for $0 < t < 1$ are mutually disjoint perfect subsets of D such that for each t, $D \in S(E_t)$. We notice that all of the sets mentioned are effectively given. The set D is the set we set out to construct.

The fact, mentioned in the introduction, that this set does not have non-zero σ-finite Hausdorff measure for any Hausdorff measure is clear. To have non-zero measure, each of the subsets E_t would have to have non-zero measure, and since there are uncountably many of them in D and they are pairwise disjoint, D would have non-σ-finite measure.

References

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On the genus of an n-connected graph

by

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1. Introduction. The *connectivity* $\varkappa(G)$ of a graph G is the smallest number of points whose removal results either in a disconnected graph or in the graph with one point and no lines. Graph G is *n-connected* if $\varkappa(G) \geqslant n$. For $n \geqslant 1$, an *n-component* of G is a maximal n-connected subgraph. Thus a 1-component of G is a (connected) component; a 2-component of G is called a *block* of G (by Battle, Harary, Kodama and Youngs [1] or Harary [3], and a nonseparable subgraph by Whitney [6]); and a 3-component of G will be called a *brick* of G.

The *genus* $\gamma(G)$ of G is the smallest integer n such that G is imbeddable in the orientable surface S_n whose genus $\gamma(S_n)$ is n. In [1], we (Battle, Harary, Kodama and Youngs) proved that the genus of any graph is the sum of the genuses of its blocks. Our present object is to study the genus of a graph in terms of its bricks, and in general of its n-components.

The problem is so complicated that we restrict our study in this note to the case where an n-connected graph G is the union of two $(n+1)$-components, B and C. We will see that the number of points in $B \cap C$ is exactly n, that $\gamma(G) \leqslant \gamma(B)+\gamma(C)+n-1$, and by an example that this inequality is best possible. Let $v_1, v_2, \ldots, v_n$ be the set of points in $B \cap C$ and call G_{ij} the graph obtained by adding line v_iv_j to G. Then we will prove that, if $\gamma(G_{ij}) > \gamma(G)$ for all $1 \leqslant i < j \leqslant n$, then $\gamma(G) = \gamma(B)+\gamma(C)+n-1$. This last equation is specialized to the case where B and C are bricks, i.e., $n = 2$.

2. Results. We will present one lemma, one theorem, one corollary, and several examples.

Remark 1. Let an n-connected graph G be the union of two $(n+1)$-components B and C. Then the number of points of $B \cap C$ is exactly n. Moreover, the set of lines of $B \cap C$ consists of all lines of G whose end points are in $B \cap C$.

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