A remark on characters of locally compact Abelian groups

by

E. Hewitt* (Seattle, Wash.)

1. Introduction. Let G be an Abelian group, and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies (1) on G such that $\mathcal{O}_1 \supsetneqq \mathcal{O}_2$ and G is a locally compact topological group (2) under both $\mathcal{O}_1$ and $\mathcal{O}_2$. Z. Semadeni has asked if there is necessarily a character of G that is continuous in the topology $\mathcal{O}_1$ and discontinuous in the topology $\mathcal{O}_2$. In the present note we answer this question affirmatively (Theorem (3.3)), finding that there are always at least $2^{\aleph_1}$ $\mathcal{O}_1$-continuous characters that are $\mathcal{O}_2$-discontinuous. We also discuss some topological questions suggested by our construction.

(1.1) Notation. The letters $G, H, A, F, J, \ldots$ will henceforth be used to denote groups *with topologies*. If G is a topological group, the symbol G_d will denote the group having the discrete topology and having the same algebraic structure as G. The symbol R denotes the additive group of real numbers with the usual topology. The symbol Q denotes the additive group of rational numbers with the usual topology as a subspace of R. The symbol Z denotes the additive group of all integers with the discrete topology. The symbol T denotes the circle group, here regarded as the multiplicative group of all complex numbers of absolute value 1, with the usual topology. For a cardinal number $\mathfrak{m} > 1$ and a topological group G, the symbol $G^{\mathfrak{m}}$ denotes the full direct product of $\mathfrak{m}$ replicas of G, with the Cartesian product topology. The symbol $\mathfrak{c}$ denotes the cardinal number of the continuum.

For the reader's convenience, we now recite some known facts.

(1.2) Theorem. *Let G be a locally compact Abelian group, with connected component of the identity C. There exist a nonnegative integer a and a topological isomorphism σ of R^a into G such that C is equal to $\sigma(R^a)\cdot E$,*

---

* Research supported by the National Science Foundation, U.S.A., under grant number NSF-G18838.

(1) Here and throughout this note, we define topologies by families of open sets.

(2) We always suppose T_0 (and hence Hausdorff) separation for topological groups.

where E is a compact connected subgroup of G. The number a is the largest of all nonnegative integers n such that G contains a topological isomorph of R^n. The group G itself is equal to $\sigma(R^a)\cdot A$, where A is a closed subgroup of G containing a compact subgroup F that is open in the relative topology of A. The subgroup A contains all $x \in G$ such that $\{x^n: n \in Z\}^-$ is compact. The relation $\sigma(R^a) \cap A = \{e\}$ obtains, and G is topologically isomorphic with the direct product $R^a \times A$.

A proof of Theorem (1.2) appears in [1], Theorem (24.29).

(1.3) THEOREM. *Let G be a σ-compact, locally compact group, and let f be a continuous homomorphism of G onto a locally countably compact group $\tilde{G}$. Then f is an open mapping.*

Proofs of Theorem (1.3) are found in [1], Theorem (5.29) and in [3], Theorem 12, p. 123.

(1.4) THEOREM. *Let G be a compact, infinite, Abelian group with character group* (3) *X; write $\mathfrak{m}$ for the cardinal number of X. Then the cardinal number of G is $2^{\mathfrak{m}}$, and G admits $2^{2^{\mathfrak{m}}}$ discontinuous characters.*

Proof. The first statement is a theorem of S. Kakutani [2]; a detailed proof appears in [1], Example (24.47). To prove the second statement, consider the discrete group G_d. By Kakutani's theorem, the (compact) character group Y of G_d has cardinal number $2^{2^{\mathfrak{m}}}$. Since only $\mathfrak{m}$ of the elements of Y belong to X, there are $2^{2^{\mathfrak{m}}}$ characters of G that are discontinuous.

(1.5) THEOREM. *The group R admits $2^{\mathfrak{c}}$ discontinuous characters.*

This is an elementary fact. It is perhaps most easily seen by noticing that R_d is isomorphic with the *weak* direct product of $\mathfrak{c}$ copies of the additive rationals Q: R_d is isomorphic with $[(Q_d)^{\mathfrak{c}*}]_d$, where the asterisk means that in each element of the direct product, we allow only a finite number of nonzero entries. The character group of Q_d is a certain solenoidal group S of cardinal number $\mathfrak{c}$. Thus the character group of R_d is topologically isomorphic with $S^{\mathfrak{c}}$ and has cardinal number $2^{\mathfrak{c}}$. Since R admits only $\mathfrak{c}$ continuous characters, there are $2^{\mathfrak{c}}$ characters of R that are discontinuous.

2. Locally compact topologies on R^a. Semadeni's question may be paraphrased as follows. Given a locally compact Abelian group, if one strengthens its topology, does one necessarily introduce new continuous characters? The group R and its powers $R^2, R^3, \ldots$ are very important locally compact Abelian groups, and it is natural to begin with

(3) By *character group of G* we mean the group of continuous characters of the topological group G. However, we will have occasion to write *discontinuous character*.

a discussion of these groups. Since R^a is σ-compact, it is obvious from (1.3) that there are no topologies on R^a *weaker* than the usual one under which R^a is a locally compact Abelian group. The locally compact group topologies on R^a *stronger* than the usual topology are completely described as follows. For an integer $a > 1$ and $b = 0, 1, 2, \ldots, a-1$, the topological group $R^b \times (R_d)^{a-b}$ plainly has a locally compact group topology strictly stronger than the usual topology for R^a. Up to topological isomorphisms, these are the only possibilities, as we will now prove.

(2.1) LEMMA. *Let R_s denote the group R with a locally compact group topology that is strictly stronger than the usual topology of R. Then R_s is the discrete group R_d.*

Proof. By (1.2), R_s is topologically isomorphic with $R^a \times A$, where a is a nonnegative integer and A is a locally compact Abelian group containing a compact open subgroup F. With the customary abuse of notation, we suppose that R^a, A, and F are subgroups of $R^a \times A$. Let τ be a topological isomorphism carrying $R^a \times A$ onto R_s. The identity mapping of $\tau(F)$ into R is continuous, and, since F is compact, this identity mapping is a homeomorphism. The only compact subgroup of R is $\{0\}$, and therefore F is a 1-element group. Therefore A is discrete, and if R_s is nondiscrete, the integer a must be positive. If $\tau(R^a) = R_s$, Theorem (1.3) shows that R_s has the topology of R^a, so that $a = 1$ and $R_s = R$. This contradicts our hypothesis. Hence R_s contains a proper open subgroup B topologically isomorphic with R^a. Regarded as a subset of the topological space R_s, B is connected. Regarded as a subset of the topological space R, B is a continuous image of B as a subset of R_s, and so B is connected in its relative topology as a subset of R. Since B is also a subgroup of R, it must be equal to R or to $\{0\}$. Thus if R_s is nondiscrete, we have a contradiction.

(2.2) THEOREM. *Let a be an integer greater than 1, and let R_s^a denote the group R^a with a locally compact group topology strictly stronger than the usual topology for R^a. Then there exist an integer $b \in \{0, 1, \ldots, a-1\}$ and a topological isomorphism σ carrying R_s^a onto $R^b \times (R_d)^{a-b}$.*

Proof. By (1.2), R_s^a contains an open subgroup D such that there is a topological isomorphism τ carrying $R^b \times F$ onto D, where b is a nonnegative integer and F is a compact Abelian group. As in the proof of (2.1), we see that $\tau(F) = \{0\}$, and so F is a 1-element group. If $b = 0$, then R_s^a contains a 1-element open subgroup and so is topologically isomorphic with $(R_d)^a$. Excluding this special case, we say suppose that b is positive; so we have a topological isomorphism τ carrying R^b into R_s^a.

We will now show that

$$\tau(a\boldsymbol{x}) = a\tau(\boldsymbol{x}) \tag{1}$$

for all $\alpha \in R$ and $\boldsymbol{x} \in R^b$, i.e. τ is linear for R^b and R_s^a regarded as linear spaces over the field R [4]. The relation (1) is trivial for $\boldsymbol{x} = \boldsymbol{0}$. For $\boldsymbol{x} \neq \boldsymbol{0}$, consider the subgroup $L_{\boldsymbol{x}} = \{\alpha\tau(\boldsymbol{x}):\ \alpha \in R\}$ of R_s^a. Being closed in R^a, $L_{\boldsymbol{x}}$ is closed in R_s^a and is therefore a locally compact group algebraically isomorphic with R and having a topology at least as strong as the usual topology of R. Since τ is a topological isomorphism and since $\tau(r\boldsymbol{x}) = r\tau(\boldsymbol{x})$ for all $r \in Q$, $L_{\boldsymbol{x}}$ does not have the discrete topology: the subspace $\{r\tau(\boldsymbol{x}):\ r \in Q\}$ of $L_{\boldsymbol{x}}$ has the topology of Q as a subspace of R. By (2.1), the topology of $L_{\boldsymbol{x}}$ is either discrete or the usual topology of R. We have thus proved that the topology of $L_{\boldsymbol{x}}$ is the usual topology of R. For a real number α, let $\{r_n\}_{n=1}^{\infty}$ be a sequence of rational numbers such that $\lim_{n\to\infty} r_n = \alpha$. Then we have $\lim_{n\to\infty} r_n\boldsymbol{x} = \alpha\boldsymbol{x}$ in R^b, and since τ is a topological isomorphism, we also have

$$\lim_{n\to\infty} \tau(r_n\boldsymbol{x}) = \lim_{n\to\infty} r_n\tau(\boldsymbol{x}) = \tau(\alpha\boldsymbol{x})\,.$$

Since $L_{\boldsymbol{x}}$ has the usual topology of R, it follows that $\lim_{n\to\infty} r_n\tau(\boldsymbol{x}) = \alpha\tau(\boldsymbol{x})$, and so (1) is proved.

Relation (1) implies that $\tau(R^b)$ is a linear subspace of R^a, and consequently we have $b \leqslant a$. If $b = a$, then $\tau(R^b) = R^a$, so that R_s^a has the topology of R^a, a contradiction to our hypothesis. Therefore we have $0 < b < a$, and $\tau(R^b)$ is a proper linear subspace of R_s^a. The quotient group $R_s^a/\tau(R^b)$ can be identified *algebraically* with the linear space R^{a-b}. Since $\tau(R^b)$ is an open subgroup of R_s^a, this quotient group is discrete. Since $\tau(R^b)$ is a divisible group, the group R_s^a is topologically isomorphic with $\tau(R^b)\times[R_s^a/\tau(R^b)]$, the second factor having the discrete topology. (This is proved in [1], Example (6.22).) Since $(R^{a-b})_d$ can be identified with $(R_d)^{a-b}$, the proof is complete.

3. The main theorem.

(3.1) LEMMA. *Let G be any topological group and let H be any open subgroup of G. Then the topology of G is completely determined by the topology of H as a subspace of G and the group structure of G: a subset Y of G is open if and only if $Y \cap (xH)$ is open in the relative topology of xH for all $x \in G$.*

Proof. Since translation is a homeomorphism of G onto itself, all cosets xH of H are open, and so $Y \cap (xH)$ is open for all $x \in G$ if Y is open in G. If $Y \cap (xH)$ is relatively open in xH for all $x \in G$, then $Y \cap (xH)$ is open in G for all $x \in G$, since xH is open in G. Since $Y = \bigcup_{x\in G} Y \cap (xH)$, it follows that Y is open in G if every set $Y \cap (xH)$ is open in xH.

[4] It is trivial that τ is linear for R^b and R_s^a regarded as linear spaces over the field Q.

(3.2) Notation. Throughout the remainder of Section 3, let G_1 and G_2 denote a single Abelian group having two locally compact group topologies for which the topology of G_1 is strictly stronger than the topology of G_2. Let φ denote the identity mapping of G_1 onto G_2; thus φ is a continuous (but *not* open) isomorphism of G_1 onto G_2.

(3.3) THEOREM. *There are at least $2^{\aleph_1}$ continuous characters of the group G_1 that are discontinuous for the group G_2* [5].

Proof. By (1.2), the group G_2 contains an open subgroup topologically isomorphic with $R^a \times F$, where F is compact and a is a nonnegative integer. With no loss of generality, we may suppose that R^a, F, and $R^a\times F$ are subgroups of G_2. Now consider the subgroup $\varphi^{-1}(R^a\times F) = \varphi^{-1}(R^a)\times\varphi^{-1}(F)$ of the topological group G_1. Plainly this subgroup is open in G_1. If $\varphi^{-1}(R^a)$ is homeomorphic to R^a and $\varphi^{-1}(F)$ is homeomorphic to F, then $\varphi^{-1}(R^a\times F)$ is topologically isomorphic to $R^a\times F$: this is a known property of direct products ([1], Theorem (6.12)). In this case, Lemma (3.1) implies that G_1 and G_2 are homeomorphic. Consequently:

(1) *a is positive and $\varphi^{-1}(R^a)$ has a locally compact group topology strictly stronger than the usual topology of R^a,*

or

(2) *$\varphi^{-1}(F)$ has a locally compact group topology strictly stronger than the topology of F as a subspace of G_2.*

Suppose first that (1) holds. The group $\varphi^{-1}(R^a)$ is closed in G_1 since R^a is closed in G_2, and so $\varphi^{-1}(R^a)$ has a locally compact group topology as a subspace of G_1. By Theorem (2.2), $\varphi^{-1}(R^a)$ is topologically isomorphic with $R^b\times(R_d)^{a-b}$ for an integer $b \in \{0, 1, \ldots, a-1\}$. Changing our notation for brevity's sake, we may write that G_1 contains an open subgroup of the form $M = R^b\times(R_d)^{a-b}\times\varphi^{-1}(F)$, where $b < a$ and $\varphi^{-1}(F)$ is a closed subgroup of G_1. Elements of M can be written as triples $(\boldsymbol{x}, \boldsymbol{y}, z)$. Let χ be a character of $(R_d)^{a-b}$ that is discontinuous in the usual topology of R^{a-b}. As noted in (1.5), there are $2^{\mathfrak{c}}$ such characters. Let χ' be the character of M such that $\chi'(\boldsymbol{x}, \boldsymbol{y}, z) = \chi(\boldsymbol{y})$, and let χ'' be any character of G_1 that is equal to χ' on M. Such a χ'' always exists: see [1], Lemma (24.4). Since χ' is continuous on the open subgroup M of G_1, χ'' is continuous on G_1. It is obvious that χ'' is discontinuous on G_2. Since there are $2^{\mathfrak{c}}$ possible choices for χ, the present theorem is true if (1) holds.

[5] We conjecture that the number $2^{\aleph_1}$ can be replaced in this theorem by $2^{\mathfrak{c}}$ without recourse to the continuum hypothesis. Our methods do not yield this result, however, and to obtain it would apparently require some quite delicate facts about the structure of compact Abelian groups having dense subgroups that are continuous isomorphic images of R^a.

We now suppose that (2) holds. Then $\varphi^{-1}(F)$ is not compact, but is clearly locally compact since it is a closed subgroup of G_1. By Theorem (1.2), $\varphi^{-1}(F)$ is topologically isomorphic with $R^c \times A$, where A is a closed subgroup of $\varphi^{-1}(F)$ containing a compact subgroup J that is open in A.

Suppose that $c = 0$. Then J is an open subgroup of $\varphi^{-1}(F)$, and since the topology of $\varphi^{-1}(F)$ is strictly stronger than the topology of F, Lemma (3.1) implies that $\varphi(J)$ is not open in F. Since J is compact, $\varphi(J)$ is compact and therefore closed in F. Consequently the group $F/\varphi(J)$ is a nondiscrete compact Hausdorff group and has cardinal number $\mathfrak{n} \geqslant \mathfrak{c}$ (see Theorem (1.4)). There are $2^{\mathfrak{n}}$ discontinuous characters of $F/\varphi(J)$, again by Theorem (1.4). Let ψ be the natural mapping of F onto $F/\varphi(J)$. If χ is a discontinuous character of $F/\varphi(J)$, it is easy to see that $\chi \circ \psi$ is a discontinuous character of F. Every extension of $\chi \circ \psi$ to a character of G_2 is discontinuous. However, J being an open subgroup of $\varphi^{-1}(F)$, every character of $\varphi^{-1}(F)$ that is identically 1 on J is continuous on $\varphi^{-1}(F)$ and admits an extension over G_1 that is a continuous character of G_1 ([1], Theorem (24.12)). We infer that G_1 admits at least $2^{\mathfrak{n}}$ continuous characters that are discontinuous on G_2.

Suppose finally that (2) holds and that $c > 0$. Let σ be a topological isomorphism carrying R^c into $\varphi^{-1}(F)$: thus $\sigma(R^c) \subset \varphi^{-1}(F)$. Then the subgroup $\varphi \circ \sigma(R^c)$ of F is a one-to-one continuous image of R^c. Clearly $\varphi \circ \sigma(R^c)$ cannot be closed in F, since then $\varphi \circ \sigma(R^c)$ would be locally compact, the mapping $\varphi \circ \sigma$ would be a topological isomorphism, and F would contain a noncompact closed subgroup. Let H denote the closure of $\varphi \circ \sigma(R^c)$ in F, and consider the subgroup $\varphi^{-1}(H)$ of $\varphi^{-1}(F)$. Obviously $\varphi^{-1}(H)$ is closed in $\varphi^{-1}(F)$. Now let C be the connected component of the identity in $\varphi^{-1}(H)$. By (1.2), C has the form $\delta(R^c) \cdot E$, where δ is a topological isomorphism and E is a compact connected subgroup of $\varphi^{-1}(H)$. Since $\sigma(R^c)$ is a closed connected subgroup of $\varphi^{-1}(H)$, we see that $\sigma(R^c) \subset \delta(R^c) \cdot E$ [6]. We also have $\varphi^{-1}(H) = \delta(R^c) \cdot A$, as in Theorem (1.2). Thus A is a closed subgroup of $\varphi^{-1}(H)$ that contains all compact subgroups of $\varphi^{-1}(H)$: in particular, we have $E \subset A$. Also A contains a compact open subgroup, say D. We write $D \cdot E = N$; then N is also a compact open subgroup of A. Finally, $\varphi^{-1}(H)$ is topologically isomorphic with $\delta(R^c) \times A$. Now consider the group A. If A were σ-compact, then $\varphi^{-1}(H)$ would also be σ-compact, and so by Theorem (1.3) φ would be an open mapping and hence a homeomorphism of $\varphi^{-1}(H)$ onto H:

(6) It need not be the case that $\sigma(R^c) = \delta(R^c)$. Consider the group $R \times T$, parametrized as $\{(x, \exp(2\pi i y)):\ x \in R,\ y \in [0, 1[\}$. For $0 \leqslant a < 1$, the subgroup $\sigma_a(R) = \{(x, \exp(2\pi i a x)):\ x \in R\}$ is a topological isomorph of R, and $\sigma_{a_1}(R) \cap \sigma_{a_2}(R)$ is a discrete group isomorphic with Z for $a_1 \neq a_2$.

this is a violation of (2). Consequently the group A/N is an uncountable discrete Abelian group. It has at least $2^{\aleph_1}$ characters. Let χ be any character of A/N different from the identity. An element of $\varphi^{-1}(H)$ can be written uniquely as $\delta(\boldsymbol{x}) \cdot y$, where $\boldsymbol{x} \in R^c$ and $y \in A$. Define $\chi'(\sigma(\boldsymbol{x}) \cdot y)$ as $\chi(yN)$. It is clear that χ' is a continuous character of $\varphi^{-1}(H)$ and that $\chi'(\sigma(\boldsymbol{x})) = 1$ for all $\boldsymbol{x} \in R^c$. Now extend χ' in any fashion to a continuous character χ'' of G_1: this can be done because $\varphi^{-1}(H)$ is closed in $\varphi^{-1}(F)$ and $\varphi^{-1}(F)$ is closed in G_1. Look at χ'' as a character of G_2. We have $\chi''(\varphi \circ \sigma(R^c)) = \{1\}$, and so if χ'' were continuous on G_2, we would have $\chi''(H) = \{1\}$. As this is not the case, χ'' is discontinuous on G_2. Since we have at least $2^{\aleph_1}$ choices for χ, the proof is complete.

4. Examples and remarks.

(4.1) Let G be any topological group whatever and let H be a nonopen normal subgroup of G. Let $\mathcal{U}$ be a complete family of neighborhoods of the identity for the given topology of G. The family of sets $\{U \cap H:\ U \in \mathcal{U}\}$ can be taken as a complete family of neighborhoods of the identity. Under the topology so defined, G is again a topological group in which H is an open subgroup, and in which the topology is strictly stronger than the original topology. If H is locally compact in its original topology, then the new topology for G makes G locally compact.

(4.2) Theorem (2.2) shows that the topology of R^a can be strengthened to a locally compact group topology only by the device of (4.1). It is tempting to conjecture that something similar is true for torus groups $T^{\mathfrak{m}}$, where $\mathfrak{m}$ is a cardinal number >1. However, much more complicated possibilities arise with torus groups, as we will now point out.

(4.3) THEOREM. *Let T_s denote the circle group with a locally compact group topology that is strictly stronger than the usual topology for T. Then this topology is discrete.*

Proof. By Theorem (1.2), T_s is topologically isomorphic with $R^a \times A$, where a is a nonnegative integer and A is a locally compact Abelian group containing a compact open subgroup F. Let φ be a continuous isomorphism of $R^a \times A$ onto T. If $a = 0$, then $\varphi(F)$ is a proper compact subgroup of T and so is finite. Thus A is discrete and so T_s is discrete also. If a is positive and A is a 1-element group, Theorem (1.3) implies that R^a is topologically isomorphic with T, which is trivially impossible. If a is positive and A is not a 1-element group, then $\varphi(R^a)$ is an infinite proper connected subgroup of T: and of course no such subgroup exists.

(4.4) Now consider T^2 and the possible ways of producing a locally compact group topology T_s^2 that is strictly stronger than the usual topology of T^2. Plainly T_s^2 is noncompact. By Theorem (1.2), T_s^2 is topologically isomorphic with a group

(1) $$R^a \times A\,,$$

where a is a nonnegative integer and A contains a compact open subgroup F. Let φ be a one-to-one continuous isomorphism of $R^a \times A$ onto T^2. Suppose first that $a = 0$. Then if T_s^2 is nondiscrete, F is an infinite compact subgroup of A, and φ restricted to F is a topological isomorphism carrying F onto a proper compact infinite subgroup of T^2. The character group of $\varphi(F)$ is a quotient group of Z^2 and hence has 1 or 2 generators. In the first case, $\varphi(F)$ is topologically isomorphic with T. In the second case, $\varphi(F)$ is topologically isomorphic with $T \times B$, where B is a finite cyclic group. (This is because T^2 contains no proper subgroup topologically isomorphic with T^2.) In the second case, therefore, F is topologically isomorphic with the product of T and a finite group, and so A contains once again a compact open subgroup topologically isomorphic with T. It is a routine matter to prove that the only topological isomorphisms carrying T into T^2 have the form

(2) $$z \to (z^m, z^n) \qquad (z \in T)\,,$$

where m and n are integers and the following possibilities exist: $m = \pm 1$, $n = 0$; $m = 0$, $n = \pm 1$; $m \neq 0$, $n \neq 0$, and m and n are relatively prime. From all this, we infer that if $a = 0$ in (1), then the topology of T_s^2 is that in which one of the groups appearing in (2) is pronounced to be open and retains its relative topology as a subgroup of T^2. Since F is a divisible group, it is algebraically a direct factor of T^2, and being open in T_s^2, it is topologically a direct factor of T_s^2. Thus T_s^2 has the form $F \times (T_s^2/F)$, where the second factor is discrete and F is one of the groups in (2). The exact structure of T_s^2/F can of course be computed from (2).

Suppose now that the integer a in (1) is positive. It is a routine matter to show that the only continuous isomorphs of R in T^2 are the dense subgroups

(3) $$\{(\exp(2\pi i x),\ \exp(2\pi i \alpha x)):\ x \in R\}\,,$$

where α is an irrational number. This implies at once that there is no one-to-one continuous isomorphism of R^2 into T^2, and so we have $a = 1$ in (1). Now if the subgroup A of (1) were nondiscrete, the group F would be infinite and $\varphi(F)$ would be an infinite compact subgroup of T^2 intersecting the group of (3) only at the identity $(1, 1)$. Since $\varphi(F)$ would contain a group of the form (2), it is easy to see that no such $\varphi(F)$ can exist. Therefore T_s^2 is topologically isomorphic with $R \times A$, where A is discrete. Since R is a divisible group, it is algebraically a direct factor of T^2, and it follows that A is isomorphic with $T^2/\varphi(R)$, where $\varphi(R)$ is a subgroup of T^2 of the form (3). Again, the exact structure of A can be readily computed. This completes the description of all possible topological groups T_s^2.

(4.5) Additional but inessential complications arise in classifying the topological groups T_s^3, T_s^4, ... The topological isomorphs of T^l in T^m $(l = 1, 2, \ldots, m-1)$ can all be identified explicitly. A useful tool in doing this is the fact that every continuous automorphism of T^l onto itself has the form

$$(z_1, \ldots, z_l) \to (z_1^{a_{11}} z_2^{a_{21}} \cdots z_l^{a_{l1}}, \ldots, z_1^{a_{1l}} \cdots z_l^{a_{ll}})\,,$$

where $(a_{jk})_{k=1}^{l}{}_{k=1}^{l}$ is an integer-valued matrix with determinant ± 1. (See [1], Example (26.18.h).) The group T^m contains continuous isomorphic images of R^k for $k = 1, 2, \ldots, m-1$ and for no larger values of k. The isomorphisms φ that map R^k into T^m can be computed explicitly, and the topological isomorphs of T^l in T^m that are disjoint from $\varphi(R^k)$ can also be identified completely. With these data, all of the groups T_s^m can be exhibited. The computations become very lengthy for large m, and we shall not reproduce them here.

(4.6) Now consider an infinite cardinal number $\mathfrak{m}$. The possibilities for groups $T_s^{\mathfrak{m}}$ are vastly increased over the finite-dimensional case. For every positive integer l, $T^{\mathfrak{m}}$ contains a continuous isomorphic image $\varphi(R^l)$ of R^l. Furthermore, given a compact Abelian group F with basis for open sets of cardinal number not exceeding $\mathfrak{m}$, it is easy to see that φ can be constructed in such wise that $T^{\mathfrak{m}}$ contains a topological isomorph of F disjoint from $\varphi(R^l)$. Consider then a group $T_s^{\mathfrak{m}}$ topologically isomorphic with $R^l \times A$, where A is a closed subgroup of $T_s^{\mathfrak{m}}$ containing a compact open subgroup F. The nonnegative integer l is arbitrary, and so is the compact subgroup F of A (subject only to the stated restriction on the cardinal number of an open basis for F). For a given φ, one can compute the group A explicitly, since R^l is divisible.

(4.7) Let G_1, G_2, and φ be as in (3.2), and let X_j be the character group of G_j $(j = 1, 2)$. Let $\varphi^\sim$ be the adjoint mapping carrying X_2 into X_1, defined as usual by $\varphi^\sim\chi_2(x) = \chi_2 \circ \varphi(x)$ for all $\chi_2 \in X_2$ and $x \in G_1$. Since G_1 and G_2 are identical as sets, we have $(\varphi \circ \chi_2)(x) = \chi_2(x)$ for all $x \in G_1$. Theorem (3.2) states that $\varphi^\sim(X_2)$ is always a proper subgroup of X_1. It is clear that $\varphi^\sim$ is a continuous isomorphism. The Pontryagin-van Kampen duality theorem shows that $\varphi^\sim(X_2)$ is dense in X_1. Note that the duality theorem does *not* yield a trivial proof of Theorem (3.2). If Theorem (3.2) were to fail for some G_1 and G_2, so that $\varphi^\sim$ was a continuous isomorphism carrying X_2 onto X_1, then $\varphi^{\sim\sim} = \varphi$ would still be a continuous isomorphism of G_1 onto G_2, and no contradiction would need to ensue.

(4.8) After this paper was written, we became aware of the paper *Uniform boundedness for groups* by Irving Glicksberg [Canadian J. Math. 14 (1962), pp. 269-276]. In this paper, Glicksberg has proved that G_1 has at least one continuous character that is G_2-discontinuous.

References

[1] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Volume I. Grundlehren der Math. Wiss., Band 115, Heidelberg 1963.

[2] S. Kakutani, *On cardinal numbers related with a compact abelian group*, Proc. Imp. Acad. Tokyo 19 (1943), pp. 366-372.

[3] L. S. Pontryagin, *Nepreryvnye gruppy* (in Russian), Moscow 1954.

THE UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON, U.S.A.

Reçu par la Rédaction le 20. 7. 1962

On compactifications allowing extensions of mappings

by

R. Engelking (Warszawa) and **E. G. Sklyarenko** (Moscow)

In this paper we shall deal with compactifications of a space X which allow extensions of some mappings of X into itself. By a topological space we always mean a completely regular topological space, by a mapping we mean a continuous function. We shall say that a compact space Y is a *compactification* of a topological space X if Y contains a dense subspace X' homeomorphic to X. Compactifications of the space X will be denoted by aX, a_iX, etc. The letter a will also denote the homeomorphism of X onto X'. Hence we have $a\colon X \to aX$ and a is a homeomorphism of X onto $a(X)$. We can define a partial order $\succcurlyeq$ in the class of all compactifications of X. Namely, we put $a_1X \succcurlyeq a_2X$ if there exists a mapping $f\colon a_1X \to a_2X$ such that $fa_1 = a_2$. In that case we shall write $f\colon a_1X \succcurlyeq a_2X$. The Čech-Stone compactification βX is the maximal element in the partially ordered class. By the *weight* of a space X we shall mean the smallest cardinality of bases of X. The weight of X will be denoted by $w(X)$. Let us notice the well-known result that $a_1X \succcurlyeq a_2X$ implies $w(a_1X) \geqslant w(a_2X)$.

Let $\Phi = \{\varphi_s\}_{s\in S}$ be a family of mappings of X into itself, i.e. $\varphi_s\colon X \to X$ for every $s \in S$. A compactification aX of the space X will be called a *Φ-compactification* if, for every $s \in S$, there exists a mapping $\tilde{\varphi}_s\colon aX \to aX$ such that $\tilde{\varphi}_s|X = \varphi$ (more exactly $\tilde{\varphi}_s a = a\varphi_s$). The notion of Φ-compactification was introduced in [7]. The paper contains some theorems on the existence of Φ-compactifications for metric spaces. Other results are in [3], [4], [12] and [19]. Of course, βX is a Φ-compactification for every family Φ. However, it is known that the weight of βX is much greater than the weight of X and one can set the following problem: *Determine the minimal weight of Φ-compactifications for a given space X and a family Φ.* The paper contains some results concerning this subject.

The paper is divided into two parts. In the first part we consider Φ-compactifications of X preserving the dimension of X. The second part is devoted to investigations of Φ-compactifications of X, where X is a peripherically compact space.