Two set-theoretical theorems in categories

by

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Introduction. The problems treated in this paper (but not completely solved) are the following.

(1) What categories can be embedded in the category $\mathcal{U}$ of all sets and all functions?

(2) What sense can be made of the idea of "category of all functors from $\mathcal{A}$ to $\mathcal{B}$"?

The problems may not be closely related, but the partial results given here depend on some common ideas. The remainder of the introduction states these results, somewhat informally.

Concerning (1), the fact that not all categories can be embedded in $\mathcal{U}$ was discovered independently by P. J. Freyd and the author; it answers a question in [5]. A necessary condition will be given here, and its nature is not complicated although the precise statement is complicated. For any two objects X, Y, supposing they are represented by sets, pairs of mappings $f: X \to Z$, $g: Y \to Z$ may be classified according to the subset K of $X \times Y$, $K = \{(x, y): f(x) = g(y)\}$. Therefore for such a representation to be possible, there must be no more than a set of distinct "equivalence classes" for each X and Y. I do not know whether this condition is sufficient.

Concerning (2), the problem is demonstrably very complicated. There are four successive obstructions to the definition of a category of functors. (A) Unless the domain $\mathcal{A}$ is a small category (i.e. it has only a set of objects), a functor on $\mathcal{A}$ is not a set and therefore not a member of a class, in the usual axiomatic set theories. This obstruction may be avoided by indexing functors with sets—necessarily metamathematically. (B) If the range $\mathcal{B}$ is quite large (e.g. the category $\mathcal{G}$ of all abelian groups), then for any one functor $F: \mathcal{A} \to \mathcal{B}$ there are more functors naturally equivalent to F than the number of sets in the universe. To avoid the difficulty, one must be satisfied with a representative of each "equivalence class". (C) Still there may be more than a universe of pairwise inequivalent functors from $\mathcal{A}$ to $\mathcal{B}$. This happens for $\mathcal{A} = \mathcal{B} = \mathcal{G}$, and it means that we must decide what restricted class of functors will suffice for the intended application. (D) The functors that are used must have the property that any two of them are related by no more than a set of natural transformations.

It will be shown that, in a suitable strong set theory, there are no more obstructions beyond (A)-(D). This suffices to correct [3], in which I handled the whole problem (2) carelessly. Three categories of functors on an arbitrary category $\mathcal{A}$ were introduced, called $\mathcal{P}(\mathcal{A})$, $\mathcal{P}^*(\mathcal{A})$, $\mathcal{R}(\mathcal{A})$. The results on $\mathcal{R}(\mathcal{A})$ are correct, if the special devices for (A) and (B) above are incorporated in the definition. $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}^*(\mathcal{A})$ do not exist, because of (C).

1. Realizable notions. Consider the notion of a *tail* of ordinals, i.e. the class of all ordinals $\geqslant \alpha$ for any fixed α. In the usual formulations of set theory there is no class of all tails; in fact, no class has a tail as an element. However, there is a notion P, where $P(X, Y)$ means that X is an ordinal α and Y is the tail of all ordinals $\geqslant \alpha$. Clearly P establishes metamathematically a one-to-one correspondence between the "non-class" of tails and the class of ordinals. We say that the notion of a tail is *realizable*, and is *realized* by P.

Cantor's theorem, or Russell's paradox, shows that the notion of a class is not realizable. There will be a metaarithmetic associated with the concept of realizability, but that is not our present concern (1). I wish only to add one more definition.

Let N be a unary notion and E a binary notion which establishes an equivalence relation on N-classes. More fully, $E(X, Y)$ implies $N(X)$ and $N(Y)$, and E is reflexive, symmetric, and transitive. The notion N is said to be *realized* (mod E) by a binary notion P if there is a class N^* such that $P(X, Y)$ implies that X is a set and an element of N^* and $N(Y)$; for each $X \in N^*$, there is a unique Y such that $P(X, Y)$; and $N(Y)$ implies that there is (i) at most one X such that $P(X, Y)$, and (ii) at least one X and one Y' such that $P(X, Y')$ and $E(Y', Y)$.

Briefly, N is realizable mod E if there is no more than a universe of E-equivalence classes. One could impose a further restriction to the effect that only one member of each E-class is used, but it seems unnecessary.

(1) These remarks suppose a strong set theory (as in [4]) or a narrowing of the definition of "realizable". In fact, Myhill has shown [6] that an axiom to the effect that the universe is a (non-standard) countable model is consistent with the usual axiom systems. Unfortunately the same result for the axioms of [4] is not known; but we assume them nevertheless.

It should be noted that the whole problem can be recast into ordinary cardinal arithmetic by the device of miniaturization: i.e., considering the mathematical universal class M as a set in a larger universal class V.

2. Representable categories. Grothendieck's definition of a *category* [2] is by now standard (2). The *category of all sets* $\mathcal{U}$ has for objects all sets and for mappings all functions, with multiplication defined by composition. A *concrete category* is a subcategory of $\mathcal{U}$. A *representable category* is a category isomorphic with a concrete category. The following proposition is well known.

2.1. *The dual of a representable category is representable.*

Proof. Let $\mathcal{C}$ be a concrete category. For each object X of $\mathcal{C}$, let X^* denote the set of all subsets of X. For each mapping $f\colon X \to Y$ of $\mathcal{C}$ define $f^*\colon Y^* \to X^*$ by $f^*(S) = f^{-1}(S)$. One verifies at once that $X \to X^*$, $f \to f^*$ is a dual isomorphism. Thus the dual of $\mathcal{C}$ or of any isomorphic category is representable.

Now, let $\mathcal{C}$ be an arbitrary category and X, Y objects of $\mathcal{C}$. Let $L(X, Y) = L$ be the class of all ordered pairs (f, g) of mappings of $\mathcal{C}$ of the form $f\colon W \to X$, $g\colon W \to Y$; similarly let R be the class of all (h, k), $h\colon X \to Z$, $k\colon Y \to Z$. For any subclass S of L, let S^* denote the class of all $(h, k) \in R$ such that $hf = kg$ for all $(f, g) \in S$; and for $T \subset R$, let $T^* = \{(f, g) \in L\colon hf = kg \text{ for all } (h, k) \in T\}$. Call S *equational* (a *left equational class* or *left equational relation*) if $S^{**} = S$; and similarly for right equational classes (relations) $T \subset R$.

In general, the notion of a left equational class for X, Y, need not be realizable. (This can be verified in Example 2.4.) We are interested in the case that this notion is realizable by a set, i.e. realized by a binary notion establishing a one-to-one correspondence between all the equational classes $S \subset L(X, Y)$ and some index set $I(X, Y)$. We not that * establishes a one-to-one correspondence between left and right equational classes; so if one of the notions is realizable by a set, the other is too. In this case we call the pair (X, Y) *classified.*

A set $A \subset L(X, Y)$ is a *left classifying* set if for every $(f, g) \in L(X, Y)$ there is exactly one $(p, q) \in A$ such that $\{(f, g)\}^* = \{(p, q)\}^*$.

2.2. *For every classified pair of objects of a category there exists a left classifying set. If every pair of objects is classified then there exists a function associating a left classifying set to each pair; there also exists a function associating to each pair (X, Y) a complete lattice $I(X, Y)$, in such a way that there is a notion simultaneously realizing all the notions of left equational class of X, Y, upon $I(X, Y)$.*

The proof of all this naturally requires the assumption that the universe can be well-ordered; and with that, it is trivial. For any well-ordering of the universe, for each X, Y, one has the class of all pairs

(2) One should note explicitly (as Grothendieck does not) that all the mappings of the category form a class, and the multiplication table forms another class.

$\pi \in L(X, Y)$ such that $\pi^* \neq \varrho^*$ for every predecessor ϱ of π; either it is a set or (X, Y) is unclassified. Moreover, even if not all pairs are classified there is a well-defined class consisting of all the left classifying sets which this construction gives. The complete lattice $I(X, Y)$ realizes the partial ordering of inclusion on left equational classes; its elements may be defined as suitable subsets of the left classifying set.

2.3. THEOREM. *In a representable category every pair of objects is classified.*

Proof. Let X, Y be objects of a concrete category $\mathcal{C}$. For $f\colon W \to X$, $g\colon W \to Y$, $h\colon X \to Z$, $k\colon Y \to Z$, define $[f, g]\colon W \to X \times Y$ by $[f, g](p) = (f(p), g(p))$; and $h \times k\colon X \times Y \to Z \times Z$ by $h \times k(x, y) = (h(x), k(y))$. If Δ is the diagonal in $Z \times Z$, the condition $hf = kg$ means just $[f, g](W) \subset (h \times k)^{-1}(\Delta)$. Thus for a left classifying set we need only (at most) take pairs (f, g) such that $[f, g](W)$ assumes all possible values.

I do not know whether the condition of 2.3 is sufficient for representability.

2.4. EXAMPLE. *There exists a category in which just one pair of objects is unclassified.*

Construction. Let X, Y be two objects, and $\{A_a\}$, $\{B_a\}$ two proper classes of objects with the same index class. There are identity mappings; for each α, there are four mappings, one each $A_\alpha \to X$, $A_\alpha \to Y$, $X \to B_\alpha$, $Y \to B_\alpha$; the product mapping $A_\alpha \to B_\alpha$ is unique, but for $\beta \neq \alpha$ the mappings $A_\alpha \to X \to B_\beta$ and $A_\alpha \to Y \to B_\beta$ are different. Then the properties are obvious.

3. Realizability for functors. It is becoming customary to speak of the category of all functors (of a fixed variance) from $\mathcal{C}$ to $\mathcal{D}$, whose objects are all such functors and whose mappings are the natural transformations. If $\mathcal{C}$ has only a set of objects, this is perfectly correct. Otherwise, in usual set theories, no functor on $\mathcal{C}$ can be a member of anything; evidently we can form categories only by means of realizations. We want the following metatheorem.

3.1. THEOREM. *Let $\mathcal{C}$, $\mathcal{D}$, be categories, N a notion of functor from $\mathcal{C}$ to $\mathcal{D}$ (of fixed variance), E an equivalence notion on N-functors such that N is realizable* (mod E), *and E implies natural equivalence. Suppose that for every two N-functors F, G, the notion of natural transformation from F to G is realizable by a set $M(F, G)$. Then there exist a category $\mathcal{N}$ and a notion Φ establishing an isomorphic correspondence of all N-functors* (mod E) *with the objects of $\mathcal{N}$ and of all natural transformations of N-functors* (mod E) *with the mappings of $\mathcal{N}$.*

In order to prove this, we need a strong set theory such as the theory of A. P. Morse presented in the appendix to [4], or else restrictions on the notion N and the notion P realizing it (mod E). Let us simply assume the Morse set theory; then any meaningful notion of sets defines the members of a class. We still need a well-ordering of the universe. Then if P realizes N (mod E) upon the class N^*, we can select a subclass N' upon which N is realized (mod E) by unique representatives; that is, different members α, β of N' correspond to non-E-equivalent functors. For any two N-functors F, G, we can realize the notion of natural transformation from F to G upon a particular set $M_0(F, G)$ by noting that these transformations must be determined by their effect on some set of objects of $\mathcal{C}$. (Using well-ordering, we can do this with a notion $M_0^{\#}$.) For α, β in N', we construct $\mathrm{Map}(\alpha, \beta)$ by attaching α and β as distinguishing tags to the elements of $M_0(F, G)$, where $P(\alpha, F)$ and $P(\beta, G)$. Then $\mathcal{N}$ is defined without difficulty. To write out the required notion Φ we need to select, for each N-functor F, a particular natural equivalence of F with its E-equivalent representative F' corresponding by P to $\alpha \in N'$. Take the natural equivalence whose index in $M_0(F, F')$ is first in the well-ordering.

In [3] I attempted to introduce categories of *proper* and of *reflexive* set functors on an arbitrary category $\mathcal{C}$. A proper set functor was defined simply as a set functor dominated by some set of objects of $\mathcal{C}$. This notion need not be realizable modulo natural equivalence.

To see this, consider the full subcategory of Example 2.4 formed by X and the B_α. For any class K of B_α's, there is a covariant set functor dominated by X which associates to X a two-point set, to the B_α in K two-point sets, and to the other B_α one-point sets. No two of these are naturally equivalent, and there are more than a universe of them.

Also, for any single non-trivial set functor F on a category larger than a set, the notion of functor naturally equivalent to F is not realizable absolutely (i.e. modulo identity); for there are more than a universe of them. The category of reflexive set functors is to be taken modulo natural equivalence. The proof that the conditions of 3.1 are satisfied breaks naturally into two parts, around the notion of classifying set for a set functor.

For the remainder of this paper, the reader must be familiar with [3]. Let F be a contravariant set functor on $\mathcal{C}$ which is dominated by a set S of objects of $\mathcal{C}$. For any objects X, Y, in S, let L (depending on X, Y, and S) denote the set of all ordered pairs (f, g) with $f\colon W \to X$, $g\colon W \to Y$, $W \in S$. Let R be $R(X, Y)$ as defined in Section 2. For $T \subset R$, let T_S^* be the intersection of this smaller L with the equational class T^* as defined earlier. Similarly for a subset U of $F(X) \times F(Y)$, U_S^* is the set of all $(f, g) \in L$ such that for each (p, q) in U, $F(f)(p) = F(g)(q)$. A *right classifying set of* X, Y, S, F is a set $A \subset R$ such that every U_S^*, $U \subset F(X) \times F(Y)$, *is* T_S^* for some $T \subset A$.

3.2. *The notion N of a contravariant set functor on $\mathcal{C}$ dominated by a set S of objects relative to which every pair of objects of S has a right classifying set, with the notion E of natural equivalence, satisfies the conditions of* 3.1. *The notion of a reflexive contravariant set functor implies N.*

Proof. This proof can be done in the usual set theories such as [1], weaker than Morse's theory. As usual, the universe is to be well-ordered.

First we must define a class N^* of sets H and a correspondence P associating to each H an N-functor $P(H)$ on $\mathcal{C}$, so that $P(H)$ determines H uniquely and every N-functor is naturally equivalent to at least one $P(H)$. We describe the construction of $P(H)$ simultaneously with the conditions for $H \epsilon N^*$. H must be an ordered triple (H_1, H_2, H_3). H_1 is a contravariant set functor on a small full subcategory $\mathcal{S}$ of $\mathcal{C}$, whose objects form a set S; S is an initial segment of the objects of $\mathcal{C}$ in the fixed well-ordering. H_2 is a function whose domain is the set of all ordered pairs (X, Y) of elements of S. The value $H_2(X, Y)$ is a set $A \subset R(X, Y)$, again an initial segment. H_3 is a "classifying" function defined for every $(p, q) \epsilon H_1(X) \times H_1(Y)$ whose value $H_3(p, q)$ is a non-empty set of subsets of $H_2(X, Y)$. $H_3(p, q)$ must be just the set of all $T \subset H_2(X, Y)$ such that $T^*_S = K$, for some $K \subset L$.

Given such an $H = (H_1, H_2, H_3)$, $P(H)$ is to be a functor extending H_1, dominated by S, with right classifying sets given by H_2 and the precise classification given by H_3. $(\{(p, q)\}^*_S = K.)$ One condition for $H \epsilon N^*$ is that such a functor F exists. Clearly the illegitimate bound variable F can be removed by circumlocution; the "instructions" H_1, H_2, H_3 tell us everything about the functor F except for naming the elements of $F(X)$ for X not in S The precise construction of $P(H)$ must be normalized, say by making the additional sets $P(H)(X)$ out of ordinals not occurring in H_1. Finally (so that $P(H)$ will determine H) for $H \epsilon N^*$ we require that none of the initial segments S or $H_2(X, Y)$ can be replaced by proper segments of themselves.

It remains to check that for two N-functors F, G, the natural transformations from F to G are realizable by a set. This is straightforward, using natural equivalences of F with some $P(H)$, of G with some $P(K)$.

Remark. If F is reflexive and S dominates both F and the conjugate functor F^*, all the required right classifying sets can be found among the mappings with range in S.

We conclude by checking the remark in the introduction that there are more than a universe of inequivalent functors on $\mathcal{G}$ to $\mathcal{C}$. First, consider functors on $\mathcal{U}$ to $\mathcal{U}$. Let K be any class of cardinal numbers >1. (There are more choices of K than the number of sets in the universe.) For each set X in $\mathcal{U}$, we enlarge X by adding a "base point" 0 and, for each cardinal $m \epsilon K$, the set of all ordered m-tuples of distinct elements of X; call the new set $F(X)$. A function f: $X \to Y$ induces $F(f)$: $F(X) \to F(Y)$ as follows. $F(f)|X = f$. An m-tuple $\{x_a\}$ goes to $\{f(x_a)\}$ if all $f(x_a)$ are distinct, otherwise to 0; and 0 goes to 0. These functors are easily seen to be inequivalent. Moreover, they give us inequivalent functors on $\mathcal{G}$ to $\mathcal{G}$ if we map $\mathcal{G} \to \mathcal{U}$ by taking each group to the set of its elements, and $\mathcal{U}$ back to $\mathcal{G}$ by taking a set X to a group freely generated by X.

References

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